Assumptions of Physics
Summer School 2025

Introduction to Physical Mathematics

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Main goal of the project

Assumptions of Physics

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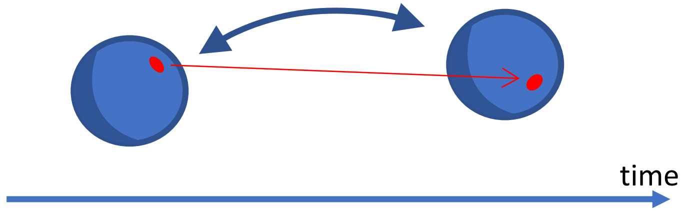
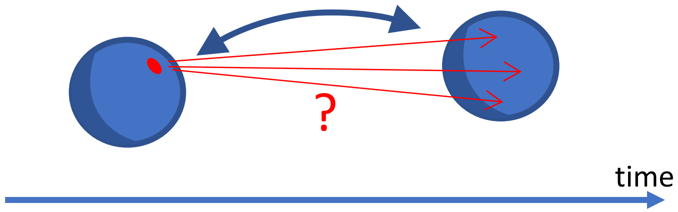
Identify a handful of physical starting points from which the basic laws can be rigorously derived

For example:

Infinitesimal reducibility ⇒ Classical state



Irreducibility ⇒ Quantum state



This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based ⇒ scientific theory must be characterized by experimentally verifiable statements ⇒ topologies and σ-algebras

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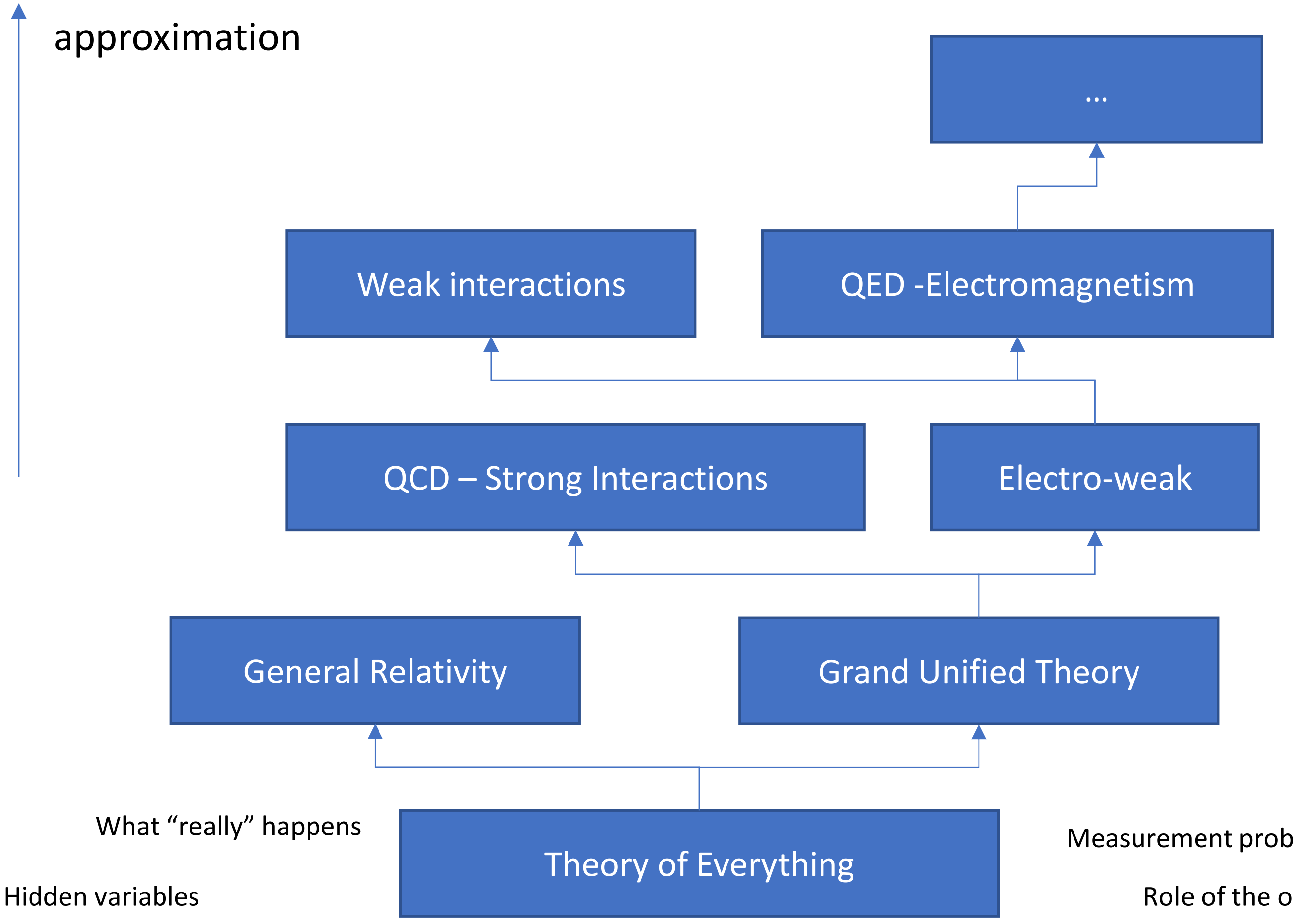
Assumptions of Physics

Standard view of the foundations of physics



What “really” happens

Hidden variables

Ontology of observables

Perfect description of the universe

Measurement problem

Role of the observer

Dark matter/energy

Goal of physics is to find the true laws of the universe!

The “real” physics!

The foundations
of physics!

Everything else
is an approximation

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We found:

Experimental verifiability $\Rightarrow$ topologies and σ-algebras

Geometrical structures $\Longleftrightarrow$ Entropic structures

Hamiltonian evolution $\Longleftrightarrow$ det-rev/isolation + DOF independence

Massive particles and potential forces $\Longleftrightarrow$ ↘ + Kinematic eq

Physical requirements and assumptions drive most of the theoretical apparatus

~~Goal of physics is to find the true laws of the universe!~~

Less productive point of view

Goal of physics is to find models that can be empirically tested

More productive point of view

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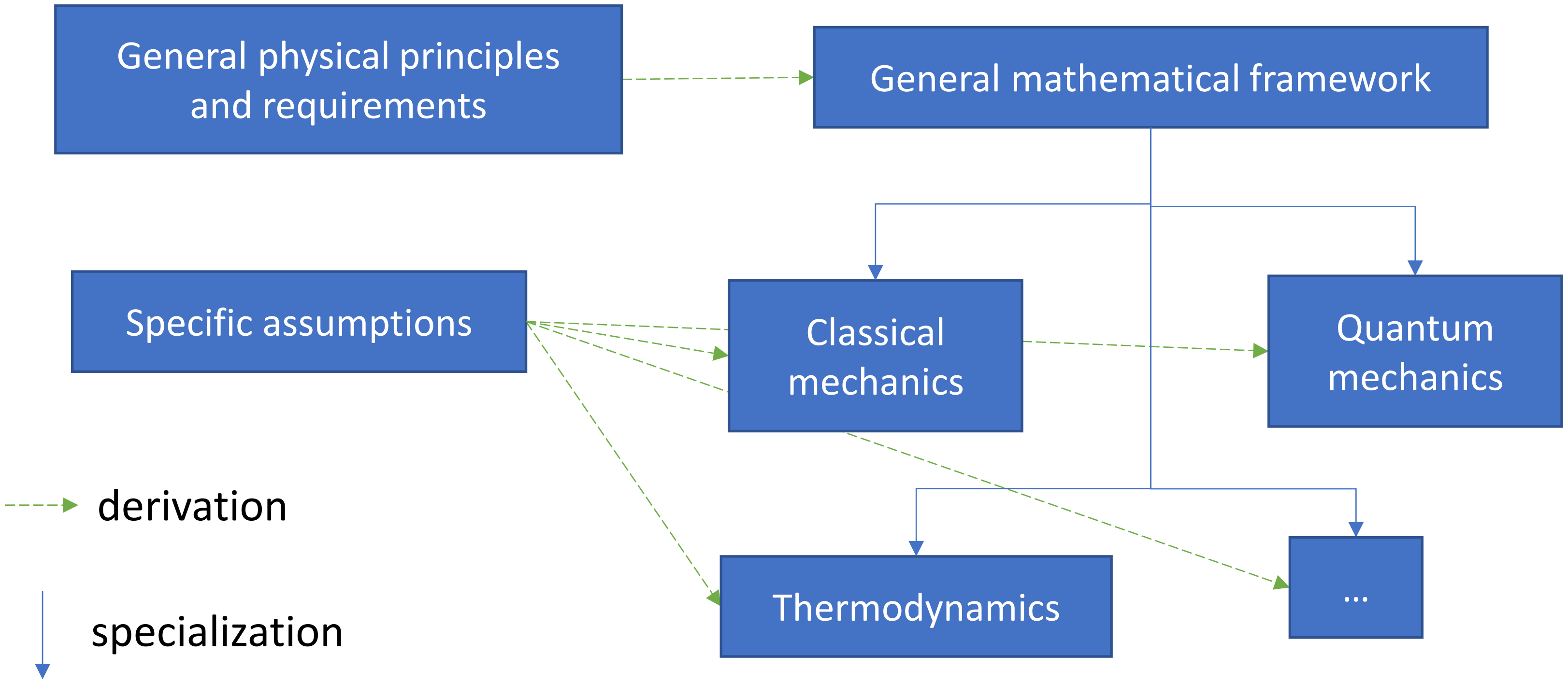
Assumptions of Physics

Our view of the foundations of physics



General physical principles and requirements

General mathematical framework

Specific assumptions

Classical mechanics

Quantum mechanics

Thermodynamics

…

derivation

specialization

Foundations of physics

⇕

The theory of physical models

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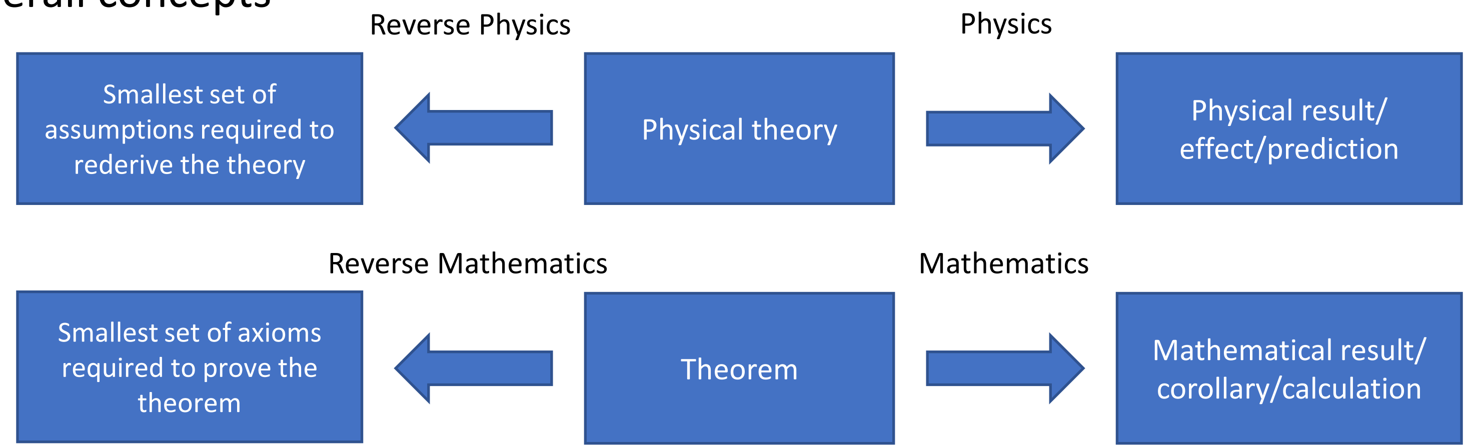
Assumptions of Physics

