Assumptions of Physics
Summer School 2025

Reverse Physics for classical mechanics

Gabriele Carcassi and Christine A. Aidala

Physics Department
University of Michigan

UNIVERSITY OF MICHIGAN

Assumptions of Physics

Overview

https://assumptionsofphysics.org/

Assumptions of Physics

Assumption IR (Infinitesimal Reducibility). *The state of the system is reducible to the state of its infinitesimal parts. That is, specifying the state of the whole system is equivalent to specifying the state of its parts, which in turn is equivalent to specifying the state of its subparts and so on.*

Classical state is a distribution over the states of infinitesimal parts (i.e. particles)

Recovers pairs of conjugate variables for each DOF, frame-invariant entropy and count of states

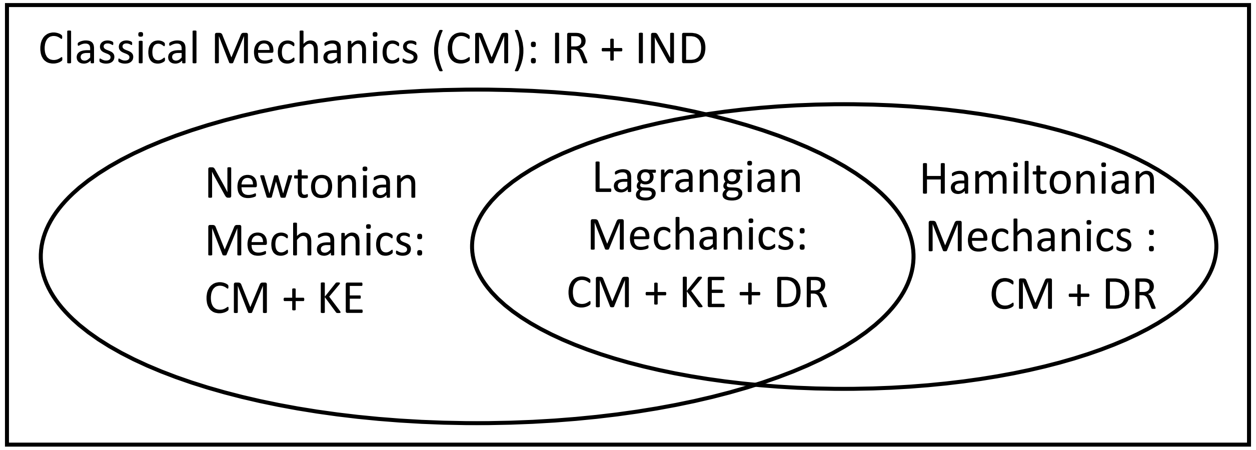
Assumption IND (Independent DOFs). *The system is decomposable into independent degrees of freedom. That is, the variables that describe the state can be divided into groups that have independent definition, units and count of states.*

Assumption DR (Determinism and Reversibility). *The system undergoes deterministic and reversible evolution. That is, specifying the state of the system at a particular time is equivalent to specifying the state at a future (determinism) or past (reversibility) time.*

Conservation of entropy/count of states and DOF independence recovers Hamiltonian evolution

Links trajectories in space with dynamics, recovering inertia, masses and scalar/vector potential forces

Assumption KE (Kinematic Equivalence). *The kinematics of the system is sufficient to reconstruct its dynamics and vice-versa. That is, specifying the motion of the system is equivalent to specifying its state and evolution.*

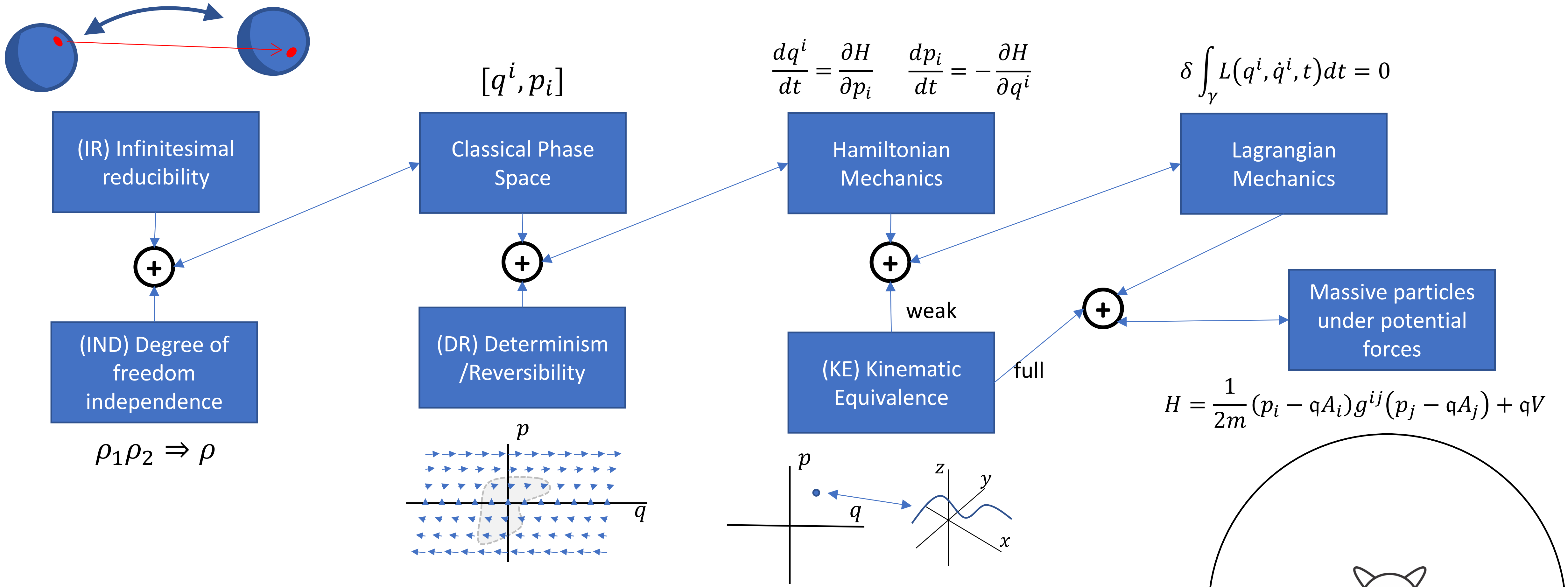


Assumptions for classical mechanics

https://assumptionsofphysics.org/

Assumptions of Physics

Assumptions of classical mechanics



https://assumptionsofphysics.org/

Assumptions of Physics

Each assumption can be expressed in multiple different but equivalent ways

Assumption DR (Determinism and Reversibility). *The system undergoes deterministic and reversible evolution. That is, specifying the state of the system at a particular time is equivalent to specifying the state at a future (determinism) or past (reversibility) time.*

