Assumptions of Physics
Summer School 2025

Open Problems in Physical Mathematics

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Overview

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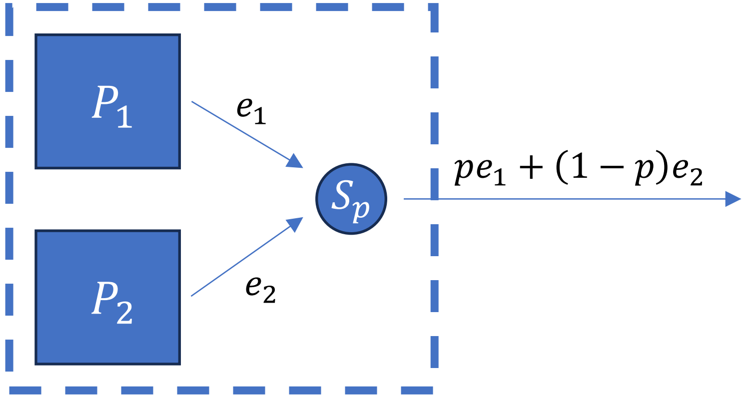
Assumptions of Physics

States and processes

Starting to come together

States based on the notion of ensembles

Process as linear transformation of ensembles



Information granularity

Only vague ideas

Need to define, in general, physical dimensions, units, …

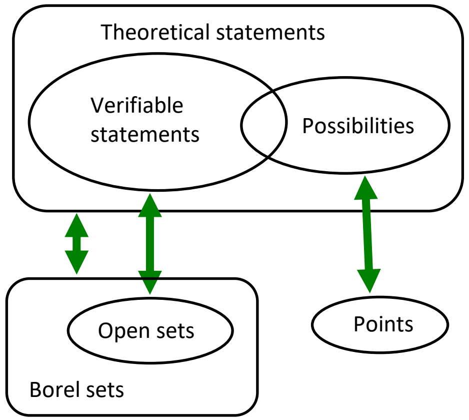
Last attempt from a few years ago

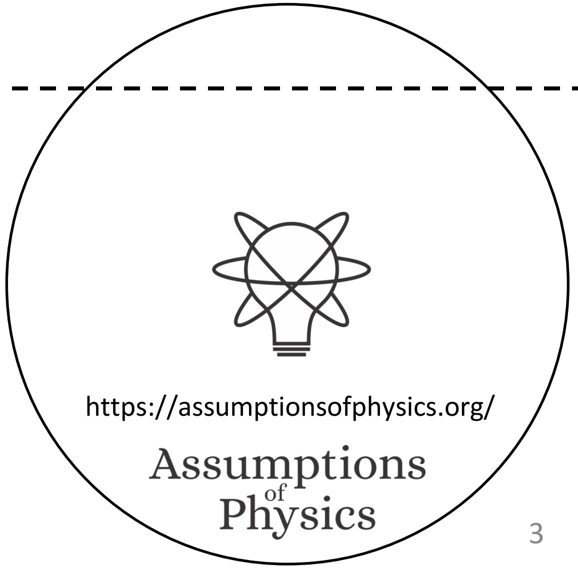
Experimental verifiability

Essentially done, though improvements are always possible

Topologies and σ-algebras recovered from empirical requirements

Real numbers recovered from idealized measurement references





Information granularity: units and information

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Status

- Goal: establish a mathematically precise notion of units, physical dimensions and their required minimal physical assumptions
  - General idea: partial order on all statements where physical dimensions define classes of statements that are comparable to each other and are order isomorphic to the reals
  - Some insights have been gained, but no general solution was found
  - Notes on github, assumptionsofphysics repo /Articles/2020-MeasureTheoryFoundation
- On the backburner as ensemble spaces are more tractable and require non-additive measures
- Need to
  - Clean up what we already have (i.e. the granularity pre-order)
  - Find necessary and sufficient conditions required to recover the appropriate σ-algebras on top of which measures can be defined
  - Understand whether there are problems regarding the non-additive measures we find in ensemble spaces

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Basic insight

Logical relationships ⟺ Topology/σ-algebra

- "The position of the object is between 0 and 1 meters" $\preccurlyeq$ "The position of the object is between 0 and 1 kilometers"
- "The fair die landed on 1" $\preccurlyeq$ "The fair die landed on 1 or 2"
- "The first bit is 0 and the second bit is 1" $\preccurlyeq$ "The first bit is 0"

The logic layer can compare statements if and only if one is "fully contained" in another

Granularity relationships ⟺ Geometry/Probability/Information

- "The position of the object is between 0 and 1 meters" $\underline{\preccurlyeq}$ "The position of the object is between 2 and 3 kilometers"
- "The fair die landed on 1" $\underline{\preccurlyeq}$ "The fair die landed on 3 or 4"
- "The first bit is 0 and the second bit is 1" $\underline{\preccurlyeq}$ "The third bit is 0"

⇒ Measure theory, geometry, probability theory, information theory, … all quantify the level of granularity of different statements

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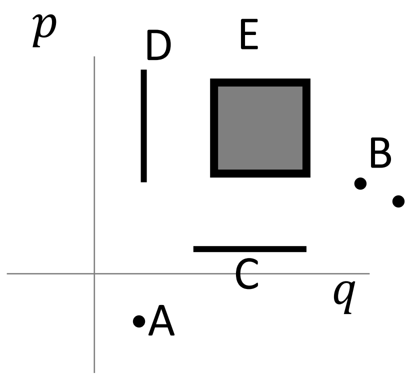
Partial order, physical dimensions and units

Statements over phase space



"The state of the system is in A"

or B, C, D, …

$$A \lesdot B \lesdot C \lesdot E$$

$$C \not\lesdot D \quad D \not\lesdot C$$

A partially ordered set allows us to compare sizes at different levels of infinity and to keep track of incommensurable quantities (i.e. physical dimensions)

Once a "unit" is chosen, a measure quantifies the granularity of another statement with respect to the unit

$$\mu_u : \bar{\mathcal{D}} \to \mathbb{R}$$

$$\mu_u(u) = 1$$

$$s_1 \lesdot s_2 \Rightarrow \mu_u(s_1) \leq \mu_u(s_2)$$

$\mu_u(s_1 \vee s_2) = \mu_u(s_1) + \mu_u(s_2)$ if s_1 and s_2 are incompatible

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Definition 2. *Fineness is a binary relationship* $\dot{\leq} : \bar{\mathcal{D}} \times \bar{\mathcal{D}} \to \mathbb{B}$ *with the following properties:*

- *boundedness: for each contingent statement* $\mathsf{s} \in \bar{\mathcal{D}}$ *we have* $\bot \lessdot \mathsf{s} \lessdot \top$
- *transitivity: if* $\mathsf{s}_1 \dot{\leq} \mathsf{s}_2$ *and* $\mathsf{s}_2 \dot{\leq} \mathsf{s}_3$ *then* $\mathsf{s}_1 \dot{\leq} \mathsf{s}_3$
- *offset monotonicity: if* $\mathsf{s} \not\asymp \mathsf{s}_1$ *and* $\mathsf{s} \not\asymp \mathsf{s}_2$ *then* $\mathsf{s}_1 \dot{\leq} \mathsf{s}_2$ *if and only if* $\mathsf{s} \vee \mathsf{s}_1 \dot{\leq} \mathsf{s} \vee \mathsf{s}_2$.