Find the right overall concepts

Reverse physics:

Start with the equations, reverse engineer physical assumptions/principles

Found Phys **52**, 40 (2022)



Reverse Physics

Physics

Smallest set of assumptions required to rederive the theory

Physical theory

Physical result/ effect/prediction

Reverse Mathematics

Mathematics

Smallest set of axioms required to prove the theorem

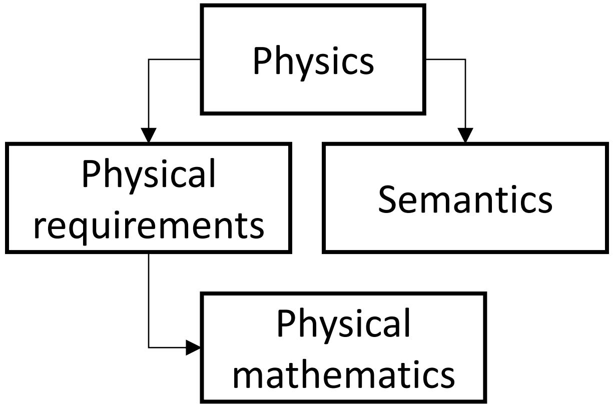
Theorem

Mathematical result/ corollary/calculation

Goal: find the right overall physical concepts, “elevate” the discussion from mathematical constructs to physical principles

Physical mathematics:

Start from scratch and rederive all mathematical structures from physical requirements



Physics

Physical requirements

Semantics

Physical mathematics

Goal: get the details right, perfect one-to-one map between mathematical and physical objects

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The fundamental misunderstanding in the foundations of physics

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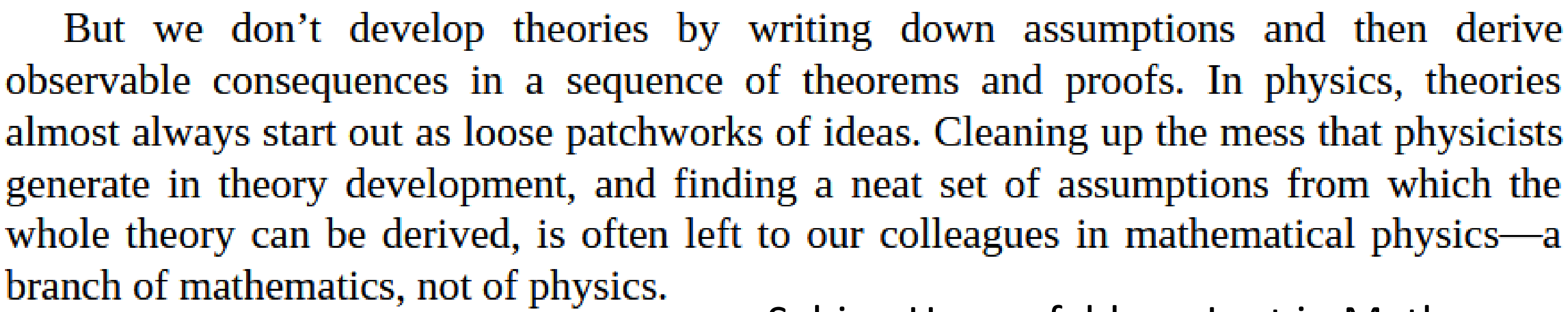
Prevalent attitude among physicists

Math is a just tool for calculation, whose technical details are better left to mathematicians



But we don't develop theories by writing down assumptions and then derive observable consequences in a sequence of theorems and proofs. In physics, theories almost always start out as loose patchworks of ideas. Cleaning up the mess that physicists generate in theory development, and finding a neat set of assumptions from which the whole theory can be derived, is often left to our colleagues in mathematical physics—a branch of mathematics, not of physics.

Sabine Hossenfelder – Lost in Math

Even those that work on the math,
they work on it as mathematicians

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Assumptions
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We use mathematics to specify our models, not just calculations, and specifying physical models is the whole point of physics

Also, there is no single way to “clean up the mess”: each axiom and definition represents a choice in mathematical modeling

Those are physical choices, which mathematicians are ill-equipped to make

So we end up with **THE WRONG MATH**

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Examples of unphysical mathematics

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In differential geometry, tangent vectors are derivations

$$v: C^\infty(X) \to C^\infty(X)$$

$$v = v^i \partial_i$$

component

basis

In polar coordinates

$$\partial_r + \partial_\theta = ???$$

$$[\mathrm{m}] \qquad [\mathrm{rad}]$$

Doesn’t work with units

In phase space

$$\partial_q + \partial_p = ???$$

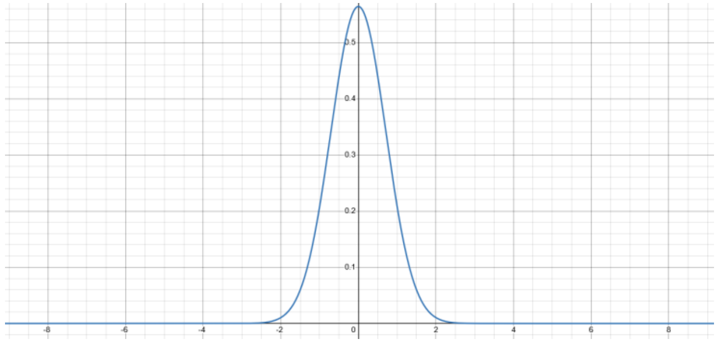
$$[\mathrm{m}] \qquad [\mathrm{Kg\ m\ s^{-1}}]$$

Mathematically precise ⇏ physically precise

Quantum states represented by L^2 Hilbert space

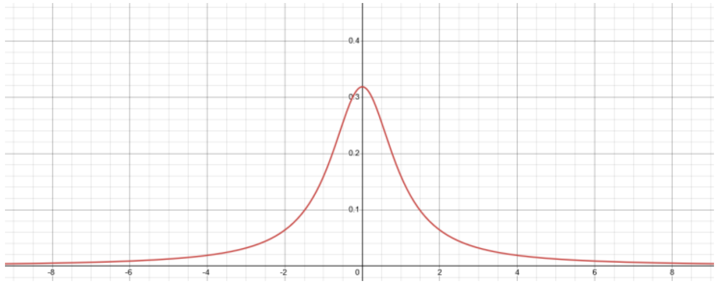
$$\psi(x) = \sqrt{\frac{e^{-x^2}}{\sqrt{\pi}}} \qquad \int |\psi|^2 dx = 1$$

$$\rho_\psi(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \qquad \langle X^2 \rangle_\psi = \frac{1}{2}$$



$$\phi(y) = \sqrt{\frac{1}{\pi(y^2+1)}} \qquad \int |\phi|^2 dx = 1$$

$$\rho_\phi(y) = \frac{1}{\pi(y^2+1)} \qquad \langle Y^2 \rangle_\phi \to \infty$$



Different observers see finite/infinite expectation

$$y = \tan\left(\frac{\pi}{2}\operatorname{erf}(x)\right)$$

$$\psi(y) = \psi(x)\sqrt{\frac{dx}{dy}}$$

Expectation can have finite-to-infinite oscillations

$$x(x_0, t) = x_0 \cos^2\frac{\pi t}{2} + \tan\left(\frac{\pi}{2}\operatorname{erf}(x_0)\right)\sin^2\frac{\pi t}{2}$$

Every continuous linear operator defined on the whole Hilbert space is bounded ⇒ position/momentum/energy/number of particles are not defined on the whole Hilbert space!!!

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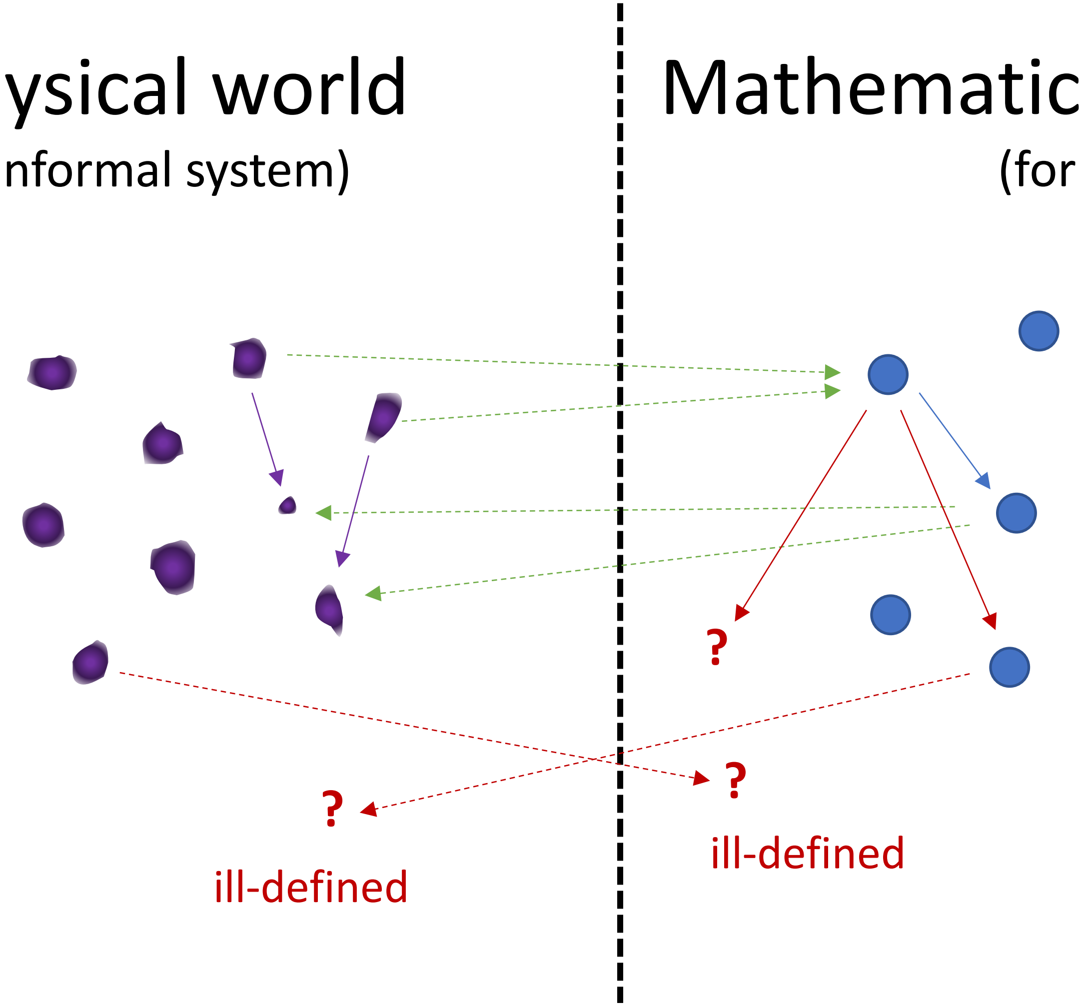
Physical world

(informal system)

Mathematical representation

(formal system)



well-defined
physical
objects

well-defined
mathematical
objects

?

?