The displacement field is divergenceless: $\partial_a S^a = 0$	(DR-DIV)
The Jacobian of time evolution is unitary: $\left\vert\partial_b \hat{\xi}^a\right\vert = 1$	(DR-JAC)
Densities are conserved through the evolution: $\hat{\rho}(\hat{\xi}^a) = \rho(\xi^b)$	(DR-DEN)
Volumes are conserved through the evolution: $d\hat{\xi}^1 \cdots d\hat{\xi}^n = d\xi^1 \cdots d\xi^n$	(DR-VOL)
The evolution is deterministic and reversible.	(DR-EV)
The evolution is deterministic and thermodynamically reversible	(DR-THER)
The evolution conserves information entropy	(DR-INFO)
The evolution conserves the uncertainty of peaked distributions	(DR-UNC)

Assumption KE (Kinematic Equivalence). *The kinematics of the system is sufficient to reconstruct its dynamics and vice-versa. That is, specifying the motion of the system is equivalent to specifying its state and evolution.*

There is a linear relationship between conjugate momentum and velocity	(FKE-LIN)
The system under study is a massive particle under scalar and vector potential forces	(FKE-POT)
The Jacobian of the transformation between state variables and kinematic variables is a non-singular function of position only.	(FKE-NSIN)
Densities over phase space can be expressed in terms of position and velocity by rescaling the value at each point: $\rho(x^i, v^j)\vert J(x^i)\vert = \rho(q^i, p_j)$.	(FKE-DEN)
Areas and volumes in phase space can be expressed in kinematic variables, and the transformation depends on position only: $dx^1 \cdots dx^n \, dv^1 \cdots dv^n = \vert J(x^i)\vert \, dq^1 \cdots dq^n \, dp_1 \cdots dp_n$.	(FKE-VOL)
The symplectic form ω_{ab} can be expressed in kinematic variables, and its components are a linear function of velocity.	(FKE-SYMP)
Density expressed in velocity at the same position is proportional to the density over states	(FKE-PROP)
At each position, there exists a local inertial frame	(FKE-INER)
The position fully defines the units of all state variables, therefore an invertible transformation between momentum and velocity	(FKE-UNIT)
Uniform distributions along momentum correspond to uniform distributions along velocity	(FKE-UNIF)

Assumption IND (Independent DOFs). *The system is decomposable into independent degrees of freedom. That is, the variables that describe the state can be divided into groups that have independent definition, units and count of states.*

The system is decomposable into independent DOFs	(IND-DOF)
The system allows statistically independent distributions over each DOF	(IND-STAT)
The system allows informationally independent distributions over each DOF	(IND-INFO)
The system allows peaked distributions where the uncertainty is the product of the uncertainty on each DOF	(IND-UNC)

Assumption IR (Infinitesimal Reducibility). *The state of the system is reducible to the state of its infinitesimal parts. That is, specifying the state of the whole system is equivalent to specifying the state of its parts, which in turn is equivalent to specifying the state of its subparts and so on.*

The state of a classical system is given by a distribution over phase space.	(IR-DIST)
A classical system can be thought of as being made of infinitesimal parts, called particles.	(IR-INF)

https://assumptionsofphysics.org/

Assumptions of Physics

Action principle
one DOF

https://assumptionsofphysics.org/

Assumptions of Physics

Required background

- Hamiltonian mechanics
  - Hamilton's equations: $\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$
- Lagrangian mechanics
  - Action is the integral of the Lagrangian over a given path: $\mathcal{A}[\gamma] = \int_\gamma L(q(t), \dot{q}(t), t)dt$
  - Actual evolutions are the paths that make the action stationary: $\delta\mathcal{A} = 0$
  - Lagrangian is the Legendre transform of the Hamiltonian: $L = p\dot{q} - H$
- Vector calculus
  - Given a vector field $\vec{v}$, a field line is a line always tangent to the field
  - The divergence $\vec{\nabla} \cdot \vec{v}$ is the flow of the field through an infinitesimal closed region
  - A divergence-free field $\vec{\nabla} \cdot \vec{B} = 0$ admits a vector potential $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$

https://assumptionsofphysics.org/

Assumptions of Physics

Action: $\mathcal{A}[\gamma] = \int_\gamma L(q(t), \dot{q}(t), t)dt$

Lagrangian: $L = T - U$

Does not always work!

Lagrangian for particle with charge q:

$$L = \frac{1}{2}mv^2 + \mathrm{q}\vec{v}\cdot\vec{A} - \mathrm{q}V$$

Kinetic ??? Potential

https://assumptionsofphysics.org/

Assumptions of Physics

Displacement field

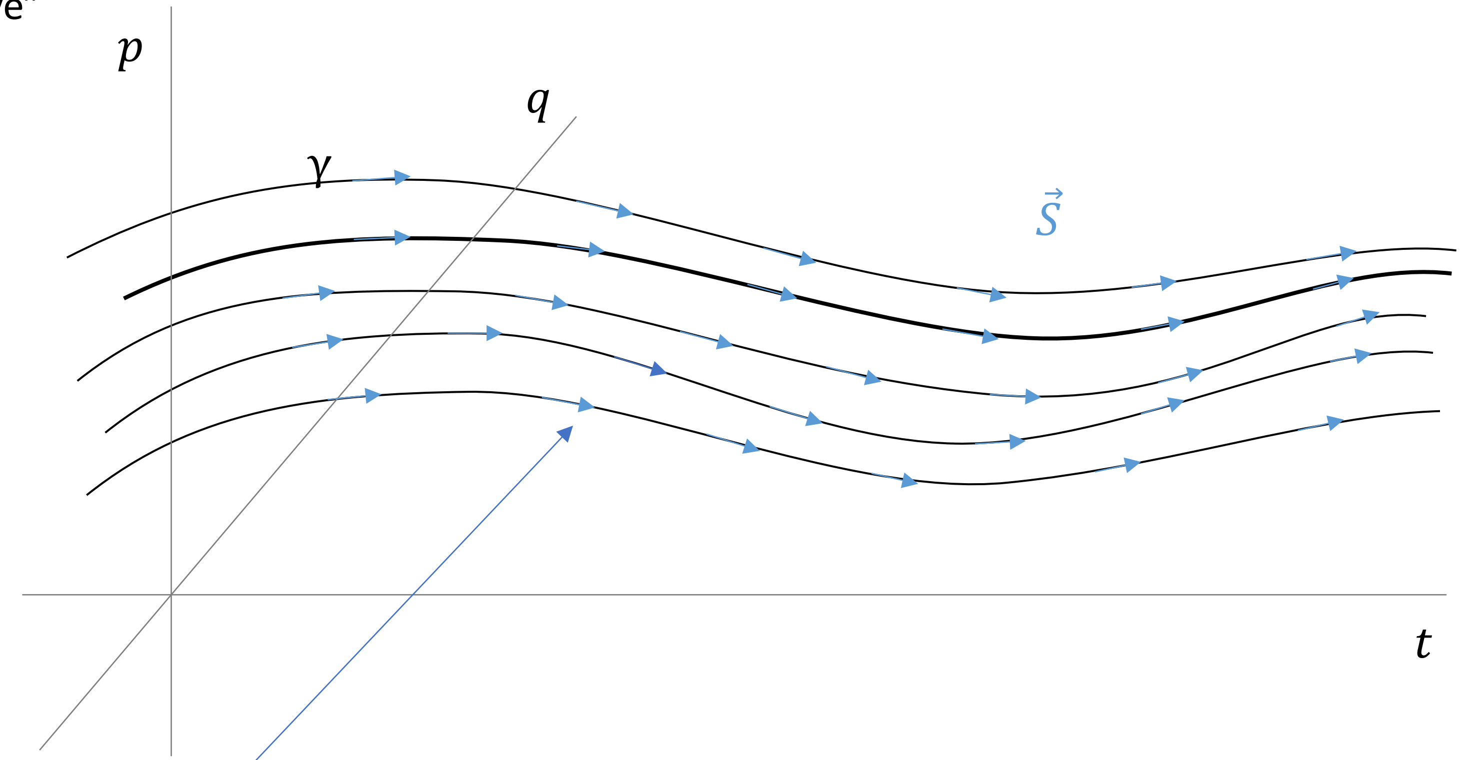
Tells us how states "move"