Minimal assumptions, many consequences

Proposition 3. *Fineness obeys the following properties:*

1. *if* $\mathsf{s} \not\asymp \mathsf{s}_1$ *and* $\mathsf{s} \not\asymp \mathsf{s}_2$ *then* $\mathsf{s}_1 \lessdot \mathsf{s}_2$ *if and only if* $\mathsf{s} \vee \mathsf{s}_1 \lessdot \mathsf{s} \vee \mathsf{s}_2$ *(these two tell us that monotony can be extend to strict fineness and equigranularity)*
2. *if* $\mathsf{s} \not\asymp \mathsf{s}_1$ *and* $\mathsf{s} \not\asymp \mathsf{s}_2$ *then* $\mathsf{s}_1 \doteq \mathsf{s}_2$ *if and only if* $\mathsf{s} \vee \mathsf{s}_1 \doteq \mathsf{s} \vee \mathsf{s}_2$
3. *if* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$ *then* $\mathsf{s}_1 \dot{\leq} \mathsf{s}_2$ *(these three tell us fineness and narrowness are monotonically ordered)*
4. *if* $\mathsf{s}_1 \prec \mathsf{s}_2$ *then* $\mathsf{s}_1 \lessdot \mathsf{s}_2$
5. *if* $\mathsf{s}_1 \equiv \mathsf{s}_2$ *then* $\mathsf{s}_1 \doteq \mathsf{s}_2$
6. $\mathsf{s}_1 \dot{\leq} \mathsf{s}_2$ *if and only if* $\mathsf{s}_1 \wedge \neg\mathsf{s}_2 \dot{\leq} \mathsf{s}_2 \wedge \neg\mathsf{s}_1$ *(these three tell us the region of overlap does not matter)*
7. $\mathsf{s}_1 \lessdot \mathsf{s}_2$ *if and only if* $\mathsf{s}_1 \wedge \neg\mathsf{s}_2 \lessdot \mathsf{s}_2 \wedge \neg\mathsf{s}_1$
8. $\mathsf{s}_1 \doteq \mathsf{s}_2$ *if and only if* $\mathsf{s}_1 \wedge \neg\mathsf{s}_2 \doteq \mathsf{s}_2 \wedge \neg\mathsf{s}_1$
9. *if* $\mathsf{s}_1 \dot{\leq} \mathsf{s}_2$ *then* $\neg\mathsf{s}_2 \dot{\leq} \neg\mathsf{s}_1$ *(these two tell us that fineness works well with negation)*
10. *if* $\mathsf{s}_1 \lessdot \mathsf{s}_2$ *then* $\neg\mathsf{s}_2 \lessdot \neg\mathsf{s}_1$
11. *if* $\top \dot{\leq} \mathsf{s}$ *then* $\top \preccurlyeq \mathsf{s}$ *and therefore* $\mathsf{s} \equiv \top$ *(these two tells us that fineness can recover narrowness for certainties and impossibilities)*
12. *if* $\mathsf{s} \dot{\leq} \bot$ *then* $\mathsf{s} \preccurlyeq \bot$ *and therefore* $\mathsf{s} \equiv \bot$

Proposition 4. *Fineness is a preorder.*

Definition 5. *We say* s_1 *is* ***equigranular*** *to* s_2 *(noted* $\mathsf{s}_1 \doteq \mathsf{s}_2$*) if* $\mathsf{s}_1 \dot{\leq} \mathsf{s}_2$ *and* $\mathsf{s}_2 \dot{\leq} \mathsf{s}_1$.

Proposition 7. *Equigranularity is an equivalence relationship.*

Definition 8. *Let* $\bar{\mathcal{D}}_{/\doteq}$ *be the granularity equivalence classes. Let* $(\cdot)_{/\doteq} : \bar{\mathcal{D}} \to \bar{\mathcal{D}}_{/\doteq}$.

Proposition 9. *The function* $(\cdot)_{/\doteq}$ *is an order homomorphism.*

Basic notions probably good

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Measure from a unit

Given $u \in \bar{\mathcal{D}}$, we want to define $\mu_u : \overline{\mathcal{D}_u} \to [0, +\infty]$ such that:

Is the algebra of comparable statements unique?

When is the measure unique?

$\mu_u(u) = 1$ Measure is one on unit statement

$s_1 \mathrel{\dot{\leq}} s_2 \Rightarrow \mu_u(s_1) \leq \mu_u(s_2)$ Monotonic

$\mu_u(s_1 \vee s_2) = \mu_u(s_1) + \mu_u(s_2)$

if s_1 and s_2 are incompatible

Additive

Are there restrictions on which statements can be taken as units or on the initial space?

Are there compatibility requirements with the topology?

Differentiability

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Status

- Goal: recover differentiability from definitions that are physically clear
  - Current definitions of tangent space and differentials in differential geometry do not work well for physics; definitions used in variational calculus are closer to what is needed
  - Good idea of what are the correct physical concepts we are trying to define
  - An initial prototype is ready
- Need to
  - Finalize the prototype and connect to current literature
  - Reinterpret/rework the use of differential geometry in physics based on the reworked definitions/concepts
  - See if the same notion of derivative can be used as a basis for multiple definitions (e.g. exterior derivative and Radon-Nikodym)

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Differentiability in math

Mathematicians have developed increasingly abstract definitions for differentials, derivatives, integrations, tangent vectors… are they suitable for physics?

Differentiable manifold	
Manifold	Differentiable structure

Changes of coordinates are differentiable

Defined on top of Fréchet derivative

Vector defined as derivation of a scalar function

$$v: C^\infty(X,\mathbb{R}) \to C^\infty(X,\mathbb{R})$$
$$v(f) = v^i \partial_i f$$

vector basis

Does not make sense physically!

- velocity is not a derivation
- momentum is not a function of a derivation
- derivations ∂_i depend on units and can't be summed (e.g. $\partial_r + \partial_\theta$)
- two mathematical notions of differentials (the new one and the one hidden in the Fréchet derivative)
- infinitesimal objects are limits of finite objects, not the other way around

Differentials defined as linear functions of vectors

$$dx: V \to \mathbb{R}$$
$$dx(v) = dx\left(v^i \partial_i\right) = v^x$$

So are covectors, like momentum

Integrals defined on top of differential forms

$$\int_\gamma dx = \Delta x$$

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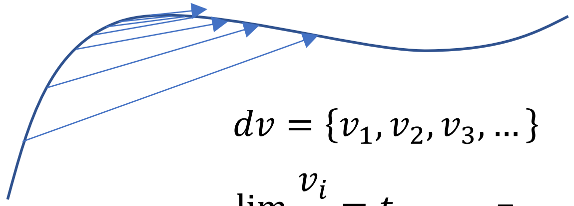
Differentiability in physics



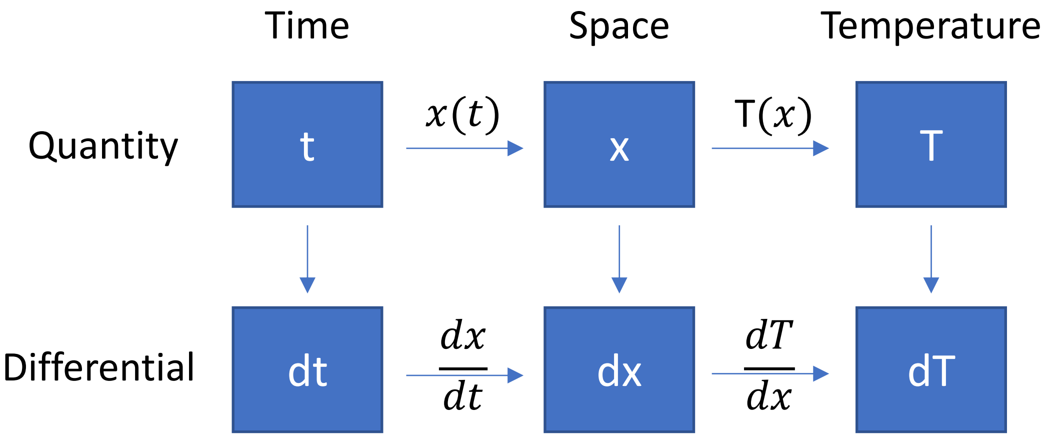
$$dv = \{v_1, v_2, v_3, \dots\}$$

$$\lim_{i\to\infty} \frac{v_i}{a_i} = t$$ Tangent vector

Convergence at all points ⇒ differentiability of curve

Infinitesimal reducibility ⇒ differentiability

General notion of differential as an infinitesimal change in ANY vector space



Differentiable function: infinitesimal changes map to infinitesimal changes

Differentiable space: infinitesimal changes are well-defined

Derivative: map between differentials

$$dx^i = \frac{dx^i}{dt} dt \qquad dT = \frac{\partial T}{\partial x^i} dx^i$$

velocity (vector) — gradient (covector)

Goal: one notion of derivative

$$dx^i e_i = dP$$

Manifold displacement (unit free)

Map between the two

Coordinate displacement (units of x^i)

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Definition 1.2 (Convergence envelope). *A* ***convergence envelope*** $\{a_i\}_{i=1}^{\infty}$ *is a sequence of non-zero elements of* $\mathbb{R}$ *that converges to* 0.