?

ill-defined

ill-defined

Current state of the art in theoretical physics

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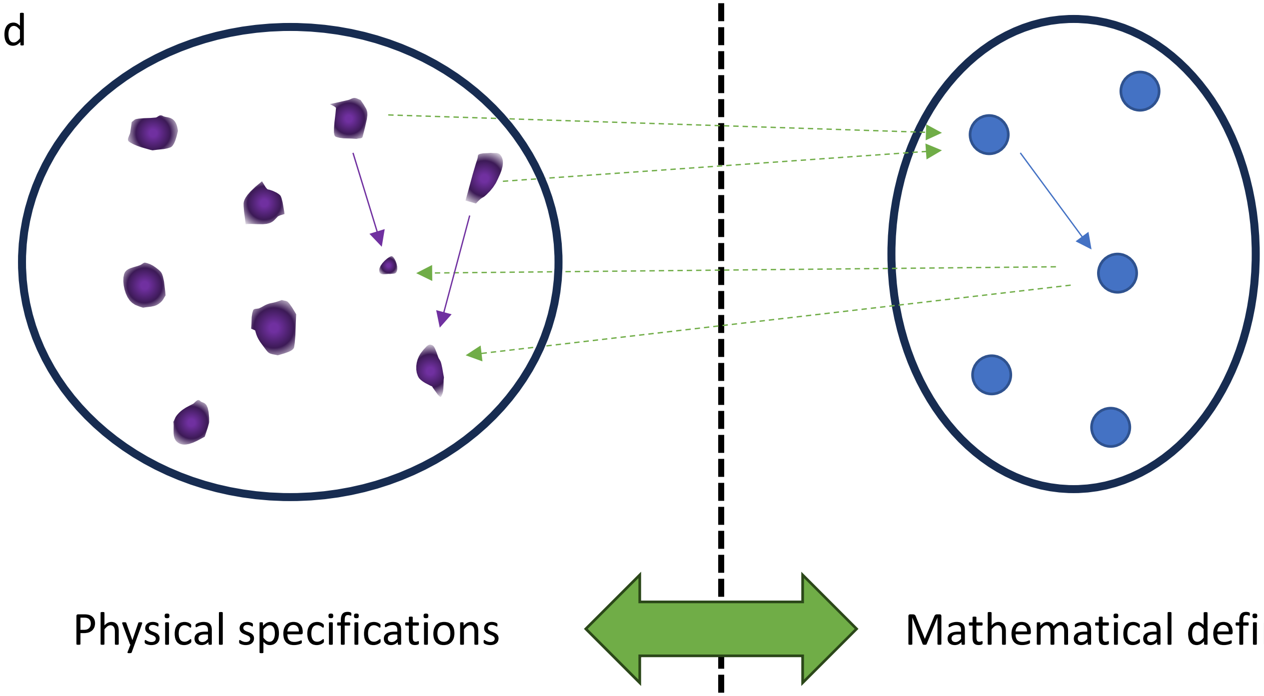
Assumptions of Physics

Physical world

(informal system)

Mathematical representation

(formal system)



well-defined
physical
objects

well-defined
mathematical
objects

Physical specifications

Mathematical definition

A mathematical definition is **physical** if it captures and only captures an aspect of the physical system

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Assumptions of Physics

Mathematicians have developed standards of rigor for their discipline

What standard of rigor should we have for physical mathematics?

For the math part, the same as mathematics

What should we do for the physics part?

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble.*

Informal intuitive statement

(something that makes sense to a physicist or an engineer)

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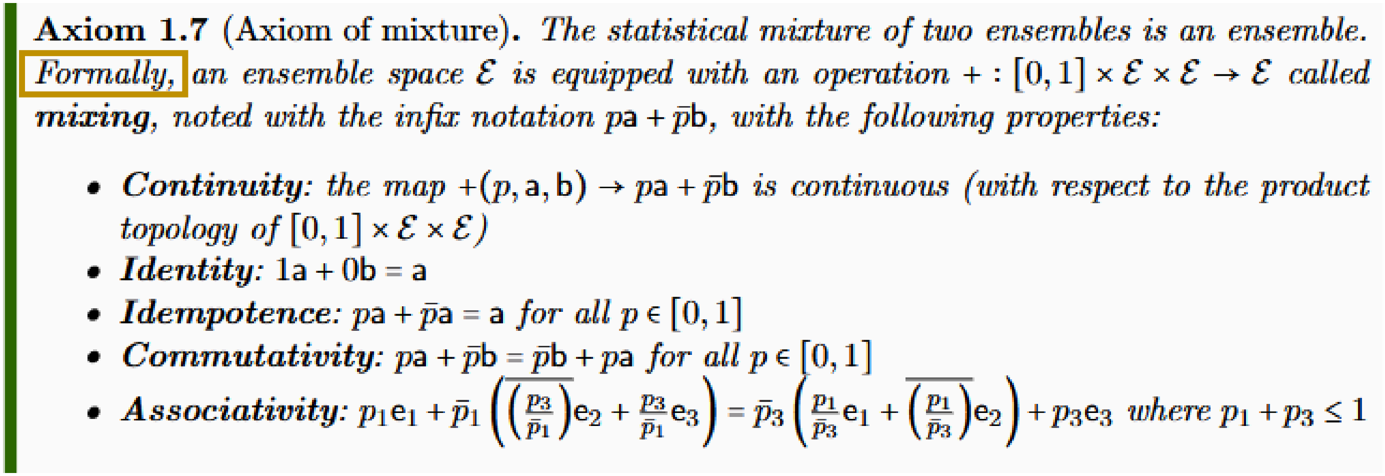
Assumptions of Physics



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- ***Continuity****: the map $+(p, \mathsf{a}, \mathsf{b}) \to p\mathsf{a} + \bar{p}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
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Informal intuitive statement

(something that makes sense to a physicist or an engineer)

Formal requirement

(something a mathematician will find precise)

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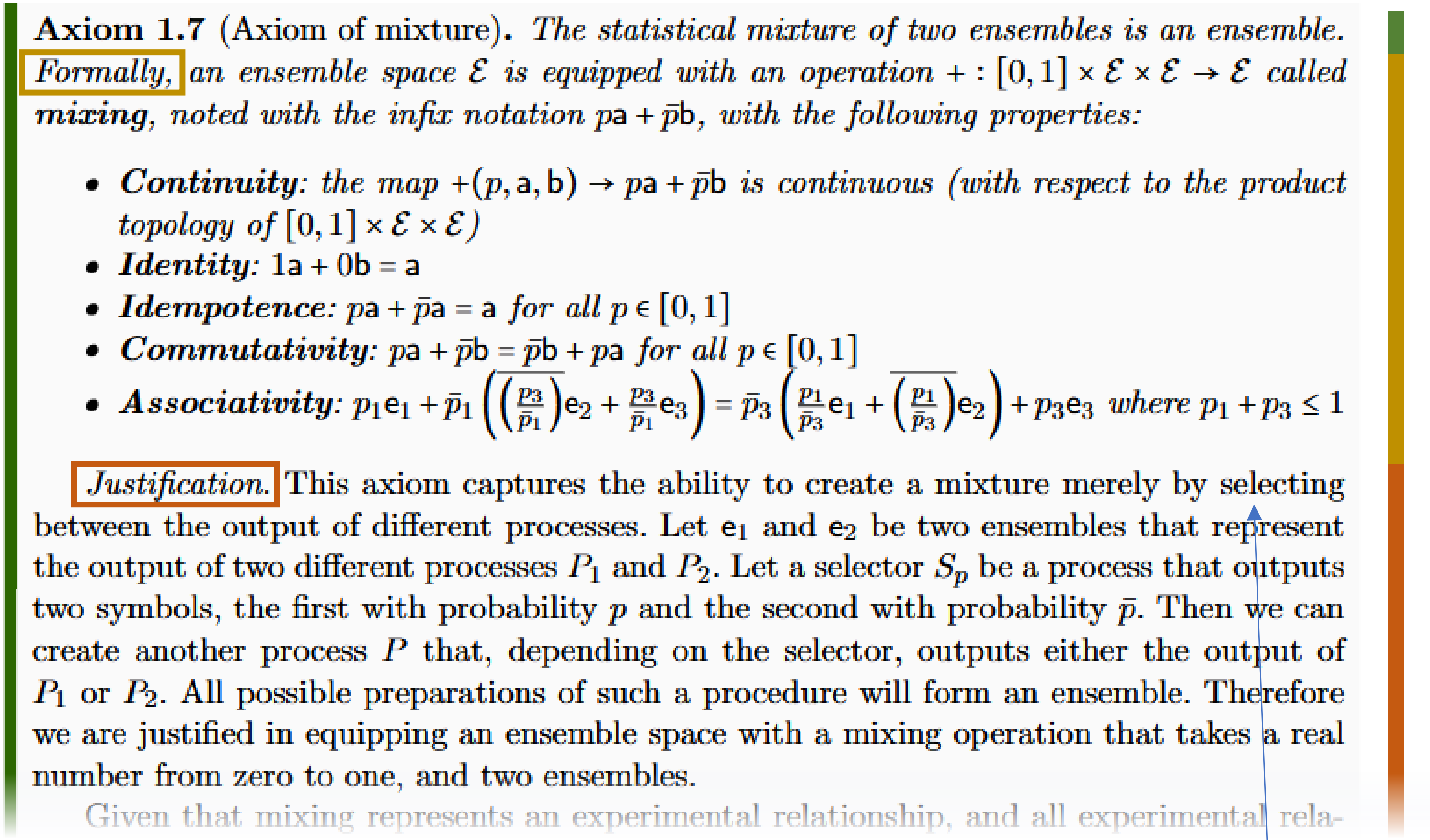
Assumptions of Physics



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Justification. This axiom captures the ability to create a mixture merely by selecting between the output of different processes. Let e_1 and e_2 be two ensembles that represent the output of two different processes P_1 and P_2. Let a selector S_p be a process that outputs two symbols, the first with probability p and the second with probability $\bar{p}$. Then we can create another process P that, depending on the selector, outputs either the output of P_1 or P_2. All possible preparations of such a procedure will form an ensemble. Therefore we are justified in equipping an ensemble space with a mixing operation that takes a real number from zero to one, and two ensembles.