$$\vec{S} = \left[\frac{dq}{dt}, \frac{dp}{dt}, \frac{dt}{dt}\right]$$

Evolution

$$\gamma = [q(t), p(t), t]$$

Time is both a parameter and a variable



Evolutions are field lines of the displacement field

Always tangent to the field

https://assumptionsofphysics.org/

Assumptions of Physics

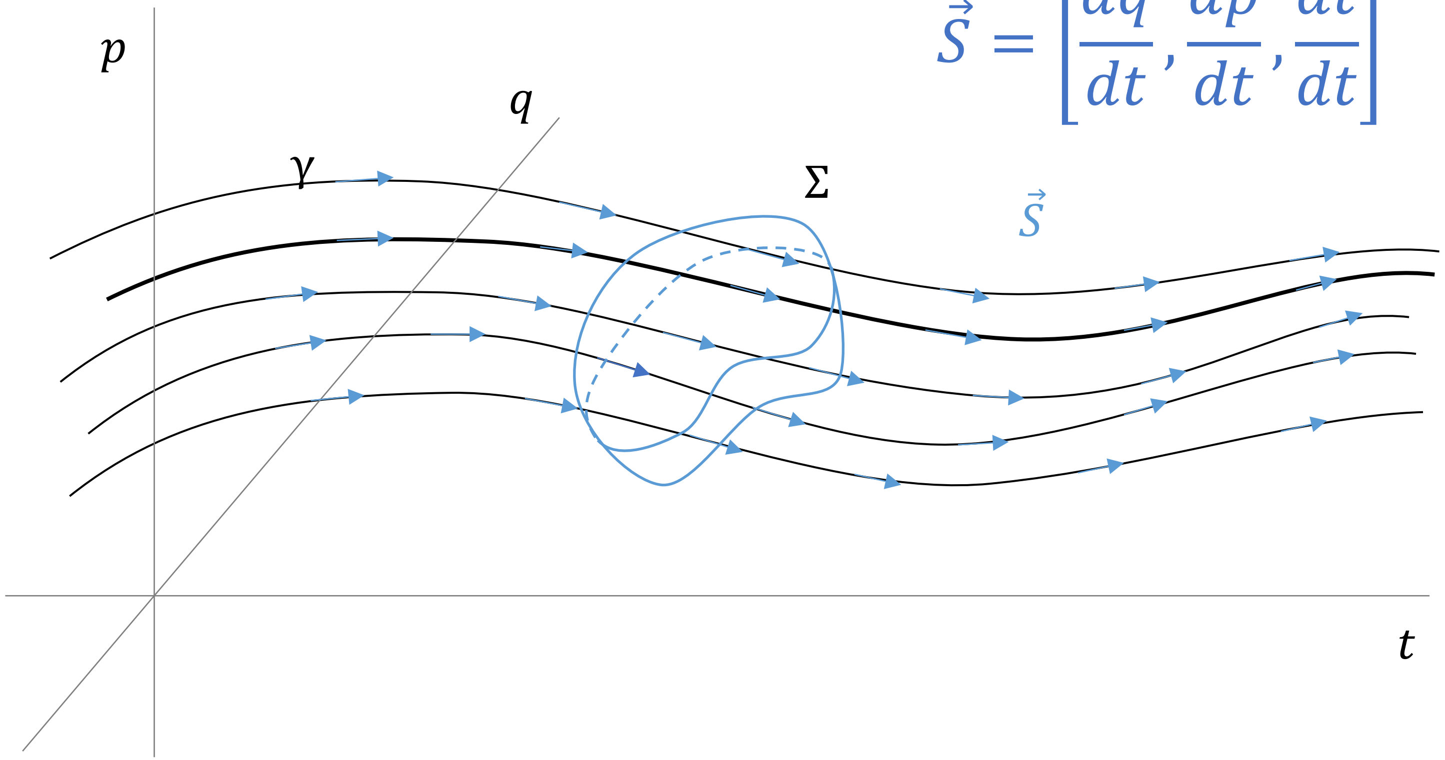
(DR) Determinism and Reversibility

No states are lost or created
(count of states is preserved in time)

$$\oint\!\!\!\!\oint_{\Sigma} \vec{S} \cdot d\vec{\Sigma} = 0$$

⇒ Divergence-free displacement

$$\vec{\nabla} \cdot \vec{S} = 0$$



$$\vec{S} = \left[\frac{dq}{dt}, \frac{dp}{dt}, \frac{dt}{dt}\right]$$

Vector potential

$$\vec{S} = -\vec{\nabla} \times \vec{\theta}$$

Minus sign added to match conventions

Displacement admits a vector potential

$$\vec{\theta} = [\theta_q, \theta_p, \theta_t]$$

Fix gauge: $\theta_a \to \theta_a - \partial_a \int_0^p \theta_p dp$

$$\vec{\theta} = [\theta_q, 0, \theta_t]$$

$S_t = \partial_p \theta_q - \partial_q \theta_p = \frac{dt}{dt} = 1$

$$\vec{\theta} = [p, 0, \theta_t]$$

Rename θ_t

$$\vec{\theta} = [p, 0, -H]$$

Without loss of generality

$$\vec{S} = -\vec{\nabla} \times \vec{\theta}$$

$$\vec{\theta} = [p, 0, -H]$$

similar to four-momentum

https://assumptionsofphysics.org/

Assumptions of Physics

$$\vec{S} = -\vec{\nabla} \times \vec{\theta} \qquad \vec{\theta} = [p, 0, -H(q,p)]$$

$$S^q = \frac{dq}{dt} = -\left(\frac{\partial}{\partial p}\theta_t - \frac{\partial}{\partial t}\theta_p\right) = \frac{\partial H}{\partial p}$$