Defines how we go to zero

(locally convex?) topological

Definition 1.3. *Let* V *be a real vector space. A* ***differential*** dv *is a sequence of vectors* $\{v_i\}_{i=1}^{\infty}$ *such that there exists a vector* $t \in V$ *and a convergence envelope* $\{a_i\}_{i=1}^{\infty}$ *for which*

$$\lim_{i \to \infty} \frac{v_i}{a_i} = t.$$

We call t *the* ***tangent vector*** *of the differential and* $\{a_i\}_{i=1}^{\infty}$ *its* ***convergence envelope****. We note* $dv[a_i\, t]$ *the differential with its tangent vector and convergence envelope.*

Only requires the (topological) vector space structure

Closer to what is used in functional analysis, and therefore usable in variational calculus, field theory, etc…

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Proposition 1.4. *Let dv be a differential. It can be expressed as*

$$v_i = a_i t_i = a_i(t + w_i)$$

where $\{t_i\}_{i=1}^{\infty}$ is a sequence of vectors that converges to t and w_i is a sequence of vectors that converges to 0.

Proposition 1.5. *Differentials respect the following property*

$$dv[a_i kt] = dv[ka_i t].$$

Some useful properties

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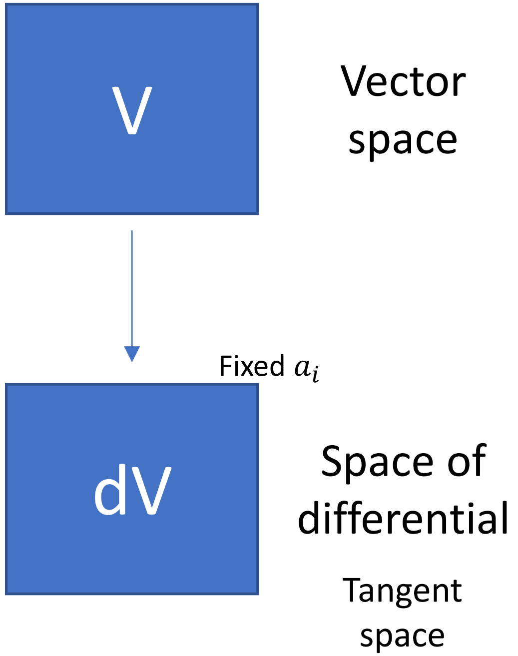
Proposition 1.6. *Differentials with the same convergence envelope form a vector space. That is*

$$b\, dv[a_i\, t] + c\, dv[a_i\, u] = dv[a_i\, (bt + cu)].$$

for any $t, u \in V$ *and* $b, c \in \mathbb{R}$.

Technically, we have a space of differentials for each a_i



In principle, these definitions work for spaces that are locally isomorphic to a vector space of any field

Definition 1.7. *Let V and W be two vector spaces. Given a map $f : V \to W$, a sequence $\{v_i\}_{i=1}^{\infty}$ that converges to some $v \in V$ and a differential $dv[a_i t]$, we define the* ***image of the differential through the map*** *as $df(v_i, dv[a_i t]) = \{f(v_i + a_i t_i) - f(v_i)\}_{i=1}^{\infty}$. The map is* ***differentiable*** *at v if there exists a map $\left.\frac{df}{dV}\right|_v : V \to W$, called* ***derivative*** *such that $df(v_i, dv[a_i t]) = dw[a_i \left.\frac{df}{dV}\right|_v (t)]$ for all $\{v_i\}_{i=1}^{\infty}$ and for all differentials. That is, df maps differentials of V to differentials of W that have the same convergence envelope and a tangent vector that depends only on the original tangent vector.*

Differentiable maps are those that map differentials to differentials

Only the tangent matters, not the convergence

$$v \mapsto f(v)$$

$$a_i(t + \epsilon_i) \mapsto a_i\left(\frac{df}{dv}(t) + \zeta_i\right)$$

Same convergence

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Proposition 1.9. *The derivative must be a linear function.*

It is automatically linear!

Proof. Recall that $dv[a_i\, kt] = dv[ka_i\, t]$, therefore $dw[a_i \left.\frac{df}{dV}\right|_v (kt)] = df(v_i, dv[a_i\, kt]) = df(v_i, dv[ka_i\, t]) = dw[ka_i \left.\frac{df}{dV}\right|_v (t)] = dw[a_i\, k \left.\frac{df}{dV}\right|_v (t)]$. Therefore $\left.\frac{df}{dV}\right|_v (kt) = k \left.\frac{df}{dV}\right|_v (t)$.

We also have

$$\begin{aligned}
dw[a_i \left.\frac{df}{dV}\right|_v (t+u)] &= df(v_i, dv[a_i\, t+u]) = \{f(v_i + a_i(t_i+u_i)) - f(v_i)\}_{i=1}^{\infty} \\
&= \{f(v_i + a_i(t_i+u_i)) - f(v_i + a_i t_i) + f(v_i + a_i t_i) - f(v_i)\}_{i=1}^{\infty} \\
&= \{f((v_i + a_i t_i) + a_i u_i) - f(v_i + a_i t_i)\}_{i=1}^{\infty} + \{f(v_i + a_i t_i) - f(v_i)\}_{i=1}^{\infty} \\
&= df(v_i + a_i t_i, dv[a_i\, u]) + df(v_i, dv[a_i\, t]) \\
&= dw[a_i \left.\frac{df}{dV}\right|_v (u)] + dw[a_i \left.\frac{df}{dV}\right|_v (t)] \\
&= dw[a_i \left.\frac{df}{dV}\right|_v (u) + \left.\frac{df}{dV}\right|_v (t)]
\end{aligned}$$

Generalizations of derivative (e.g. Fréchet) are DEFINED to be linear

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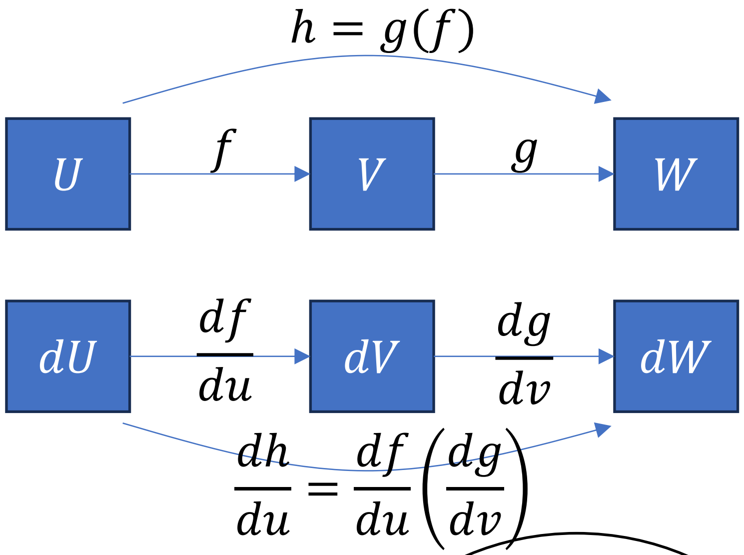
Proposition 1.14 (Chain rule). *Let U, V and W be three vector spaces. Let $f : U \to V$ and $g : V \to W$ be two differentiable maps and $h = g \circ f$ their composition. Then* $\frac{dh}{dU} = \frac{dg}{dV} \circ \frac{df}{dU}$.