Given that mixing represents an experimental relationship, and all experimental rela-

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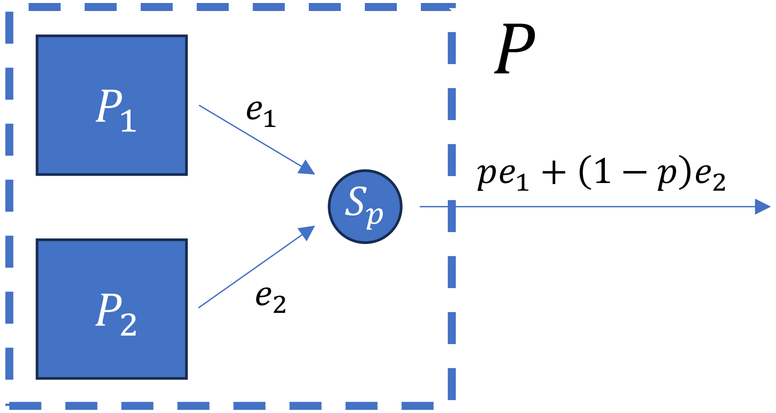
(something that makes sense to a physicist or an engineer)

Formal requirement

(something a mathematician will find precise)

Show that the formal requirement follows from the intuitive statement



Clear idea of what is being modelled

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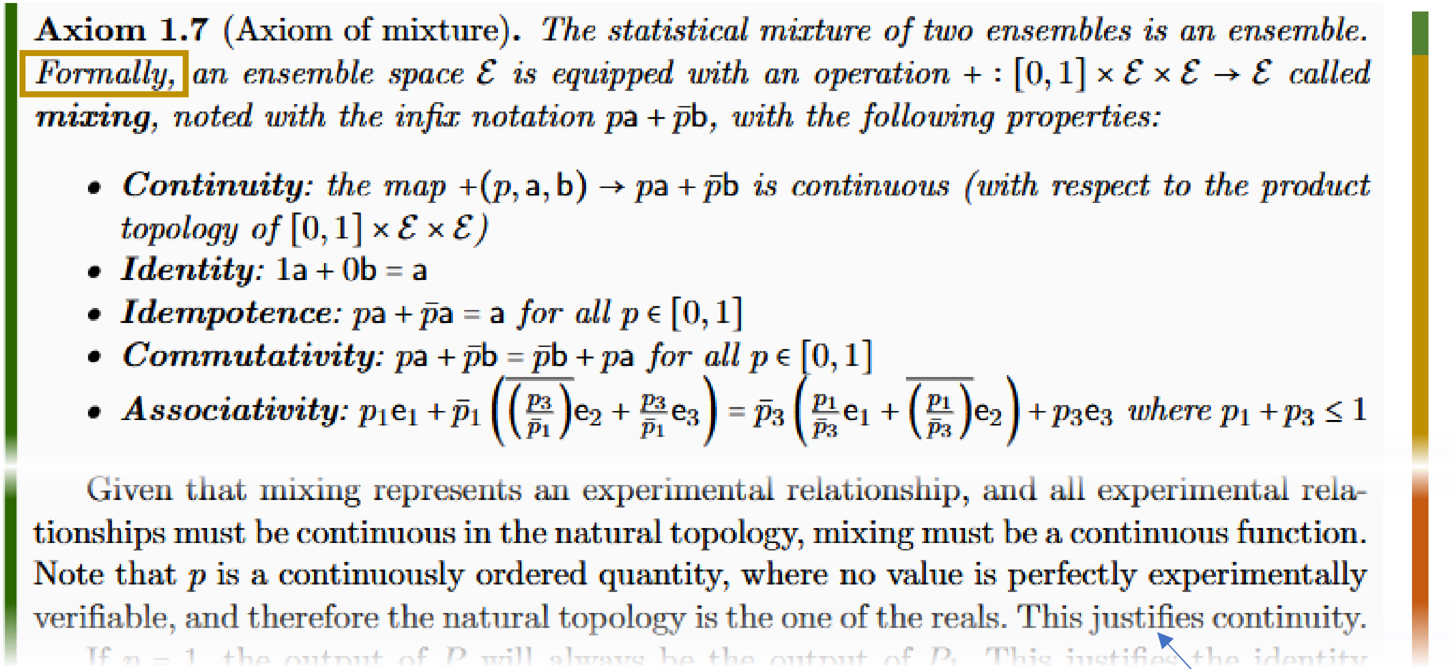
Assumptions of Physics



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Given that mixing represents an experimental relationship, and all experimental relationships must be continuous in the natural topology, mixing must be a continuous function. Note that p is a continuously ordered quantity, where no value is perfectly experimentally verifiable, and therefore the natural topology is the one of the reals. This justifies continuity.

If $p = 1$, the output of P will always be the output of P. This justifies the identity

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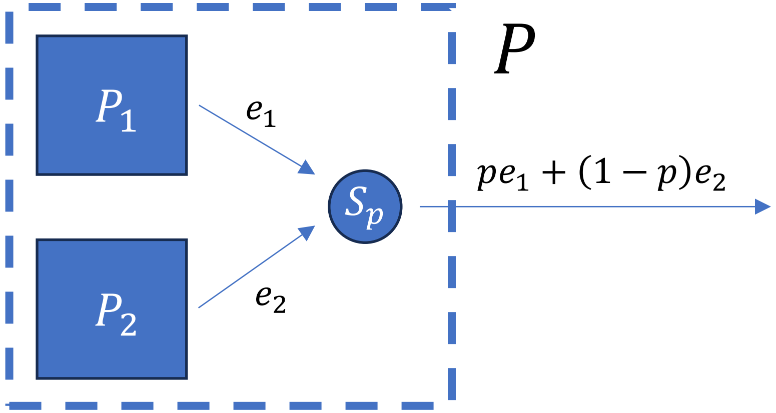
(something that makes sense to a physicist or an engineer)

Formal requirement

(something a mathematician will find precise)

Show that the formal requirement follows from the intuitive statement

Justification uses previous findings



Physical mathematics must start with most basic structures

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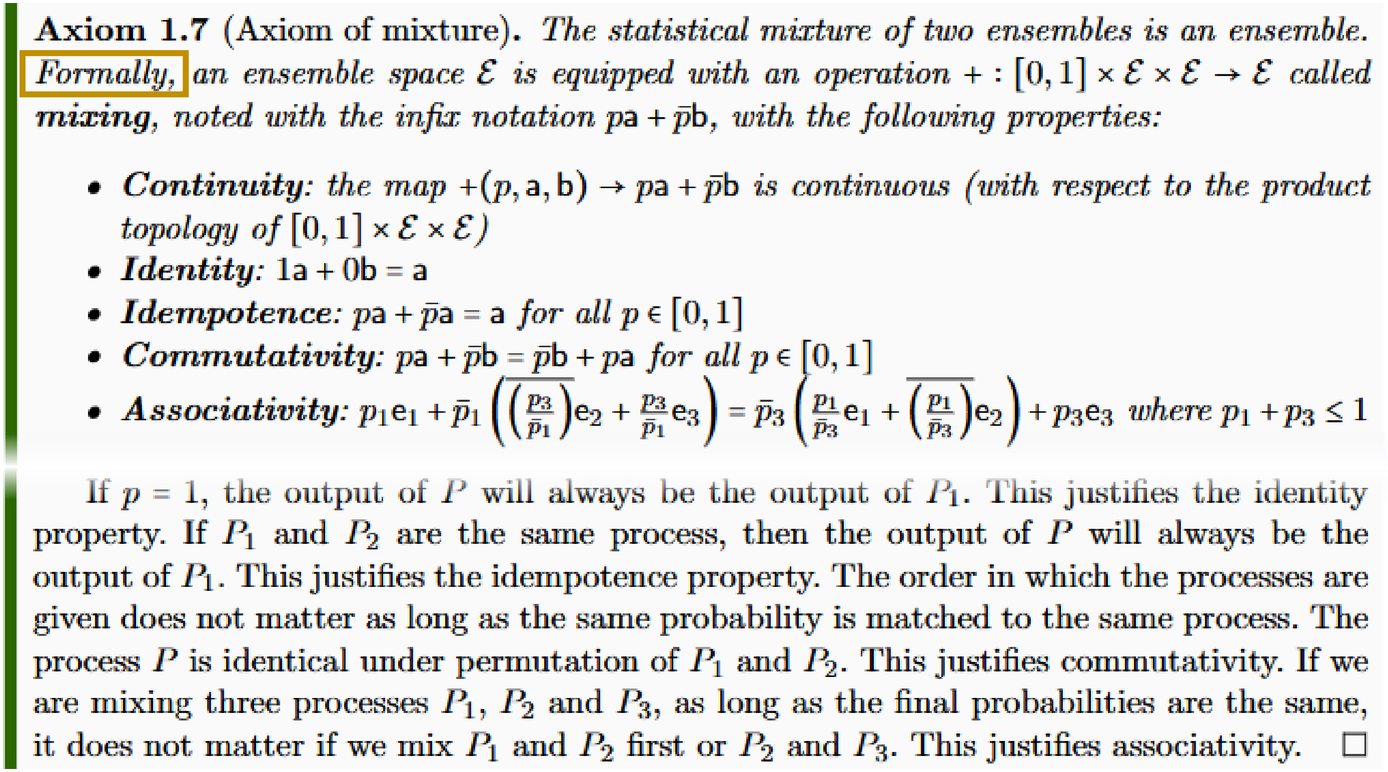

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Informal intuitive statement

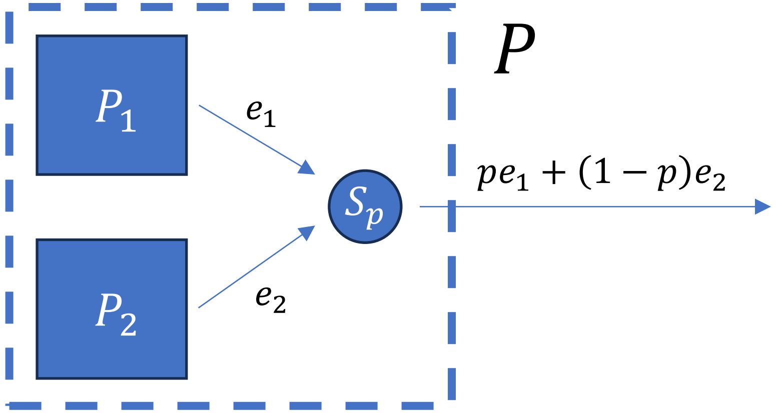
(something that makes sense to a physicist or an engineer)

Formal requirement

(something a mathematician will find precise)

Show that the formal requirement follows from the intuitive statement

$$(1 - p_1)\left(1 - \frac{p_3}{1 - p_1}\right) = 1 - p_1 - p_3 = (1 - p_3)\left(1 - \frac{p_1}{1 - p_3}\right)$$



Properties justified by understanding the model

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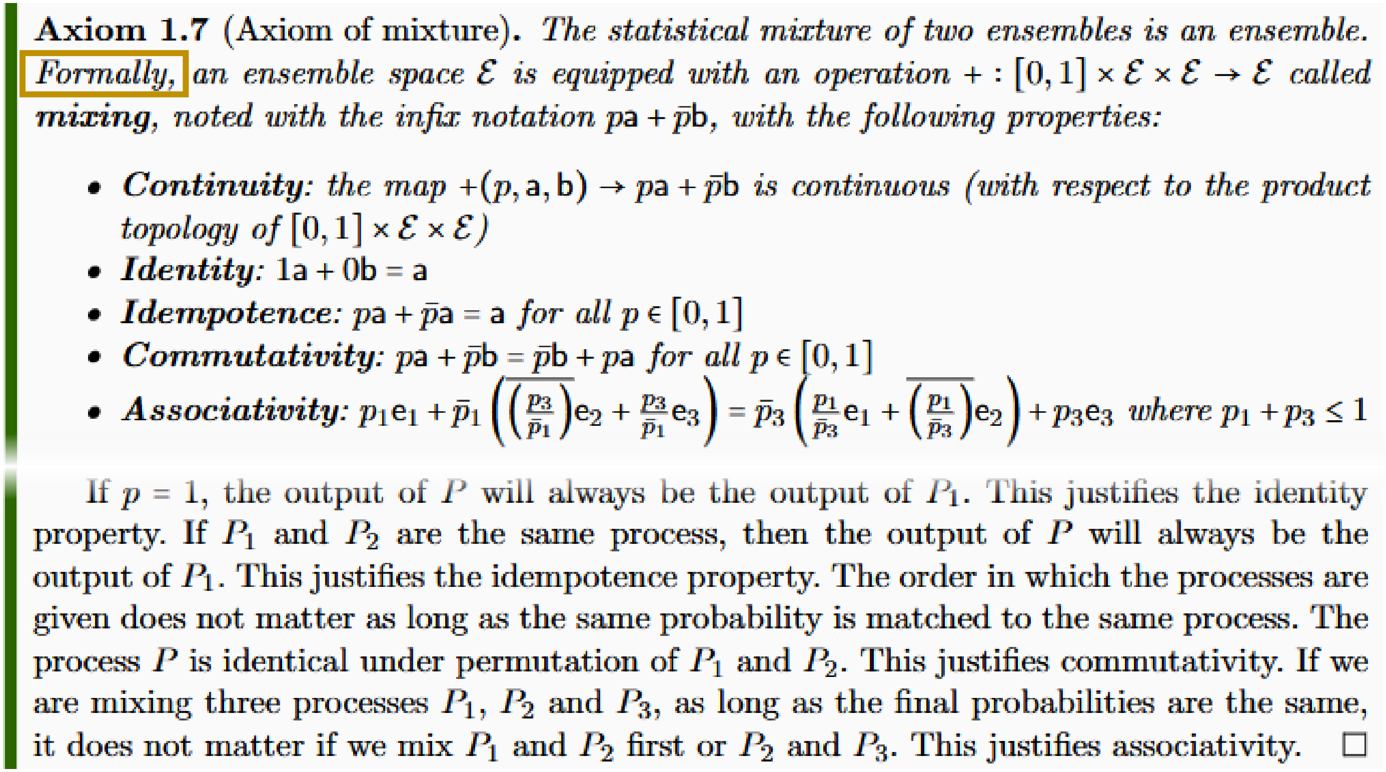