Hamilton's equations

$$S^p = \frac{dp}{dt} = -\left(\frac{\partial}{\partial t}\theta_q - \frac{\partial}{\partial q}\theta_t\right) = -\frac{\partial H}{\partial q}$$

$$S^t = \frac{dt}{dt} = -\left(\frac{\partial}{\partial q}\theta_p - \frac{\partial}{\partial p}\theta_q\right) = 1$$

https://assumptionsofphysics.org/

Assumptions of Physics

Line integral of $\vec{\theta}$

$$\int_\gamma \vec{\theta} \cdot d\vec{\gamma}$$

Express in components

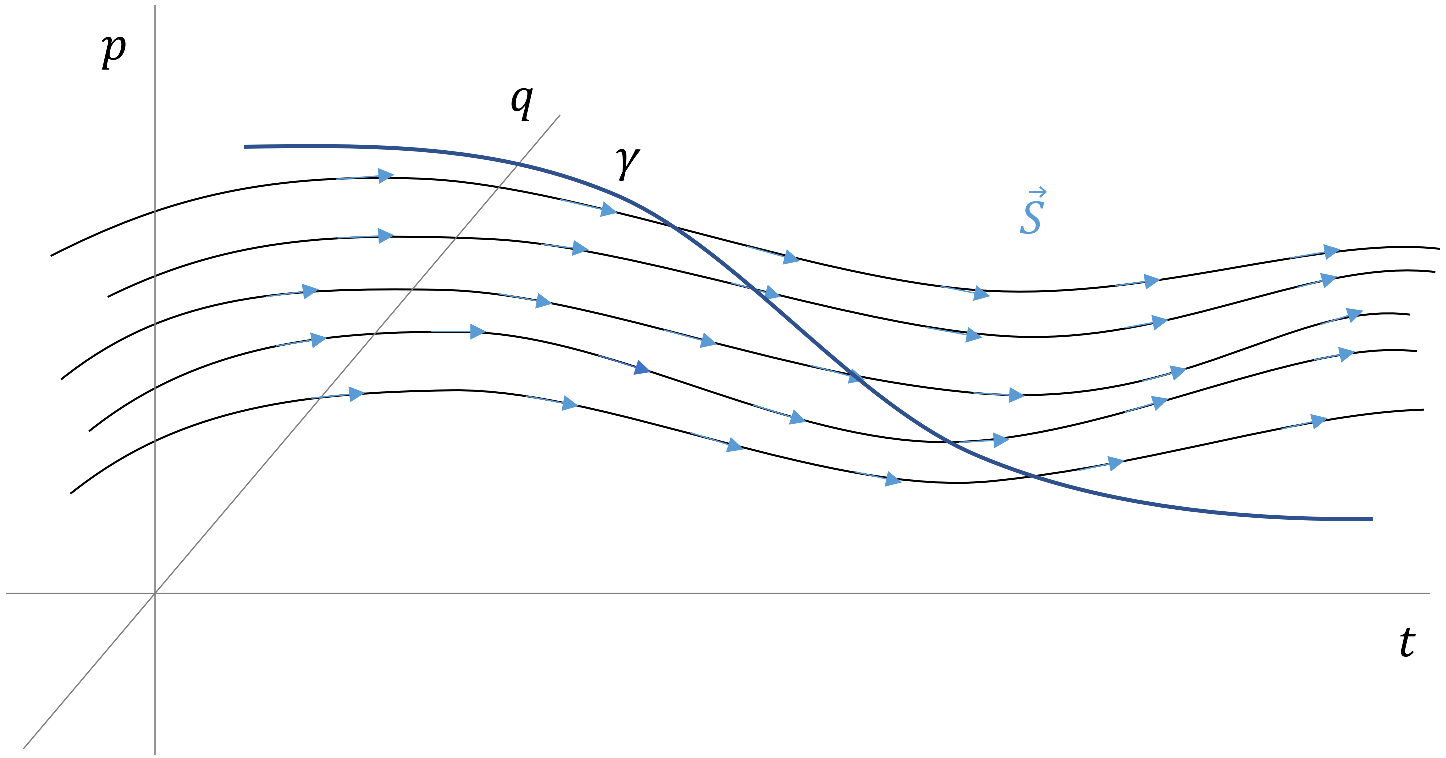
$$= \int_\gamma [p, 0, -H] \cdot \left[\frac{dq}{dt}, \frac{dp}{dt}, \frac{dt}{dt}\right] dt$$

$$= \int_\gamma \left(p \frac{dq}{dt} - H\right) dt$$

$$= \int_\gamma L(q, \dot{q}, t)\, dt$$

$$= \mathcal{A}[\gamma]$$

Assuming (KE) Kinematic Equivalence:
$p = p(q, \dot{q}, t)$



The action is the line integral
of the vector potential of the flow of states

https://assumptionsofphysics.org/

Assumptions of Physics

Flow between a path γ and its variation γ'