Chain rule is simply function composition



Proof. Since f is differentiable, we have $df(u_i, du[a_i t]) = dv[a_i \left.\frac{df}{dU}\right|_u (t)]$ for all $u_i \to u$, convergence envelopes a_i and tangent vectors t. Since g is differentiable, we have $dg(v_i, dv[a_i t]) = dw[a_i \left.\frac{dg}{dV}\right|_v (t)]$ for all $v_i \to v$, convergence envelopes a_i and tangent vectors t. In particular, we have

$$\begin{aligned}
dw[a_i \left.\frac{dg}{dV}\right|_v \left(\left.\frac{df}{dU}\right|_u (t)\right)] &= dg(f(u_i), dv[a_i \left.\frac{df}{dU}\right|_u (t)]) \\
&= dg(f(u_i), df(u_i, du[a_i t])) \\
&= dg(f(u_i), df(u_i, \{a_i t_i\})) \\
&= dg(f(u_i), \{f(u_i + a_i t_i) - f(u_i)\})) \\
&= \{g(f(u_i) + f(u_i + a_i t_i) - f(u_i)) - g(f(u_i))\} \\
&= \{g(f(u_i + a_i t_i)) - g(f(u_i))\} \\
&= \{h(u_i + a_i t_i) - h(u_i)\} \\
&= dh(u_i, du[a_i t]).
\end{aligned} \tag{1.15}$$

Again, proof is half a page

Therefore the image of the differential through h is a differential with convergence envelope a_i and tangent vector $\left.\frac{dg}{dV}\right|_v \left(\left.\frac{df}{dU}\right|_u (t)\right)$. The derivative of h is

$$\frac{dh}{dU} = \frac{dg}{dV} \circ \frac{df}{dU} \tag{1.16}$$

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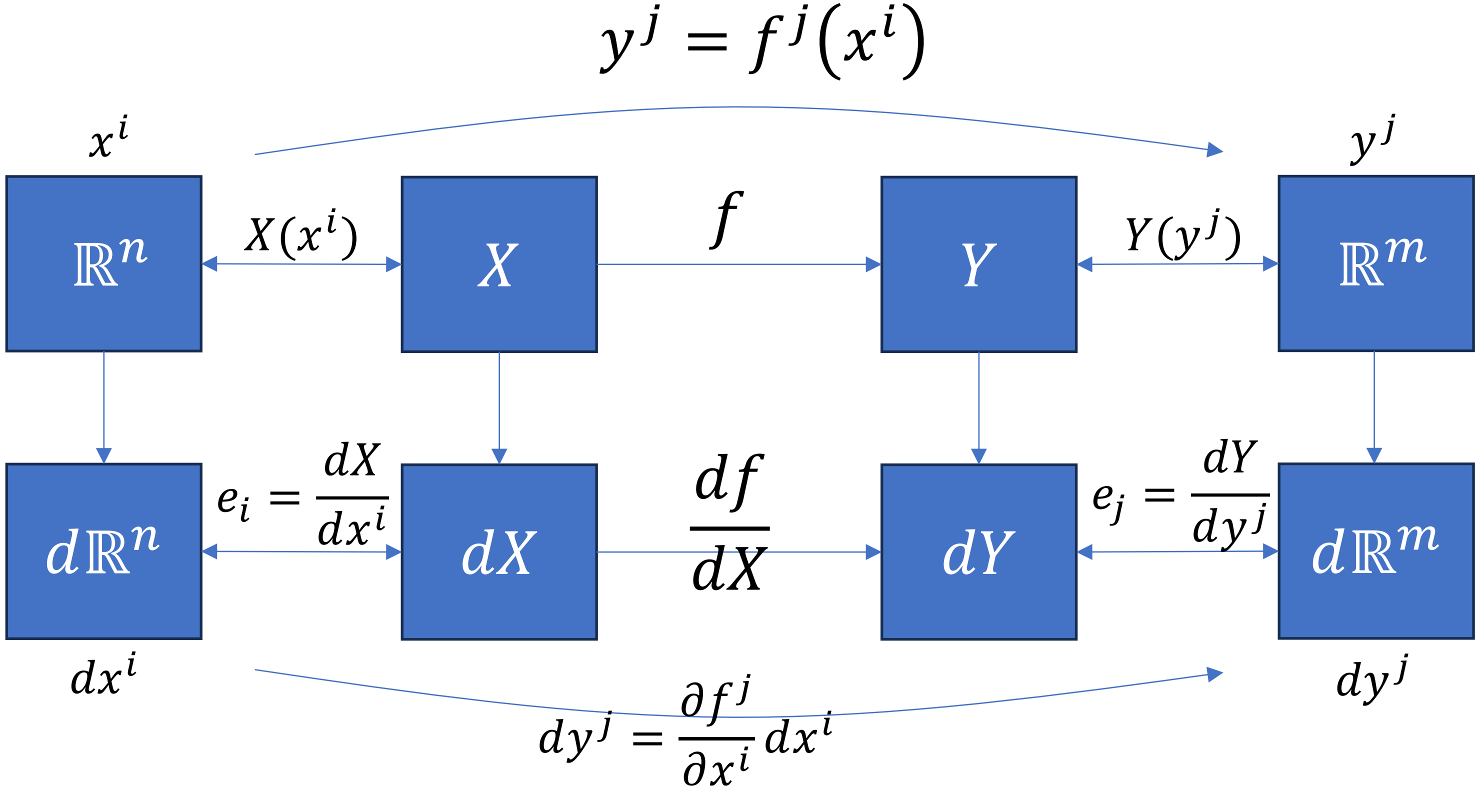
Proposition 1.13. *Let $f : \mathbb{R} \to \mathbb{R}$. Then the standard analytical notions of differentiability and derivative coincide.*

Proposition 1.19. *Let V and W be two normed vector spaces. Then the notion of Fréchet derivative and the new derivative coincide.*

Recovers the standard notion of derivatives in standard cases



"Coordinate basis vectors" e_i are actually derivatives of the coordinate functions

$$dX = e_i dx^i$$

Coordinate/unit independent

Map from unit dep to unit indep

Coordinate/unit dependent

Partial derivatives, Jacobian, tensors, … all find a place in this framework

$$y^j = y^j\left(f\left(X(x^i)\right)\right)$$

$$dy^i = \frac{dy^i}{dY}\frac{df}{dX}\frac{dX}{dx^i}dx^i$$

Ideas are clear, need to clean up all details

Go through all definitions, make sure the notation is actually good for calculation, make sure it works in infinite dimensions (why local convexity?), see what other work exists that it can be integrated with, generalize to non-real vector spaces(?), …

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Differentiability: forms and linear functionals

Starting point: finite values defined on finite regions

k-functional, k-surface, k-form, k-vector

$$f_k(\sigma^k) = \int \theta_k(d\sigma^k)$$

Thinking in terms of relationships between finite objects leads to better physical intuition

Physically measurable quantities

zero-form

Differential forms: infinitesimal limit

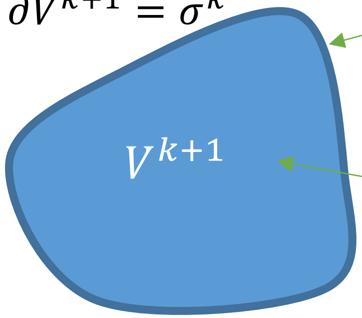
Temperature:	$T(P)$		zero-form
Work:	$W(\gamma) = \sum_i W(\gamma_i) = \int f(d\gamma)$	$f = dW/d\gamma$	one-form
Magnetic flux:	$\Phi(\sigma) = \sum_i \Phi(\sigma_i) = \iint B(d\sigma)$	$B = d\Phi/d\sigma$	two-form
Mass:	$m(V) = \sum_i m(V_i) = \iiint \rho(dV)$	$\rho = dm/dV$	three-form