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Informal intuitive statement

(something that makes sense to a physicist or an engineer)

Formal requirement

(something a mathematician will find precise)

Show that the formal requirement follows from the intuitive statement

There is no question as to what the math describes

The properties are justified by, are a consequence of, what the model describes

Every math proof can be understood physically

⇒ The math describes and only describes physically meaningful concepts

It's physical mathematics

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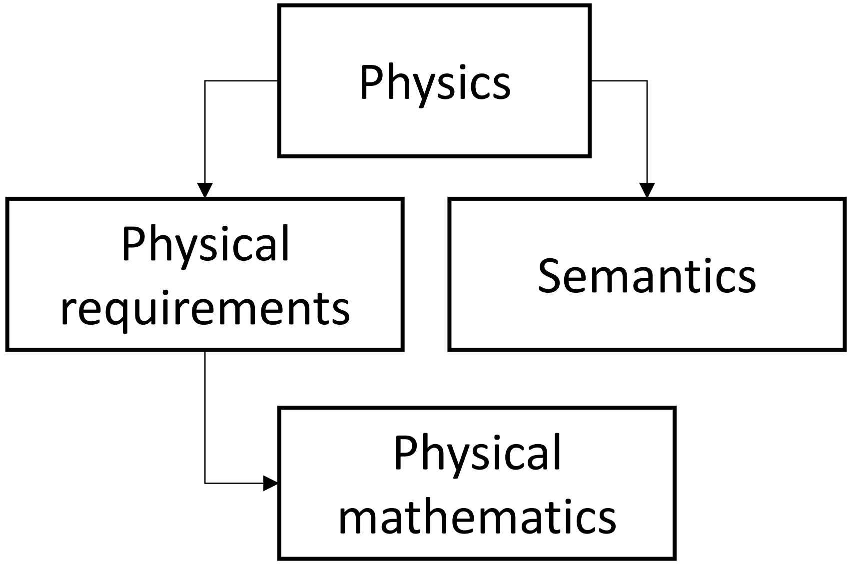
Assumptions of Physics

The goal of physical mathematics is to recover ALL mathematical structures used in physics from clear physical requirements


Physics

Physical requirements

Semantics

Physical mathematics

Clarify realm of applicability of each mathematical structure

Perfect map between math and physics

Provide a generalized structure for all physical theories

It’s a better way to do physics

It forces you to think a lot deeper about physics, what it means to have an experimentally based theory, what it means to define a state, what is entropy or energy, …

It’s not just a “math thing”

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Assumptions of Physics

Takeaway

- It is possible to define the starting points of our physical theories so that they are both mathematically precise and physically meaningful (and philosophically consistent)
- Physical mathematics: mathematical structures justified by the physics
  - Justifications provide a new standard of rigor for physical theories
- Only mathematical structures that are justified by unavoidable physical requirements can serve as truly foundational structures
  - **All** physical theories must satisfy those requirements
- ⇒ Foundations of physics is not “guessing” what the physical world is “made of,” but articulating in a precise way what physical theories are

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Logic of experimental verifiability

$$\Downarrow$$

topologies and σ-algebras

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Assumptions
of
Physics

Principle of scientific objectivity. Science is universal, non-contradictory and evidence based.

⇒ Science is about statements that are associated to experimental tests

Statements must be either true or false for everybody

two-valued logic

Tests may or may not terminate (i.e. may be inconclusive)

three-valued logic

Statement	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

Verifiable statement	Test Result
T	SUCCESS (in finite time)
F	UNDEFINED
	FAILURE (in finite time)

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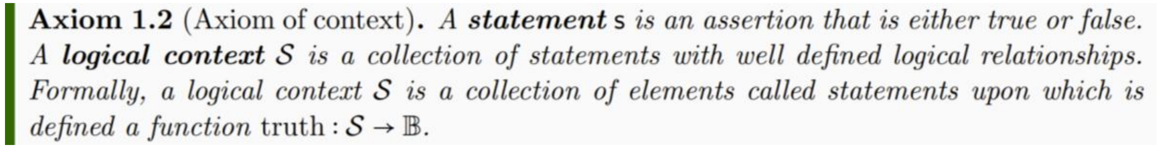
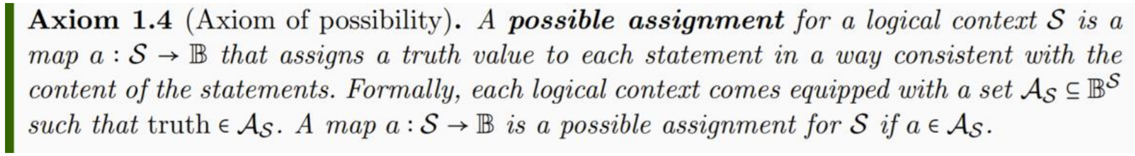
Assumptions of Physics

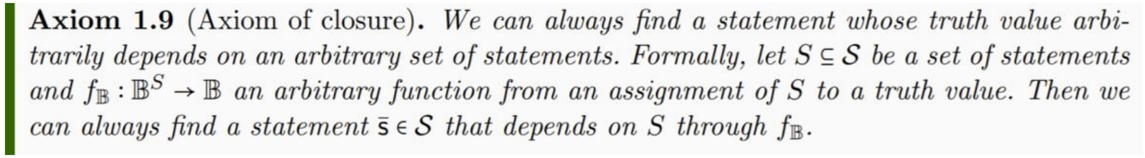
Axioms of logic



Axiom 1.2 (Axiom of context). *A* ***statement*** s *is an assertion that is either true or false. A* ***logical context*** $\mathcal{S}$ *is a collection of statements with well defined logical relationships. Formally, a logical context* $\mathcal{S}$ *is a collection of elements called statements upon which is defined a function* $\text{truth} : \mathcal{S} \to \mathbb{B}$.



Axiom 1.4 (Axiom of possibility). *A* ***possible assignment*** *for a logical context* $\mathcal{S}$ *is a map* $a : \mathcal{S} \to \mathbb{B}$ *that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set* $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$ *such that* $\text{truth} \in \mathcal{A}_{\mathcal{S}}$. *A map* $a : \mathcal{S} \to \mathbb{B}$ *is a possible assignment for* $\mathcal{S}$ *if* $a \in \mathcal{A}_{\mathcal{S}}$.

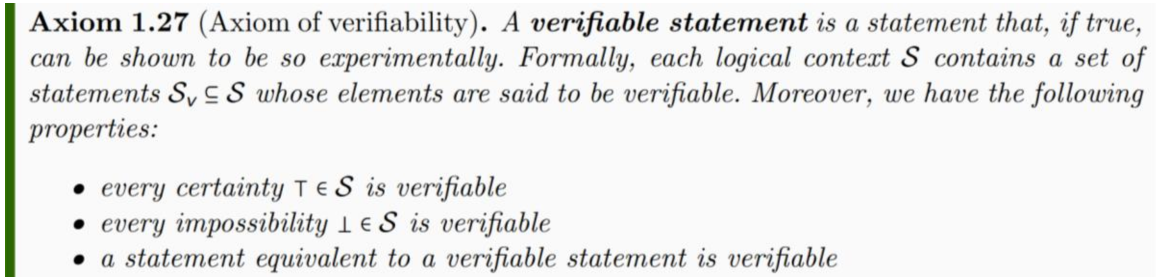


Axiom 1.9 (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let* $S \subseteq \mathcal{S}$ *be a set of statements and* $f_{\mathbb{B}} : \mathbb{B}^{S} \to \mathbb{B}$ *an arbitrary function from an assignment of* S *to a truth value. Then we can always find a statement* $\bar{\mathsf{s}} \in \mathcal{S}$ *that depends on* S *through* $f_{\mathbb{B}}$.

Lead to standard logic
(i.e. Boolean algebra)

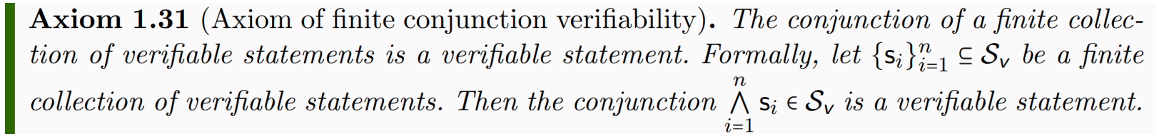
two-valued logic

Axioms of verifiability

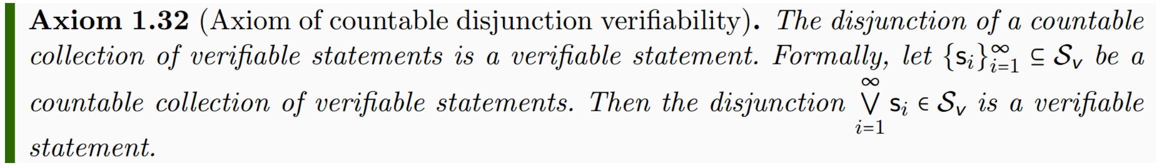


Axiom 1.27 (Axiom of verifiability). *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, each logical context* $\mathcal{S}$ *contains a set of statements* $\mathcal{S}_{\mathsf{v}} \subseteq \mathcal{S}$ *whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty* $\top \in \mathcal{S}$ *is verifiable*
- *every impossibility* $\bot \in \mathcal{S}$ *is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*



Axiom 1.31 (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a finite collection of verifiable statements. Then the conjunction* $\bigwedge\limits_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*



Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a countable collection of verifiable statements. Then the disjunction* $\bigvee\limits_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Lead to intuitionist logic
(i.e. Heyting algebra)

three-valued logic

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Assumptions
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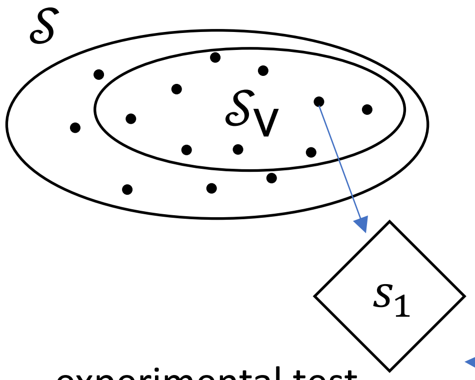
Axiom 1.27 (Axiom of verifiability). *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_{\mathsf{v}} \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



experimental test

s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED

Tests are not part of the formal system

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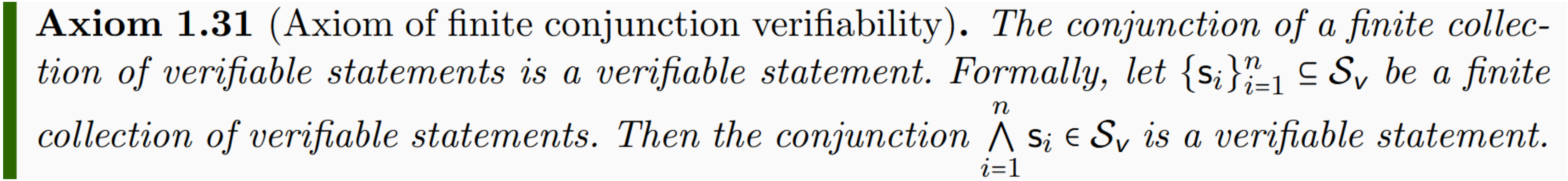
Assumptions of Physics