$$\Phi[\Sigma] = -\iint_\Sigma \vec{S} \cdot d\vec{\Sigma}$$

Hamilton equations

$$= \iint_\Sigma \vec{\nabla} \times \vec{\theta} \cdot d\vec{\Sigma}$$

Stokes' theorem

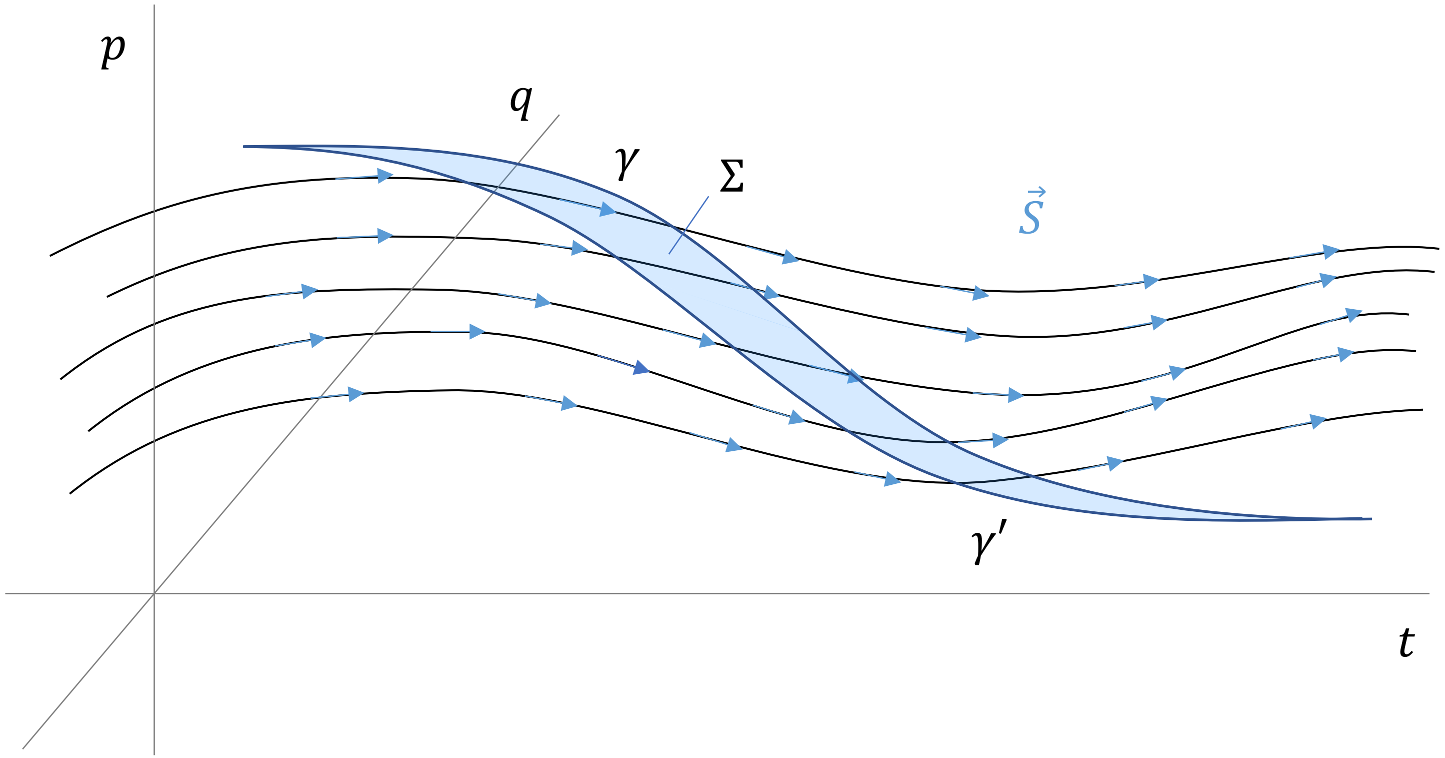
$$= \oint_{\partial\Sigma} \vec{\theta} \cdot d\vec{\gamma}$$

$$= \int_\gamma \vec{\theta} \cdot d\vec{\gamma} - \int_{\gamma'} \vec{\theta} \cdot d\vec{\gamma}$$

$$= \delta \int_\gamma \vec{\theta} \cdot d\vec{\gamma}$$

$$= \delta \int_\gamma L\, dt$$

$$= \delta \mathcal{A}[\gamma]$$



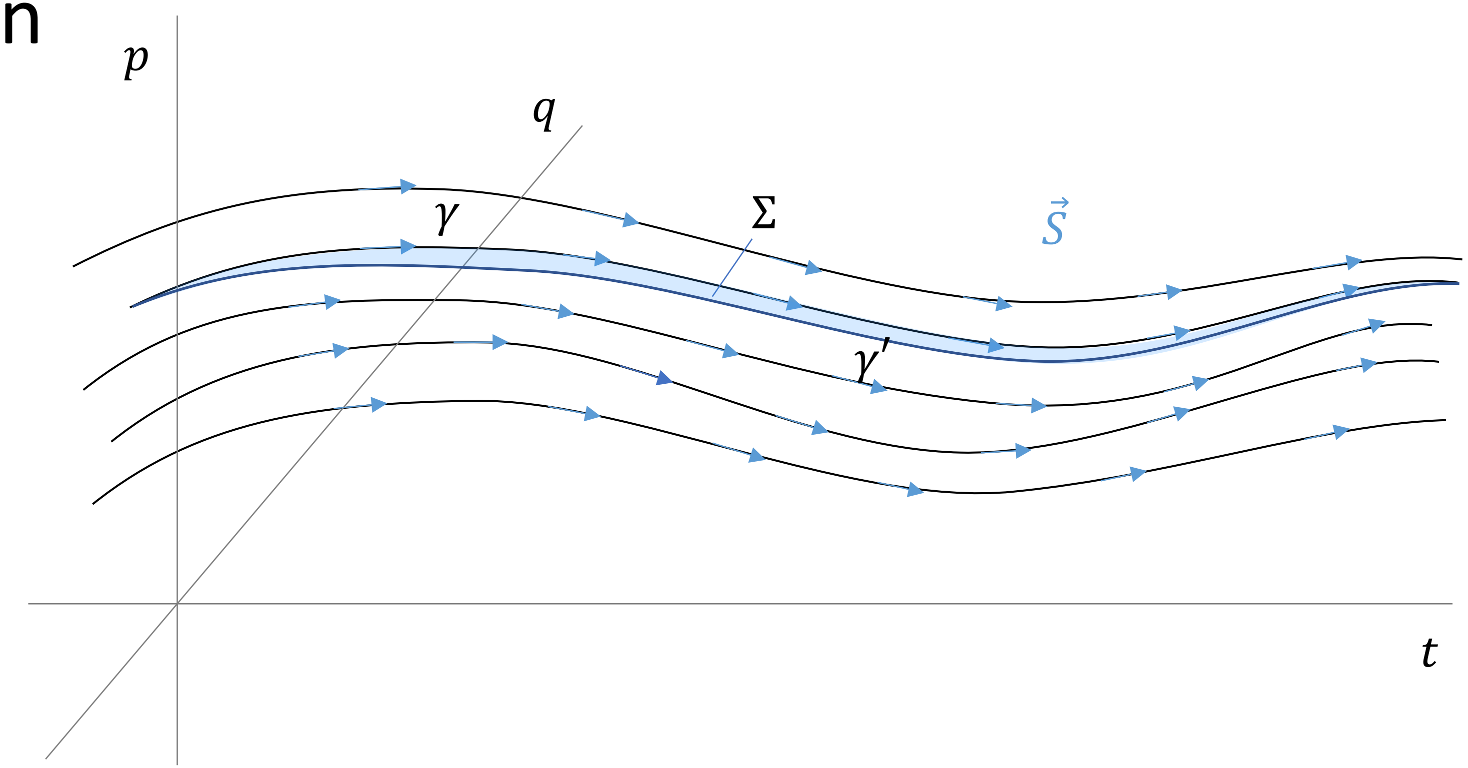
The variation of the action is the flow between path and its variation

https://assumptionsofphysics.org/
Assumptions of Physics

If the path γ is an evolution of the system

$$\Phi[\Sigma] = -\iint_{\Sigma} \vec{S} \cdot d\vec{\Sigma}$$
$$= \delta \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma} = 0$$

flow is tangent



If the path γ is not an evolution of the system, we can always find a variation γ' such that

$$-\iint_{\Sigma} \vec{S} \cdot d\vec{\Sigma} = \delta \int_{\gamma} \vec{\theta} \cdot d\vec{\gamma} \neq 0$$