Assume additivity over disjoint regions

The mathematics is contingent upon the assumption of infinitesimal reducibility (e.g. mass in volumes sums only if boundary effects can be neglected)

We can define functionals that act on boundaries

$\partial V^{k+1} = \sigma^k$



f^k Given a functional

Define higher dimensional functional that acts on the boundary

$$g^{k+1}(V^{k+1}) \equiv f^k(\sigma^k)$$

Exterior functional

$$\partial\partial f^k(\sigma^{k+2}) = f^k(\partial\partial\sigma^{k+2}) = f^k(\emptyset) = 0$$

Boundary of a boundary is the empty set ⇒ exterior derivative of exterior derivative is zero

$g_{k+1} \leftarrow f_k$ (exterior functional)

$g_{k+1} \xrightarrow{d/dV^{k+1}} \omega_{k+1}$, $\omega_{k+1} \xrightarrow{\int_{V^{k+1}}} g_{k+1}$

$f_k \xrightarrow{d/d\sigma^k} \theta_k$, $\theta_k \xrightarrow{\int_{\sigma^k}} f_k$

$\omega_{k+1} \xleftarrow{\partial} \theta_k$ (exterior derivative)

Reversing the exterior derivative is finding a (non-unique) potential

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Topological convex spaces

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Status

- Goal: develop a theory of topological convex spaces
  - Ensemble spaces can be guaranteed to be convex spaces with a second countable T_0 topology
  - Topological vector spaces are important in functional analysis; convex spaces have been studied as abstract objects and in applied optimization theories
  - Little work on topological convex spaces
- Need to
  - Create a self-contained theory
  - Understand when topological convex spaces embed continuously in a topological vector space
  - Understand which results from topological vector spaces can be generalized to topological convex spaces

Axiom 1.4 (Axiom of ensemble). *The state of a system is represented by an* ***ensemble****, which represents all possible preparations of equivalent systems prepared according to the same procedure. The set of all possible ensembles for a particular system is an* ***ensemble space****. Formally, an ensemble space is a T_0 second countable topological space where each*

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.*

Axiom 1.21 (Axiom of entropy). *Every element of the ensemble is associated with an* ***entropy*** *which quantifies the variability of the preparations of the ensemble. Formally, an*

Convex combination: $pa + \bar{p}b$

Ensemble spaces defined from three minimal **necessary** requirements on physical theories

How do convexity and topology interact?

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Definition 1.13 (Infinite mixture). *Let* $\{e_i\}_{i=1}^{\infty} \subseteq \mathcal{E}$ *be a sequence of ensembles and* $\{p_i\} \in [0,1]$ *be a sequence of coefficients such that* $\sum_{i=1}^{\infty} p_i = 1$*, then the ensemble* $\mathsf{a} = \sum_{i=1}^{\infty} p_i \mathsf{e}_i$ *is, if it exists, the topological limit of sequence of finite mixtures* $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{p_i}{p_n} \mathsf{e}_i$*, where* $p_n = \sum_{i=1}^{n} p_i$*.*

Use the topology to extend finite convex combinations to infinite ones

Definition 1.21. *Let* $\mathcal{E}$ *be an ensemble space. We say* $A \subseteq \mathcal{E}$ *is a* ***convex subset*** *of* $\mathcal{E}$ *if it contains all possible mixtures, including infinite ones, of its elements. Formally, it is a topologically closed set that is also closed under convex combinations. If we want to stress that we are closing only under finite mixing, we say that* A *is closed under finite convex combination.*

Extend the notion of convex closure to include infinite convex combinations (i.e. topological closure)

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Many conjectures to be proven or disproven

Conjecture 1.14 (Convergence of submixtures (finite)). *If* $\sum_{i=1}^{\infty} p_i \mathsf{e}_i$ *converges, then* $\sum_{i=2}^{\infty} \frac{p_i}{\bar{p}_1} \mathsf{e}_i$ *converges.*

Conjecture 1.15 (Convergence of submixtures (infinite)). *All the submixtures of an infinite mixture converge. That is, if* $\sum_{i=1}^{\infty} p_i \mathsf{e}_i$ *converge, then* $\sum_{i \in I} p_i \mathsf{e}_i$ *for all* $I \subseteq \mathbb{N}$.

Conjecture 1.27. *Let* $A \subset \mathcal{E}$ *be a set of ensembles and let* $\mathsf{e} \in \mathrm{hull}(A)$. *Then we can find* $\{\mathsf{a}_i\} \subseteq A$ *and* $p_i \in [0,1]$ *such that* $\mathsf{e} = \sum_i p_i \mathsf{a}_i$.

Conjecture 1.28. *The topological closure of the finite convex closure is closed under convex combinations.*

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Assumptions of Physics

Definition 1.44. *A convex space X is* ***cancellative*** *if* $p\mathsf{a} + \bar{p}\mathsf{e} = p\mathsf{b} + \bar{p}\mathsf{e}$ *for some* $p \in (0,1)$ *implies* $\mathsf{a} = \mathsf{b}$.

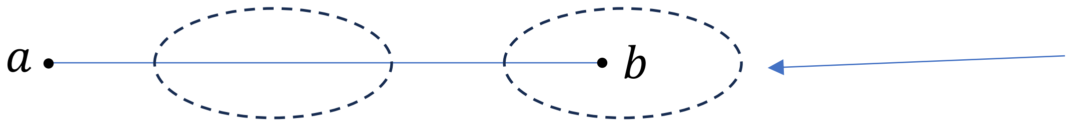
Theorem 1.45 (Ensemble spaces are cancellative)**.** *Let $\mathcal{E}$ be an ensemble space. Let* $\mathsf{a}, \mathsf{b}, \mathsf{e} \in \mathcal{E}$ *such that* $p\mathsf{a} + \bar{p}\mathsf{e} = p\mathsf{b} + \bar{p}\mathsf{e}$ *for some* $p \in (0,1)$*. Then* $\mathsf{a} = \mathsf{b}$.

Cancellative convex spaces embed into vector spaces

When is the embedding continuous? For example, can a T_1 second countable cancellative topological convex space embed non-continuously in a topological vector space?

Finite dimensional topological vector spaces with the same dimension are isomorphic (i.e. they have a unique topology). Does something similar hold for topological convex spaces?

The edges of the topological convex space create topologically "different regions"



Open sets with extreme points are different from open sets with only internal points

Is this the only problem? For example, do two internal points always have two neighborhoods that are isomorphic?

Any second countable T_1 topological vector space is automatically metrizable

Does something similar hold for topological convex spaces?

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Assumptions of Physics

Entropic geometry

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Assumptions of Physics

Status

- Goal: derive all geometric structure from a definition of entropy
  - From Reverse Physics, we know all geometric structures in physics are related to the entropy, so there should be a general theory to recover all results more rigorously
- Need to
  - Understand the role of entropy in defining limits, and therefore the topology of ensemble spaces
  - Understand what properties can be generalized from classical and quantum spaces, and how can everything work in field theories (connects to the work on differentiability)

How much does the entropy increase during mixture?

$$MS(\mathsf{a},\mathsf{b}) = S\left(\frac{1}{2}\mathsf{a} + \frac{1}{2}\mathsf{b}\right) - \left(\frac{1}{2}S(\mathsf{a}) + \frac{1}{2}S(\mathsf{b})\right)$$

1. *non-negativity:* $MS(\mathsf{a},\mathsf{b}) \geq 0$
2. *identity of indiscernibles:* $MS(\mathsf{a},\mathsf{b}) = 0 \iff \mathsf{a} = \mathsf{b}$
3. *unit boundedness:* $MS(\mathsf{a},\mathsf{b}) \leq 1$
4. *maximality of orthogonals:* $MS(\mathsf{a},\mathsf{b}) = 1 \iff \mathsf{a} \perp \mathsf{b}$
5. *symmetry:* $MS(\mathsf{a},\mathsf{b}) = MS(\mathsf{b},\mathsf{a})$

Recovers the Jensen-Shannon divergence (JSD)
(both classical and quantum)

Pseudo-distance defined from the entropy

(does not satisfy the triangle inequality)

In classical and quantum spaces, it is the square of a distance function.
Is it true in general? Is it true for orthogonally decomposable spaces?