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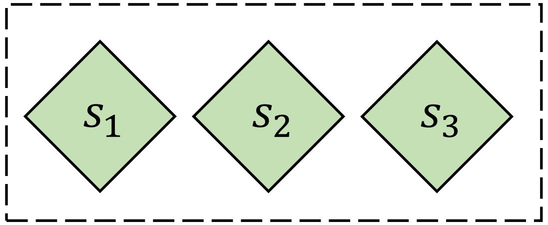
Conjunction (AND) of verifiable statements:
check that all tests terminate successfully

⇒ Only finite conjunction is guaranteed to terminate

$$\bigwedge_{i=i}^{n} s_i$$



All tests must succeed

$\wedge\,(e_i)$:
1. Run all e_i
2. If all succeed, return SUCCESS
3. Return FAILURE

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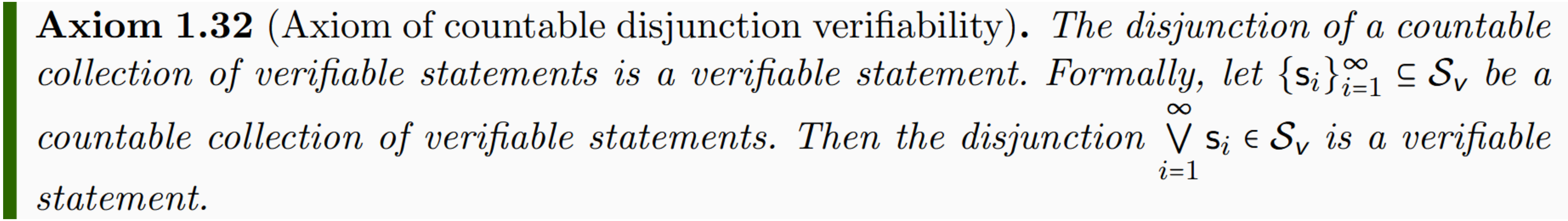
Assumptions of Physics

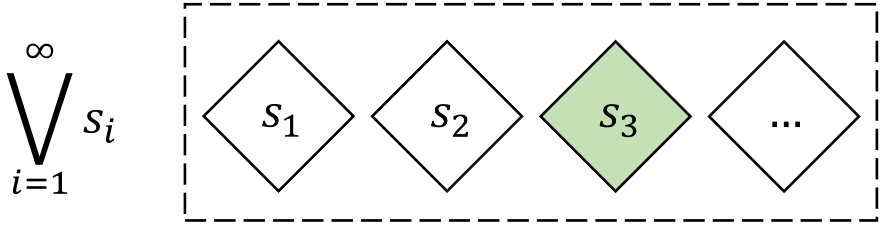


Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a countable collection of verifiable statements. Then the disjunction* $\bigvee\limits_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Disjunction (OR) of verifiable statements:
check that ONE test terminates successfully

watch out for non-termination!

⇒ Only countable disjunction can reach all tests



One successful test is sufficient

```
∨ $(e_i)$:
1.  Initialize $n$ to 1
2.  For each $i = 1 \dots n$
      a)  Run $e_i$ for $n$ seconds
      b)  If $e_i$ succeeds, return SUCCESS
3.  Increment $n$ and go to 2
```

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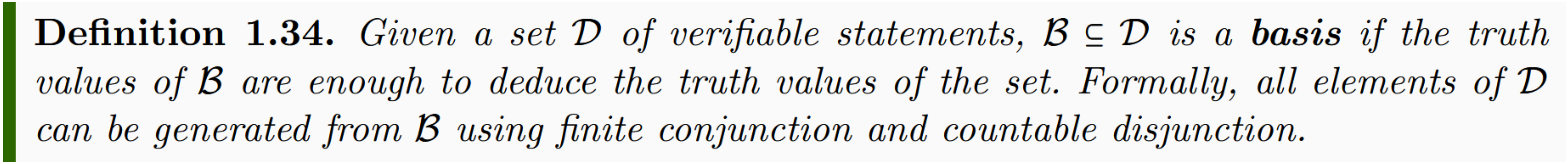
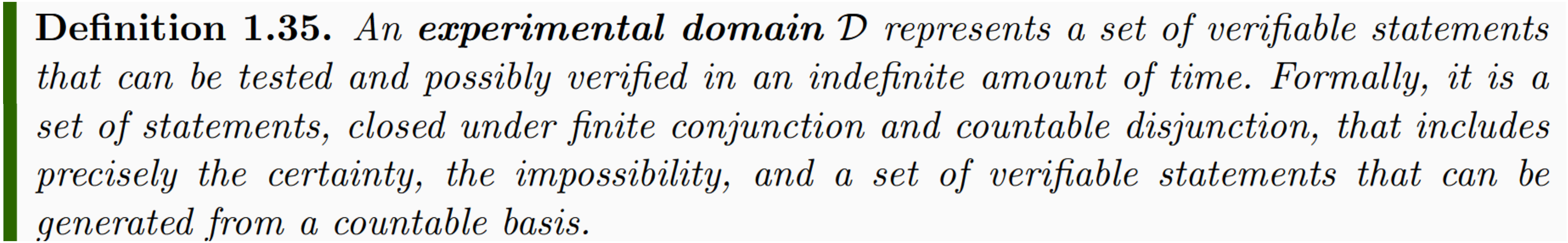
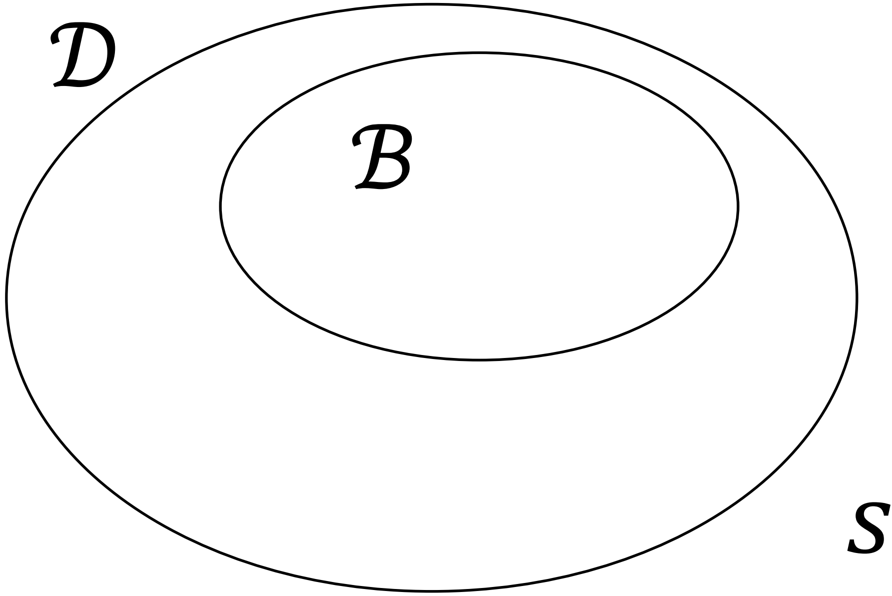
Assumptions of Physics



Definition 1.34. *Given a set $\mathcal{D}$ of verifiable statements, $\mathcal{B} \subseteq \mathcal{D}$ is a* **basis** *if the truth values of $\mathcal{B}$ are enough to deduce the truth values of the set. Formally, all elements of $\mathcal{D}$ can be generated from $\mathcal{B}$ using finite conjunction and countable disjunction.*



Definition 1.35. *An* ***experimental domain*** *$\mathcal{D}$ represents a set of verifiable statements that can be tested and possibly verified in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the certainty, the impossibility, and a set of verifiable statements that can be generated from a countable basis.*



$$\mathcal{B} = \{e_1, e_2, e_3, \dots\}$$

Countable basis

Only finite conjunction and countable disjunction

$$s_1 = (e_1 \vee e_3) \wedge e_2 \dots$$

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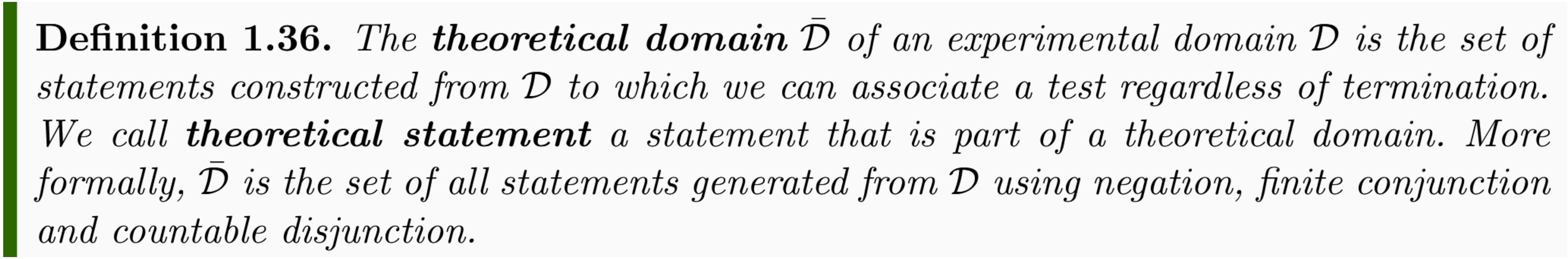
Assumptions of Physics



Definition 1.36. *The **theoretical domain** $\bar{\mathcal{D}}$ of an experimental domain $\mathcal{D}$ is the set of statements constructed from $\mathcal{D}$ to which we can associate a test regardless of termination. We call **theoretical statement** a statement that is part of a theoretical domain. More formally, $\bar{\mathcal{D}}$ is the set of all statements generated from $\mathcal{D}$ using negation, finite conjunction and countable disjunction.*

Extend the domain to include all statements that are associated with a test, regardless of termination.