Action is stationary over and only over evolutions

https://assumptionsofphysics.org/

Assumptions of Physics

1 DOF of classical phase space

+

Determinism/Reversibility

+

Kinematic equivalence

$\Updownarrow$

Principle of stationary action

https://assumptionsofphysics.org/

Assumptions of Physics

Action principle multiple DOF

https://assumptionsofphysics.org/

Assumptions of Physics

Required background

- Elements of differential geometry
  - A differential n-form takes n vectors and returns a number that depends on the volume of the parallelepiped (must be anti-symmetric):
    $W[\gamma] = \int_\gamma dW = \int_\gamma F_a dx^a$
    $\Phi[\Sigma] = \int_\Sigma d\Phi = \int_\Sigma B_{ab} v^a w^b$
    $m[V] = \int_V dm = \int_{\mathrm{V}} \rho_{abc} v^a w^b u^c$
  - Pseudo-vectors are really two-forms in 3D: $\vec{B} \to B_{ij}$
  - The exterior derivative generalizes gradient, curl, divergence:
    $\nabla f \to \partial_a \wedge f \qquad \nabla \times \vec{\theta} \to \partial_a \wedge \theta_b \qquad \nabla \cdot \vec{B} \to \partial_a \wedge B_{bc}$
- Generalized properties
  - Generalized Stokes theorem: $\int_\Sigma \partial_a \wedge \theta_b \, d\Sigma^{ab} = \int_{\partial\Sigma} \theta_b d\gamma^a$
  - Define closed form as having zero exterior derivative: $\partial_a \wedge \omega_{ab} = 0$
  - A closed form (locally) admits a vector potential:

$$\omega_{ab} = \partial_a \wedge \theta_b = \partial_a \theta_b - \partial_b \theta_a$$

https://assumptionsofphysics.org/

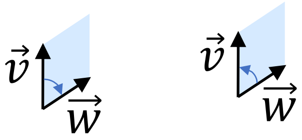
Assumptions of Physics

Characterize configuration flow

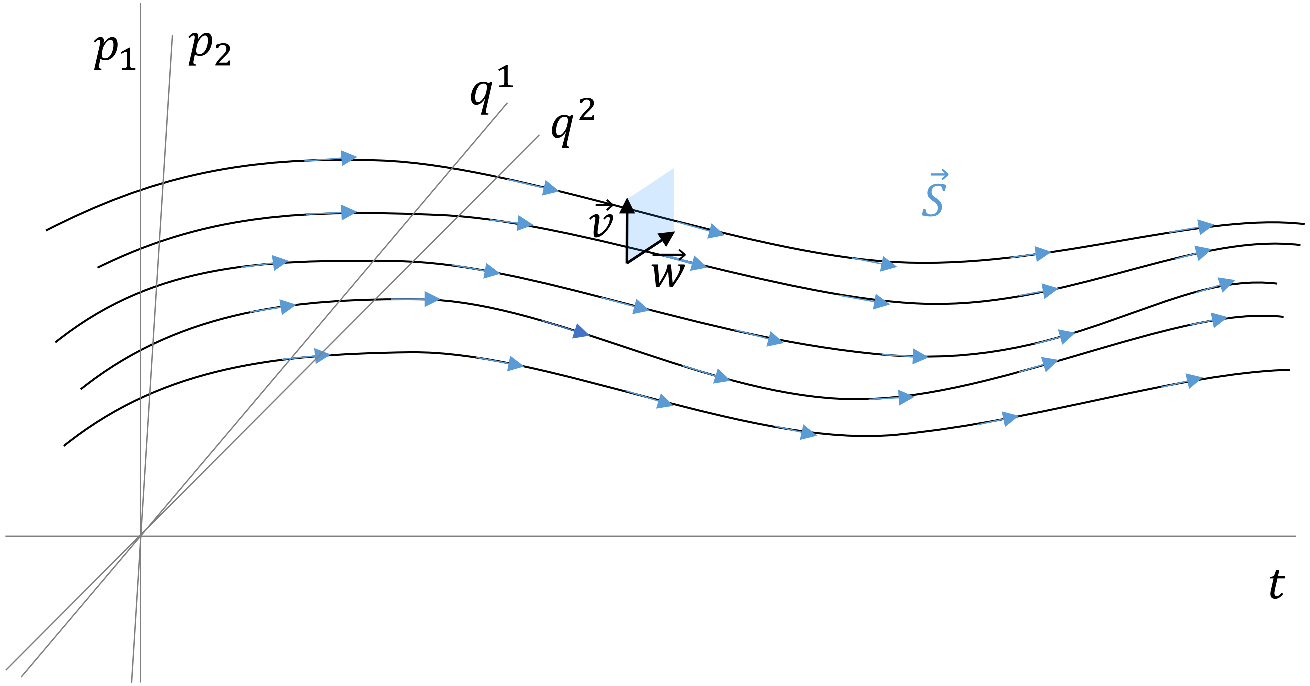
Flow of configurations through an infinitesimal surface

$$d\Phi = v^a \omega_{ab} w^b$$

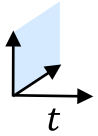
Function of surface: $\omega_{ab} = -\omega_{ba}$



change of order = change of orientation

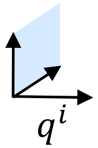


(DR) Determinism and reversibility: $\partial_t \wedge \omega_{ab} = 0$



Configurations are conserved through time

(IND) DOF independence: $\partial_{q^i} \wedge \omega_{ab} = 0$ $\partial_{p_i} \wedge \omega_{ab} = 0$



Configurations are conserved across DOF

IND + DR: ω_{ab} closed two-form $(\partial_a \wedge \omega_{bc} = 0)$

https://assumptionsofphysics.org/

Assumptions of Physics

$$\partial_a \wedge \omega_{bc} = 0$$

Closed form (DR) + (IND)

$$\omega_{ab} = -\partial_a \wedge \theta_b = -(\partial_a\theta_b - \partial_b\theta_a)$$