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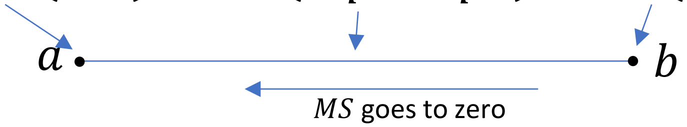
Proposition 1.89. *Let* $\mathsf{a}, \mathsf{b} \in \mathcal{E}$. *Then* $MS(\mathsf{a}, \mathsf{b}) \geq MS(\mathsf{a}, p\mathsf{a} + \bar{p}\mathsf{b})$ *with the equality holding if and only if* $p = 0$ *or* $\mathsf{a} = \mathsf{b}$.

$$0 = MS(a, a) \leq MS(a, pa + \bar{p}b) \leq MS(a, b)$$



Does the mixing entropy fully specify the criteria of convergence and therefore the topology?

Conjecture 1.98. *For every sequence* $\mathsf{a}_i \in \mathcal{E}$, $\mathsf{a}_i \to \mathsf{a}$ *if and only if* $MS(\mathsf{a}, \mathsf{a}_i) \to 0$.

Proof. Let $\mathsf{a}_i \to \mathsf{a}$, fix $r \in \mathbb{R}$ and consider $B_r(\mathsf{a})$. This is an open set, therefore we will be able to find some k such that $\mathsf{a}_i \in B_r(\mathsf{a})$ for all $i \geq k$. But this means that $MS(\mathsf{a}, \mathsf{a}_i) < r$ for all $i \geq k$. Therefore $MS(\mathsf{a}, \mathsf{a}_i) \to 0$.

Second part?

Definition 1.95. *Given* $\mathsf{a} \in \mathcal{E}$ *and* $r \in [0, 1]$, *an entropic open ball is the set of all ensembles for which the mixing entropy from* a *is within* r. *That is,* $B_r(\mathsf{a}) = \{\mathsf{e} \in \mathcal{E} \mid MS(\mathsf{a}, \mathsf{e}) < r\}$.

Equivalently: is the topology generated by entropic open balls?

Is the subtopology generated by the entropic open balls metrizable?
Does it make the space metrizable?

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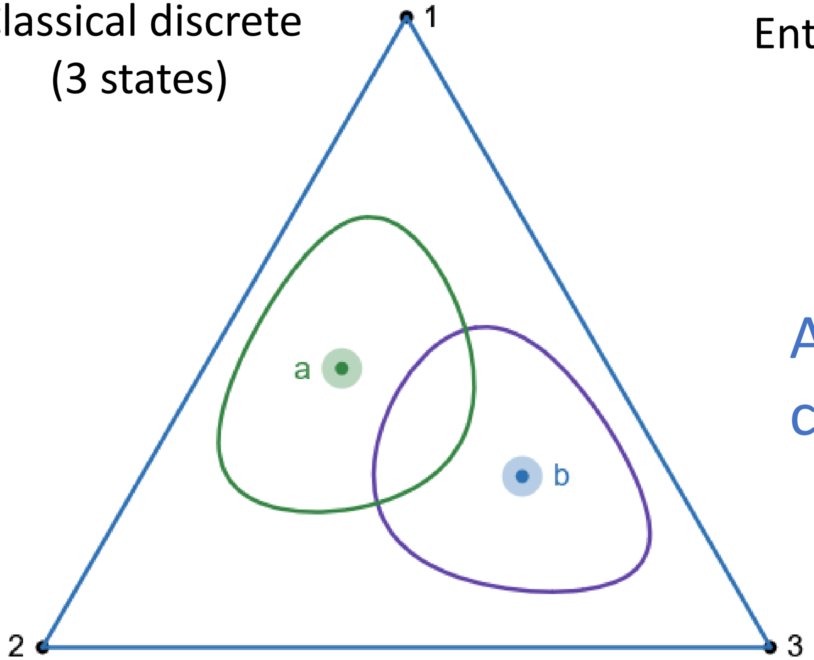
Assumptions of Physics

Classical discrete
(3 states)

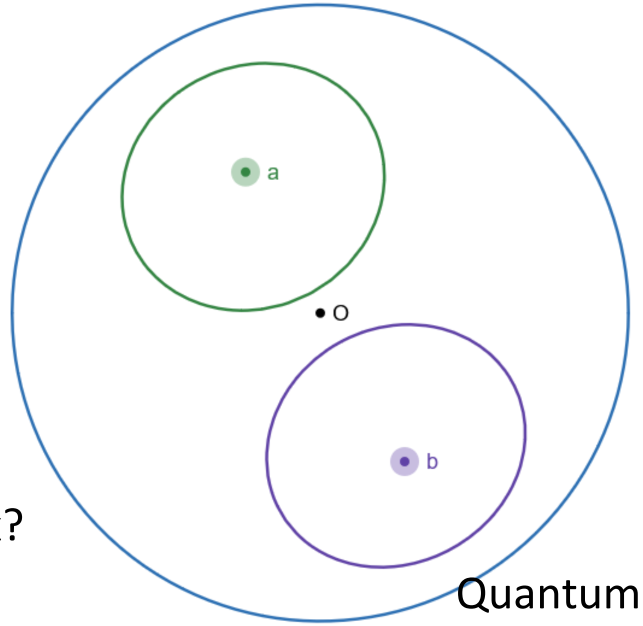


Entropic open balls of radius 0.04

They appear to be convex

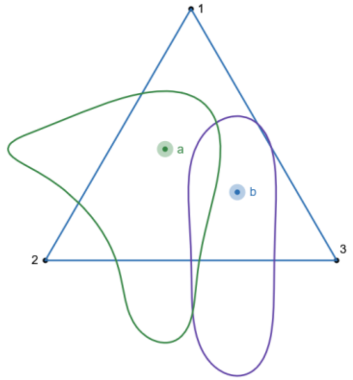
Are entropic open balls convex in classical and quantum spaces?

Is this what makes the spaces locally convex?



Quantum
(slice of Bloch sphere)

Not enough in the axiom to make them convex.
In the example to the right, the entropy is given by $-(x^4 + y^4)$,
which is concave and satisfies the entropic bounds
in appropriate units (without ever reaching the upper one)



Are the entropic open balls convex if the space is orthogonally decomposable?

Does it matter?

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Assumptions of Physics

Entropy imposes a metric on the ensemble space

$$\|\delta \mathsf{e}\|_{\mathsf{e}} = \sqrt{8MS(\mathsf{e}, \mathsf{e} + \delta \mathsf{e})}$$

$$g_{\mathsf{e}}(\delta \mathsf{e}_1, \delta \mathsf{e}_2) = \frac{1}{2}\left(\|\delta \mathsf{e}_1 + \delta \mathsf{e}_2\|_{\mathsf{e}}^2 - \|\delta \mathsf{e}_1\|_{\mathsf{e}}^2 - \|\delta \mathsf{e}_2\|_{\mathsf{e}}^2\right)$$

$$\Rightarrow \quad g_{\mathsf{e}}(\delta \mathsf{e}_1, \delta \mathsf{e}_2) = -\frac{\partial^2 S}{\partial \mathsf{e}^2}(\delta \mathsf{e}_1, \delta \mathsf{e}_2).$$

Entropy strict concavity means the Hessian is negative definite

Recovers Fisher-Rao information metric (both classical and quantum)

Make sure this works properly in infinite dimensional cases. Links to the definition of differentiability and notion of information in field theories.