All statements depend on the verifiable statements
(which depend on the basis)

No new information is captured

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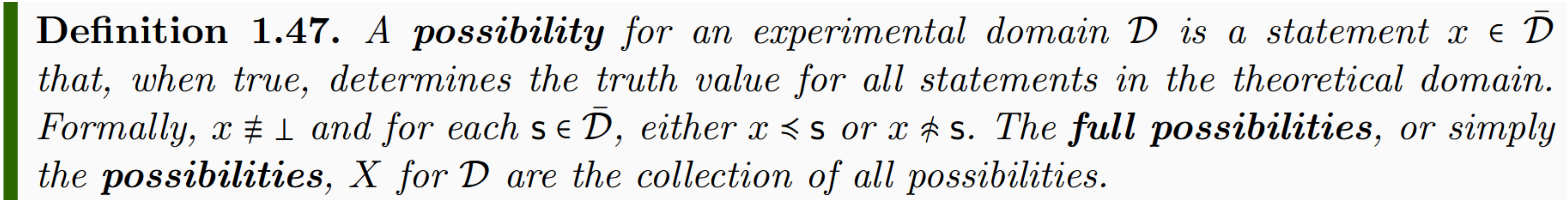
Assumptions of Physics

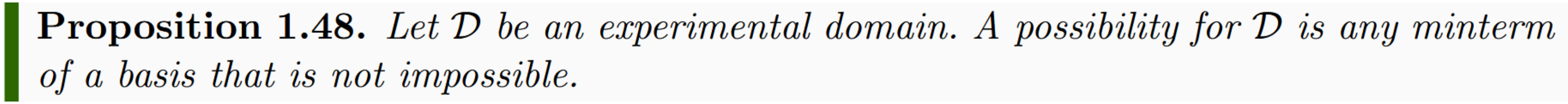


Definition 1.47. *A **possibility** for an experimental domain $\mathcal{D}$ is a statement $x \in \bar{\mathcal{D}}$ that, when true, determines the truth value for all statements in the theoretical domain. Formally, $x \not\equiv \perp$ and for each $\mathsf{s} \in \bar{\mathcal{D}}$, either $x \preccurlyeq \mathsf{s}$ or $x \not\curlywedge \mathsf{s}$. The **full possibilities**, or simply the **possibilities**, X for $\mathcal{D}$ are the collection of all possibilities.*

A possibility of a domain is a statement that picks one assignment

Possibilities: experimentally defined alternative cases defined by the verifiable cases

s_1	s_2	s_3	…	x_1	x_2	x_3	x_4
T	T	F	T	T	F	F	F
F	F	T	T	F	T	F	F
F	T	T	F	F	F	T	F
T	F	F	T	F	F	F	T



Proposition 1.48. *Let $\mathcal{D}$ be an experimental domain. A possibility for $\mathcal{D}$ is any minterm of a basis that is not impossible.*

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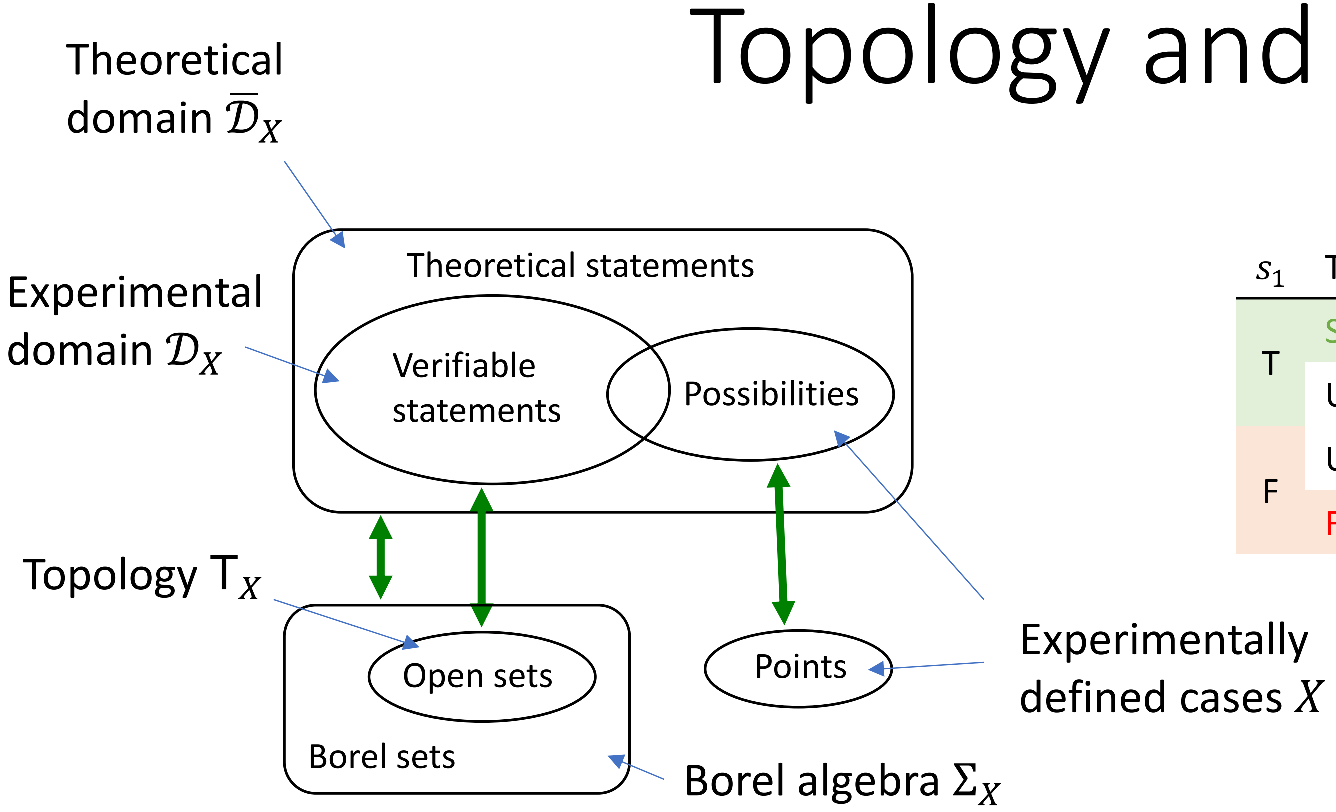
Assumptions of Physics

Topology and σ-algebra



Theoretical domain $\overline{\mathcal{D}}_X$

Experimental domain $\mathcal{D}_X$

Topology T_X

Theoretical statements

Verifiable statements

Possibilities

Open sets

Borel sets

Points

Experimentally defined cases X

Borel algebra Σ_X

s_1	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

Perfect map between math and physics

Open set $(509.5, 510.5)$ ⟺ Verifiable “the mass of the electron is 510 $\pm$ 0.5 KeV”

Closed set $[510]$ ⟺ Falsifiable “the mass of the electron is exactly 510 KeV”

Borel set $\mathbb{Q}$ $(int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset)$ ⟺ Theoretical “the mass of the electron in KeV is a rational number” (undecidable)

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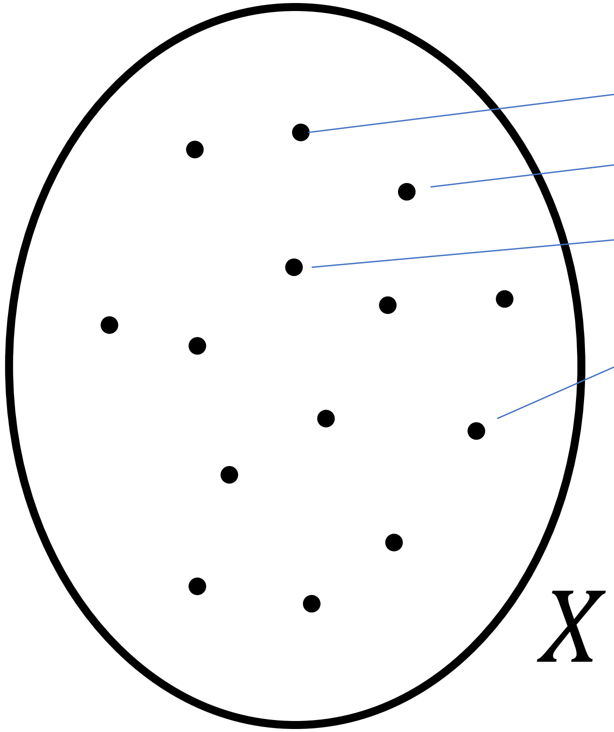
Assumptions of Physics

Maximum cardinality of distinguishable cases

Set of distinguishable cases



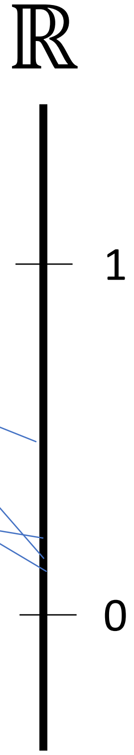
Each point identified by truth of countably many verifiable stmts

FTFFFTTTFTFTT…
TFFTTFTTFFFTF…
FTFFFTTFTFFTF…
FTTFTFTTFTFFT…

Most we can test over arbitrarily long time

Correspond to binary expansion

0.0100011101011…
0.1001101100010…
0.0100011010010…
0.0110101101001…



Correspondence to binary sequence

0100011101011…
1001101100010…
0100011010010…
0110101101001…

$$|X| \leq |\mathbb{R}|$$

- Sets with greater cardinality (e.g. the set of all discontinuous functions from $\mathbb{R}$ to $\mathbb{R}$) cannot represent physical objects
- Issues about higher infinities (e.g. large cardinals) are not relevant, but those surrounding the continuum hypothesis may be

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Assumptions of Physics

Power set vs Borel algebra

Using the set $2^{\mathbb{R}^n}$ of all possible subsets for $\mathbb{R}^n$ is problematic

Notion of size (i.e. measure) cannot be defined on all sets

Using non-measurable sets leads to the Banach-Tarski paradox

wikipedia

These problems are avoided if we restrict ourselves to Borel sets

⇒ If we restrict ourselves to experimentally definable objects, these paradoxes are avoided

Physical mathematics can give insight
to these foundational issues in mathematics

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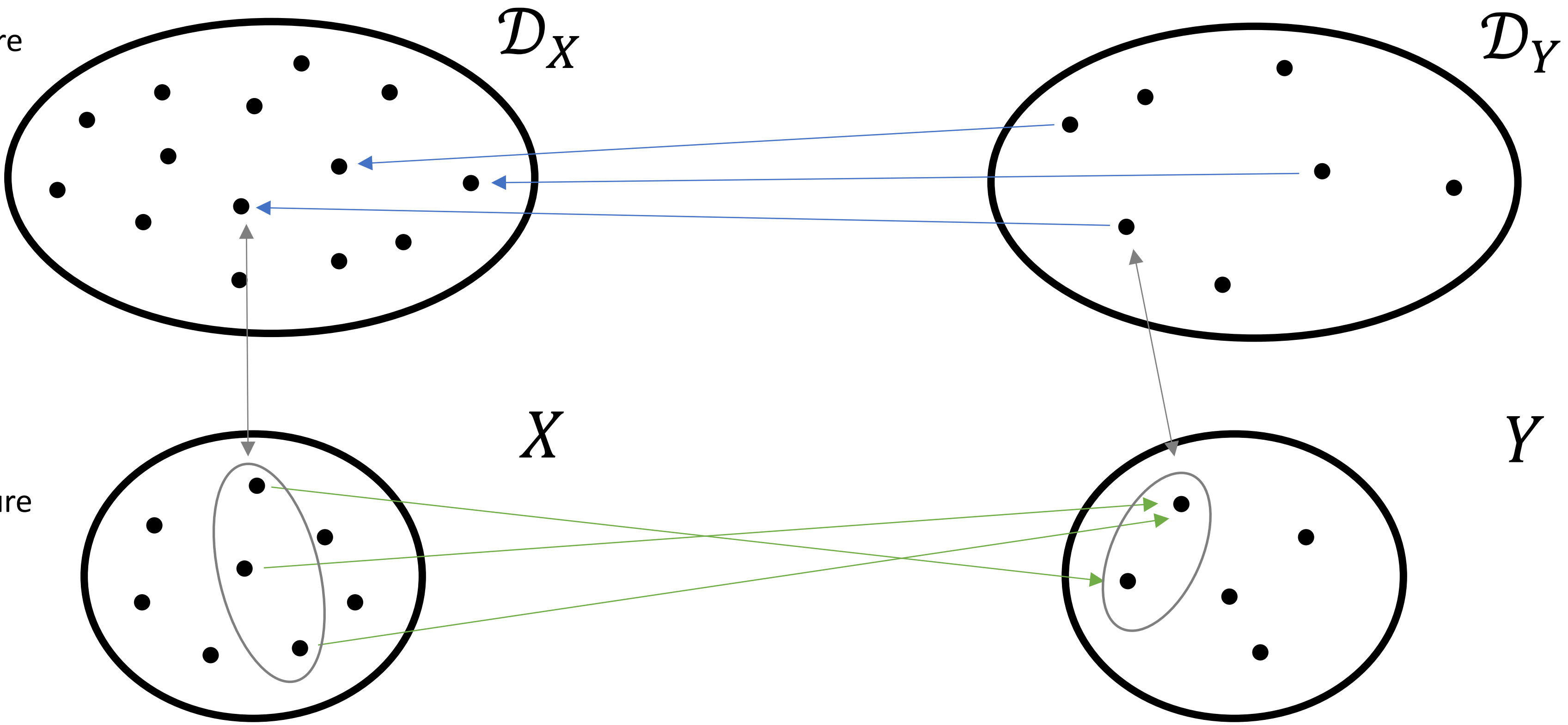
Assumptions of Physics