Vector potential

θ is the potential of ω

$$\theta_a = \begin{bmatrix} p_i & 0 & -H \end{bmatrix}$$

without loss of generality

$$\begin{aligned}
\partial_a\theta_b - \partial_b\theta_a &= \begin{bmatrix} \partial_{q^i}\theta_{q^j} - \partial_{q^j}\theta_{q^i} & \partial_{q^i}\theta_{p_j} - \partial_{p_j}\theta_{q^i} & \partial_{q^i}\theta_t - \partial_t\theta_{q^i} \\ \partial_{p_i}\theta_{q^j} - \partial_{q^j}\theta_{p_i} & \partial_{p_i}\theta_{p_j} - \partial_{p_j}\theta_{p_i} & \partial_{p_i}\theta_t - \partial_t\theta_{p_i} \\ \partial_t\theta_{q^j} - \partial_{q^j}\theta_t & \partial_t\theta_{p_j} - \partial_{p_j}\theta_t & \partial_t\theta_t - \partial_t\theta_t \end{bmatrix} \\
&= \begin{bmatrix} \partial_{q^i}p_j - \partial_{q^j}p_i & \partial_{q^i}0 - \partial_{p_j}p_i & \partial_{q^i}(-H) - \partial_t p_i \\ \partial_{p_i}p_j - \partial_{q^j}0 & \partial_{p_i}0 - \partial_{p_j}0 & \partial_{p_i}(-H) - \partial_t 0 \\ \partial_t p_j - \partial_{q^j}(-H) & \partial_t 0 - \partial_{p_j}(-H) & \partial_t(-H) - \partial_t(-H) \end{bmatrix} \\
&= \begin{bmatrix} 0 - 0 & 0 - \delta_i^j & -\partial_{q^i}H - 0 \\ \delta^i_j - 0 & 0 - 0 & -\partial_{p_i}H - 0 \\ 0 + \partial_{q^j}H & 0 + \partial_{p_j}H & -\partial_t H + \partial_t H \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\delta_i^j & -\partial_{q^i}H \\ \delta^i_j & 0 & -\partial_{p_i}H \\ \partial_{q^j}H & \partial_{p_j}H & 0 \end{bmatrix}.
\end{aligned}$$

$$\omega_{ab} = \begin{bmatrix} \omega_{q^iq^j} & \omega_{q^ip_j} & \omega_{q^it} \\ \omega_{p_iq^j} & \omega_{p_ip_j} & \omega_{p_it} \\ \omega_{tq^j} & \omega_{tp_j} & \omega_{tt} \end{bmatrix} = \begin{bmatrix} 0 & \delta^i_j & \partial_{q^i}H \\ -\delta_i^j & 0 & \partial_{p_i}H \\ -\partial_{q^j}H & -\partial_{p_j}H & 0 \end{bmatrix}$$

https://assumptionsofphysics.org/

Assumptions of Physics

Displacement kills the flow

Displacement only direction that "kills" the flow

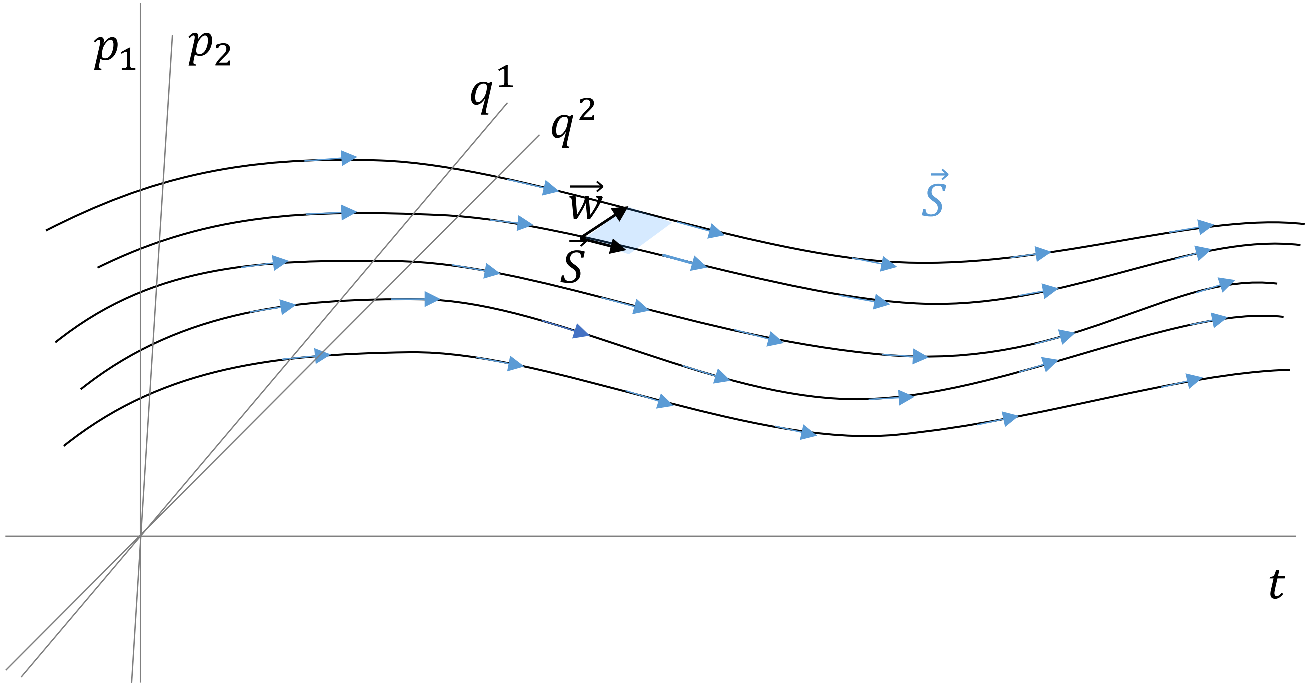
$$S^a \omega_{ab} w^b = 0$$

For all $\vec{w}$

Flow is tangent (and only tangent) to the displacement

$$S^a \omega_{ab} = 0$$

Let's calculate!