More ill-defined problems

Finding the right problem definition often IS the problem

How does the geometry of the space of ensembles relate to the geometry of the space of the states?

Given that $0 \leq MS(a,b) \leq 1$ with 0 only when $a = b$ and 1 only when $a \perp b$, can MS be turned into an inner product? Can we show that ensemble spaces are, in general, inner product spaces?

Can we generalize some results from entropic geometry (e.g. Holevo bound) to ensemble spaces?

Generalized measure theory

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Assumptions of Physics

Status

- Goal: find a structure that generalizes classical probability and state count that works on the generalized ensemble space
  - A non-additive generalization of probability naturally arises by asking what fraction of an ensemble can be recovered from a set of other ensembles
  - A non-additive generalization of state count naturally arises by asking how many distinguishable states is an ensemble spread over
  - The idea is to develop a version of non-additive calculus that restricts those measures to a set of “limit states” and recovers the notion of probability density, expectation values, etc…
- Need to
  - Understand how to get the space of limit points
  - Understand the proper generalization of integral/derivative

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Assumptions of Physics

Probability space: $(X, \Sigma_X, p: \Sigma_X \to [0,1])$

Sample space – What can happen

Events – What can be tested

Probability measure – Probability of positive outcome of a test

Probability in classical mechanics

State count: $\mu: \Sigma_X \to [0, +\infty]$

Count of states corresponding to each event

Absolutely continuous
Zero states ⇒ zero probability

$$\Rightarrow \rho = \frac{dp}{d\mu}$$

Probability density – Radon-Nikodym derivative

Random variable: $f: X \to \mathbb{R}$

Assigns a value to every possible outcome

$$\Rightarrow E[f] = \int f \rho d\mu$$

Expectation – Average value of the random variable

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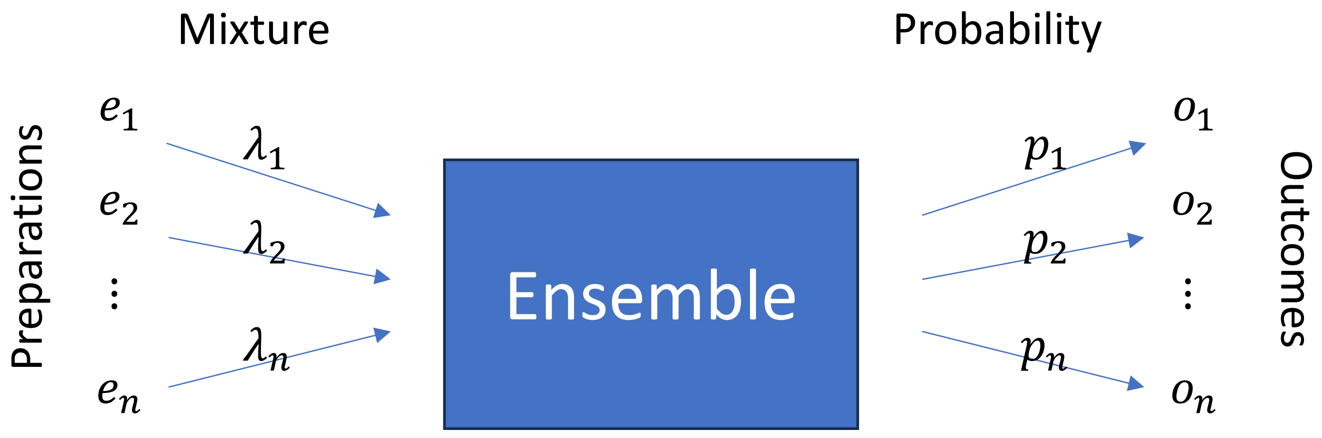
Assumptions of Physics



In classical mechanics, mixtures of preparations and probability of outcomes always coincide

In quantum mechanics, they do not

⇒ quantum ensemble spaces not simplexes (i.e. classical probability fails)

Can we have common measure theoretic tools on the preparation side?

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Assumptions of Physics

Given a set of possible ensembles A, the count of configurations is the exponential of the maximum entropy reachable using mixtures. If A is the set of classical distributions over a particular support U, the maximum entropy is given by the uniform distribution ⇒ recovers the usual count of states! If A is the set of density matrices that has zero eigenvalues outside of a subspace H, the state capacity recovers the dimensionality of the space ⇒ count of distinguishable states!

Definition 4.133. *Let $A \subseteq \mathcal{E}$ be a subset of an ensemble space. The **state capacity** of A is defined as* $\text{scap}(A) = \sup(2^{S(\text{hull}(A))} \cup \{0\})$.

capacity also name of a non-additive measure

Proposition 4.134. *The state capacity is a set function that is*

1. *non-negative:* $\text{scap}(A) \in [0, +\infty]$
2. *monotone:* $A \subseteq B \implies \text{scap}(A) \leq \text{scap}(B)$
3. *subadditive:* $\text{scap}(A \cup B) \leq \text{scap}(A) + \text{scap}(B)$
4. *additive over orthogonal sets:* $A \perp B \implies \text{scap}(A \cup B) = \text{scap}(A) + \text{scap}(B)$

fuzzy measure

State capacity is a non-additive measure

additive over orthogonal sets

How much of e is a mixture of other ensembles?

$e = p(\sum_i \lambda_i a_i) + \bar{p}b$

biggest p

Definition 1.83. *Let* $\mathsf{e}, \mathsf{a} \in \mathcal{E}$ *be two ensembles. The* ***fraction*** *of* a *in* e *is the greatest mixing coefficient for which* e *can be expressed as a mixture of* a*. That is,* $\mathrm{frac}_{\mathsf{e}}(\mathsf{a}) = \sup(\{p \in [0,1] \mid \exists\, \mathsf{b} \in \mathcal{E} \text{ s.t. } \mathsf{e} = p\mathsf{a} + \bar{p}\mathsf{b}\})$.

Definition 1.85. *Let* $\mathsf{e} \in \mathcal{E}$ *be an ensemble and* $A \subseteq \mathcal{E}$ *a Borel set. The* ***fraction capacity*** *of* A *for* e *is the biggest fraction achievable with convex combinations of* A*. That is,* $\mathrm{fcap}_{\mathsf{e}}(A) = \sup(\mathrm{frac}_{\mathsf{e}}(\mathrm{hull}(A)) \cup \{0\})$.