An inference relationship is a map $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

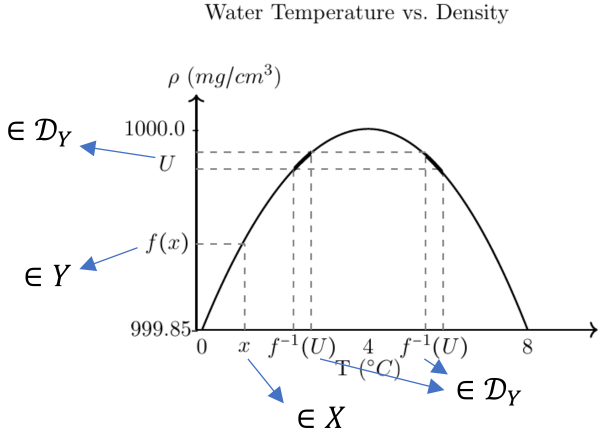


e.g. the water temperature is between 0 and 0.52 Celsius or between 7.6 and 9.12 Celsius

e.g. the water density is between 999.8 and 999.9 kg/m^3

e.g. the water temperature is exactly 4 Celsius

e.g. the water density is exactly 1 kg/m^3



A causal relationship is a map $f: X \to Y$ such that $x \preccurlyeq f(x)$

1) Two domains admit an inference relationship if and only if they admit a causal relationship
2) The causal relationship must be a continuous map in the natural topology

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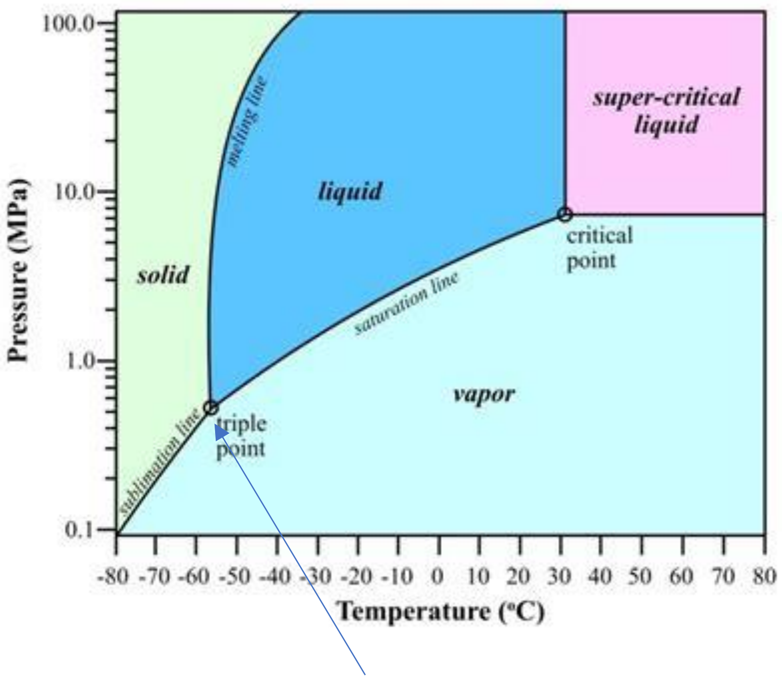
Assumptions of Physics

Why functions are well-behaved

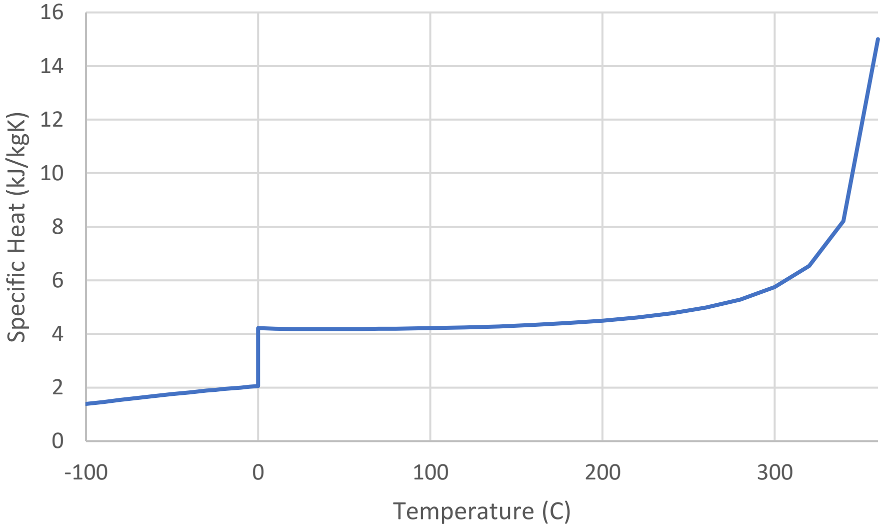


CO2 Phase diagram

Figure from https://www.researchgate.net/publication/326492803_Implications_of_Permeability_Uncertainty_During_Three-phase_CO2_Flow_in_a_Basalt_Fracture_Network

Phase transition regions are experimentally decidable ⟺ Topologically isolated regions

Water heat capacity

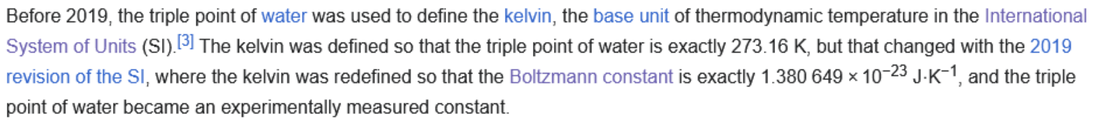


data from https://www.engineeringtoolbox.com

Topological continuity ≠ Analytical continuity

We can verify we are at the triple point

We measure the equilibrium of three phases, not the pressure/temperature



Before 2019, the triple point of water was used to define the kelvin, the base unit of thermodynamic temperature in the International System of Units (SI).[3] The kelvin was defined so that the triple point of water is exactly 273.16 K, but that changed with the 2019 revision of the SI, where the kelvin was redefined so that the Boltzmann constant is exactly 1.380 649 × 10^{-23} J⋅K^{-1}, and the triple point of water became an experimentally measured constant.

from https://en.wikipedia.org/wiki/Triple_point

Analytical discontinuity can only happen in regions that are experimentally decidable

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Assumptions of Physics

Takeaway

- Requiring experimental verifiability in a physical theory leads to topologies and σ-algebras
  - Open sets correspond to verifiable statements, continuous functions preserve experimental verifiability, Borel sets correspond to statements associated with tests, …
- All proofs can be understood as describing arguments on experimental verifiability
  - Limits (truth sequences of verifiable statements become constants), topological distinguishability (experimental distinguishability), interior/exterior/boundary (verifiable/falsifiable/undecidable), …
- Further constructions become more meaningful
  - Probability measure defined on σ-algebra: we assign probability to statements with a test; topological groups: transformations we can experimentally identify/define; …

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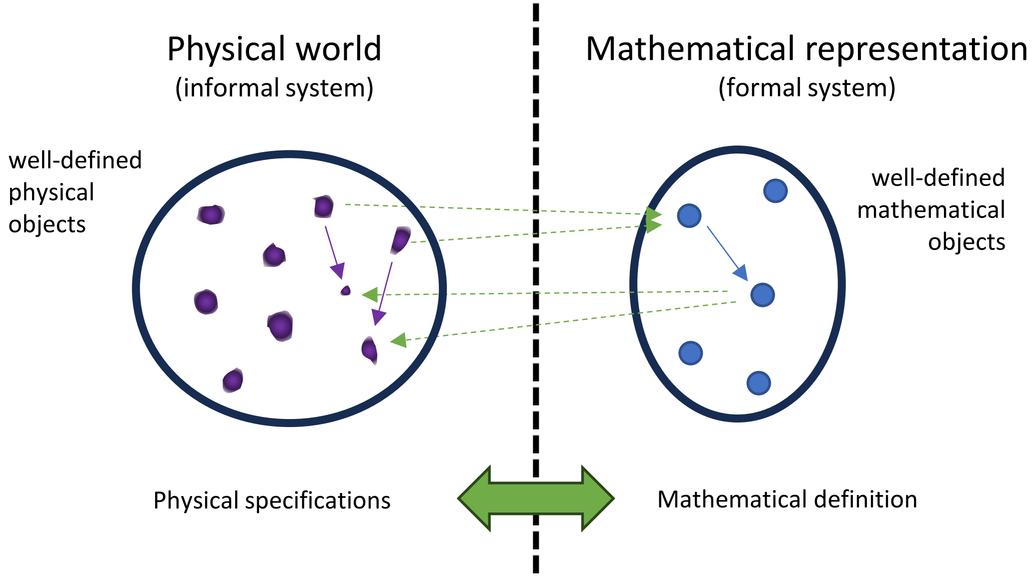
Assumptions of Physics

It is possible to develop a foundation of physics that is both mathematically rigorous and physically meaningful

The mathematical definitions ARE the physical requirements and assumptions



Physical world
(informal system)

Mathematical representation
(formal system)

well-defined
physical
objects

well-defined
mathematical
objects

Physical specifications

Mathematical definition

No issues of “interpretations”

Clear realm of applicability of mathematical tools

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Assumptions of Physics

There is no “just math”

Either the math represents physical objects, then it’s describing physics

Or it doesn’t, and therefore it should be stripped away from the physical theory

Only by understanding the full details of the math and physics (and philosophy) can you make that determination

If you do not know what the well-ordering of the reals is, you are precisely a person that cannot determine whether it is physically significant or not

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Assumptions
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Physics

Wrapping it up

- Assumptions of Physics: different approach to the foundations of physics
  - No interpretations, no theories of everything: physically meaningful starting points from which we can rederive the laws and the mathematical frameworks they need
  - Physical theories are models
    - Need to clarify exactly what the realm of applicability of each model is
- Physical mathematics: derive the math required from physical requirements
  - In physics, mathematics is used to model physical systems, therefore we need mathematics that is designed specifically for that purpose
- You need to start at the lowest level of mathematics
  - Rigor, precision, meaning, correctness cannot be “sprinkled on top”
  - “Big systems that work evolve from small systems that work, never from big systems that do not work” (Gall’s law)

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Assumptions of Physics

To learn more

- Project website
  - https://assumptionsofphysics.org for papers, presentations, …
  - https://assumptionsofphysics.org/book for our open access book (updated every few years with new results)
- YouTube channels
  - https://www.youtube.com/@gcarcassi
    Videos with results and insights from the research
  - https://www.youtube.com/@AssumptionsofPhysicsResearch
    Research channel, with open questions and livestreamed work sessions
- GitHub
  - https://github.com/assumptionsofphysics
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