Proposition 1.87. *The fraction capacity for an ensemble is a set function that is*

1. *non-negative and unit bounded:* $\mathrm{fcap}_{\mathsf{e}}(A) \in [0,1]$
2. *monotone:* $A \subseteq B \implies \mathrm{fcap}_{\mathsf{e}}(A) \leq \mathrm{fcap}_{\mathsf{e}}(B)$
3. *subadditive:* $\mathrm{fcap}_{\mathsf{e}}(A \cup B) \leq \mathrm{fcap}_{\mathsf{e}}(A) + \mathrm{fcap}_{\mathsf{e}}(B)$
4. *continuous from below:* $\mathrm{fcap}_{\mathsf{e}}(\lim\limits_{i\to\infty} A_i) = \lim\limits_{i\to\infty} \mathrm{fcap}_{\mathsf{e}}(A_i)$ *for any increasing sequence* $\{A_i\}$
5. *continuous from above:* $\mathrm{fcap}_{\mathsf{e}}(\lim\limits_{i\to\infty} A_i) = \lim\limits_{i\to\infty} \mathrm{fcap}_{\mathsf{e}}(A_i)$ *for any decreasing sequence* $\{A_i\}$

fuzzy measure

Fraction capacity is a non-additive probability measure

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Assumptions of Physics

Statistical quantities (i.e. expectation values on ensembles)

Definition 1.126. *A* ***statistical property****, or simply property, is an attribute that allows statistical handling. Formally, it is a continuous map $F : \mathcal{E} \to \mathcal{Q}$ where $\mathcal{Q}$ is a convex topological space such that $F(p\mathsf{e}_1 + \bar{p}\mathsf{e}_2) = pF(\mathsf{e}_1) + \bar{p}F(\mathsf{e}_2)$.*

A ***statistical quantity****, or statistical variable, or simply variable, is a numerical statistical property. That is, it is a continuous linear real valued operator $F : \mathcal{E} \to \mathbb{R}$.*

Since ensembles are basic objects, we define quantities as real linear functions of ensembles: they correspond to expectation values of random variables

$$F(e) = \int f \rho_e d\mu$$

Classical case

$$F(e) = tr[O_f \rho_e]$$

Quantum case

Show that statistical quantities are one-to-one with respect to classical random variables and quantum observables

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Assumptions of Physics

Both fraction capacity and state capacity are defined on the ensemble space

$$\text{scap}: \Sigma_{\mathcal{E}} \to [0, +\infty] \qquad \text{fcap}_e: \Sigma_{\mathcal{E}} \to [0,1]$$

Need to:

Construct the space of “perfect/pure” states from the ensemble space

E.g. extreme points if they exist (e.g. classical discrete and quantum)

limit distributions – delta Diracs – if there are no extreme points (e.g. classical mechanics)

Confirm the fraction capacity uniquely represents each ensemble

Define a notion of derivative and integral such that we can write

$$\rho_e = \frac{d\text{fcap}_e}{d\text{scap}}$$

$$F = E[f] = \int f \rho_e d\text{scap}$$

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Assumptions of Physics

Note: there are already non-additive measure theoretic tools!

In mathematics, **fuzzy measure theory** considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure (also *capacity*, see [1]), which was

A **Choquet integral** is a subadditive or superadditive integral created by the French mathematician Gustave Choquet in 1953.[1] It was initially used in statistical mechanics and potential theory,[2] but found its way into decision theory in the 1980s,[3] where it is used

Then the Choquet integral of f with respect to ν is defined by:

$$(C)\int f d\nu := \int_{-\infty}^{0} (\nu(\{s|f(s) \geq x\}) - \nu(S))\, dx + \int_{0}^{\infty} \nu(\{s|f(s) \geq x\})\, dx$$

In mathematics, the **Sugeno integral**, named after M. Sugeno,[1] is a type of integral with respect to a fuzzy measure.

The **Sugeno integral over the fuzzy set** $\tilde{A}$ of the function h with respect to the fuzzy measure g is defined by:

$$\int_A h(x) \circ g = \int_X [h_A(x) \wedge h(x)] \circ g$$

Unfortunately, they do not seem to be the right ones…

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Assumptions of Physics

Generalized Poisson structure

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Assumptions of Physics

Status

- Goal: find a single structure that generalizes classical Poisson brackets and quantum commutators that works on the generalized ensemble space
  - Saying that "the difference between classical and quantum mechanics is whether observables commute" is nonsensical: the Poisson bracket, which defines classical spaces, is formally equivalent to the commutation relationship, which defines quantum spaces
  - The algebraic structures are the same, but "implemented" on different spaces
  - It is likely that this requires another axiom
- Need to
  - Understand how to define this common structure
  - Create a theory of processes/transformations that is common on all ensemble spaces
  - Show that this reduces to the one of classical/quantum mechanics in the appropriate conditions

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Assumptions of Physics

$s(t) \to s(t + dt)$

H - generator of the transformation

t - parameter of the generated transformation

$$\frac{df}{dt} = \{f, H\} \qquad \frac{dO}{dt} = \frac{[O, H]}{\imath\hbar}$$

A skew-symmetric bracket that tells you how a variable/observable changed during the transformation

This is what we need to generalize

In ensemble space, we have statistical quantities

Definition 1.126. *A* ***statistical property****, or simply property, is an attribute that allows statistical handling. Formally, it is a continuous map $F : \mathcal{E} \to \mathcal{Q}$ where $\mathcal{Q}$ is a convex topological space such that $F(p\mathsf{e}_1 + \bar{p}\mathsf{e}_2) = pF(\mathsf{e}_1) + \bar{p}F(\mathsf{e}_2)$.*

A ***statistical quantity****, or statistical variable, or simply variable, is a numerical statistical property. That is, it is a continuous linear real valued operator $F : \mathcal{E} \to \mathbb{R}$.*

$$F(e) = \int f \rho_e d\mu$$

Classical case

$$F(e) = tr[O_f \rho_e]$$

Quantum case

Define bracket as $\{F, G\}_{\mathcal{E}}(e) =$

$$\int \{f, g\}_P \rho_e d\mu$$

$$tr\left[[O_f, O_g]\rho_e\right]$$

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$$\{F, G\}_{\mathcal{E}}(e) =$$

$$\int \{f, g\}_P \rho_e d\mu \qquad\qquad tr\left[[O_f, O_g]\rho_e\right]$$

- *Bilinearity*,

  $$[ax + by, z] = a[x, z] + b[y, z],$$
  $$[z, ax + by] = a[z, x] + b[z, y]$$

  for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.
- The *Alternating* property,

  $$[x, x] = 0$$

  for all x in $\mathfrak{g}$.
- The *Jacobi identity*,

  $$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

  for all x, y, z in $\mathfrak{g}$.

It’s a Lie algebra!

Since both integral and trace are linear operators, properties are inherited from Poisson bracket/commutator

$$\{F, G\}_{\mathcal{E}}(e) =$$

$$\int \{f, g\}_P \rho_e d\mu \qquad\qquad tr\left[[O_f, O_g]\rho_e\right]$$

The Poisson bracket acts as a derivation of the associative product ·,

Is it a Poisson bracket?

What would the product be?

It’s not the product of statistical variables because $E[fg] \neq E[f]E[g]$

Can it be defined in terms of the generator of the composed transformation?

Can we define non-commutativity of variables based on non-commutativity of composed transformations? Can we find a relationship between the expectation of the product and the commutativity, or lack thereof?

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Assumptions of Physics

What new axiom is needed (if at all)?

$$\frac{df}{dt} = \{f, H\} \qquad \frac{dO}{dt} = \frac{[O, H]}{\imath\hbar}$$

These are both deterministic and reversible transformations: they leave the entropy unchanged

Also, this structure guarantees us that there are always stationary ensembles under each deterministic and reversible transformation (e.g. Boltzmann distributions)

Can we turn it around? We already assumed ensembles must be equilibria to have an entropy well-defined. Is assuming that every ensemble is a stationary state of some time evolution enough?

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Assumptions of Physics

Wrapping it up

- We strongly believe it is possible to create a solid mathematical framework for all of physics that starts from physical requirements
  - Mathematicians can be guaranteed those axioms are satisfied
  - We have a clear idea of what we are talking about
- To do so, we need to construct all the needed mathematics from the ground up
  - Sometimes this means reorganizing core mathematical ideas in a different way that may seem confusing/pointless to a mathematician
  - On the other hand: new interesting math to be developed!
- This is a huge undertaking that spans multiple different branches of math, physics, information theory, …
  - We need a coordinated effort from a group of very heterogeneous experts

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Assumptions of Physics

How to contribute to Physical Mathematics

- Passive contribution: make yourself available as a “consultant”
  - Don’t have to follow the project, called only if there is something relevant that matches your background/expertise
  - Occasional discussion/review of material to make sure things make sense from multiple perspectives
- Active contribution: case-by-case
  - Need to have sufficient background or be able to get it independently
  - Much better chance of success if already working/expert in a research area

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