https://assumptionsofphysics.org/

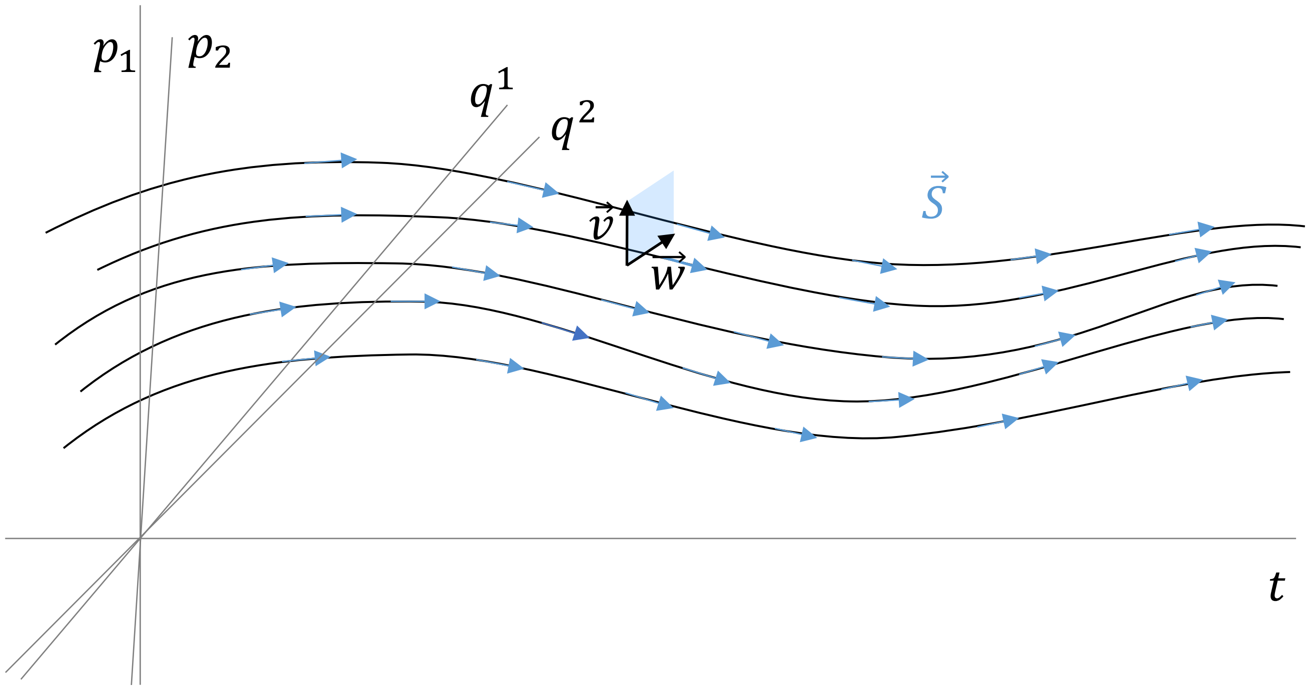
Assumptions of Physics

Displacement kills the flow

$$S^a \omega_{ab} = 0$$

$$\omega_{ab} = \begin{bmatrix} 0 & \delta^i_j & \partial_{q^i} H \\ -\delta^j_i & 0 & \partial_{p_i} H \\ -\partial_{q^j} H & -\partial_{p_j} H & 0 \end{bmatrix}$$

$$\begin{aligned}
S^a \omega_{aq^j} &= S^{q^i}\omega_{q^i q^j} + S^{p_i}\omega_{p_i q^j} + S^t \omega_{tq^j} \\
&= -S^{p_j} - S^t \partial_{q^j} H = -S^{p_j} - \partial_{q^j} H = 0 \\
S^a \omega_{ap_j} &= S^{q^i}\omega_{q^i p_j} + S^{p_i}\omega_{p_i p_j} + S^t \omega_{tp_j} \\
&= S^{q^j} - S^t \partial_{p_j} H = S^{q^j} - \partial_{p_j} H = 0 \\
S^a \omega_{at} &= S^{q^i}\omega_{q^i t} + S^{p_i}\omega_{p_i t} + S^t \omega_{tt} \\
&= S^{q^i}\partial_{q^i} H + S^{p_i}\partial_{p_i} H \\
&= \partial_{p_i} H \partial_{q^i} H - \partial_{q^i} H \partial_{p_i} H = 0.
\end{aligned}$$



$$S^{p_i} = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

$$S^{q^i} = \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

Recovers Hamilton's equations

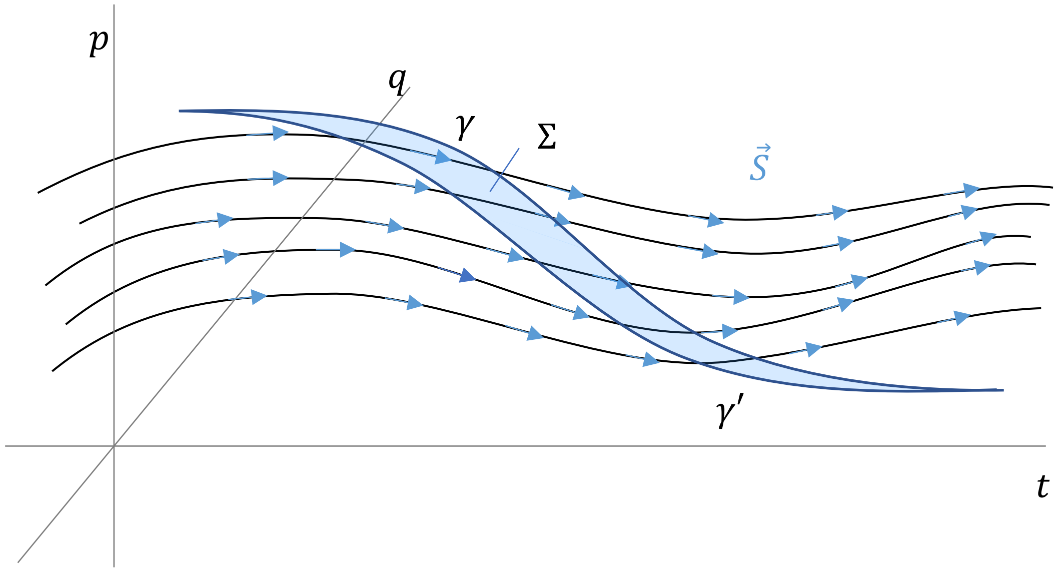
https://assumptionsofphysics.org/

Assumptions of Physics

$$\xi^a = [q^i, p_i, t] \qquad \theta_a = [p_i, 0, -H]$$

$$\int_\gamma \theta_a d\gamma^a = \int_\gamma \theta_a d_t \xi^a dt = \int_\gamma \left(p_i d_t q^i - H\right) dt = \int_\gamma L dt = \mathcal{A}[\gamma]$$

Action is the line integral of the vector potential



https://assumptionsofphysics.org/

Assumptions of Physics

$$\delta\mathcal{A}[\gamma] = \delta\int_\gamma L dt = \delta\int_\gamma \theta_a d_t\xi^a dt = \oint_{\partial\Sigma}\theta_a d\xi^a = \int_\Sigma \partial_a \wedge \theta_b d\xi^a d\eta^b = -\int_\Sigma d\xi^a \omega_{ab} d\eta^b = -\Phi[\Sigma]$$

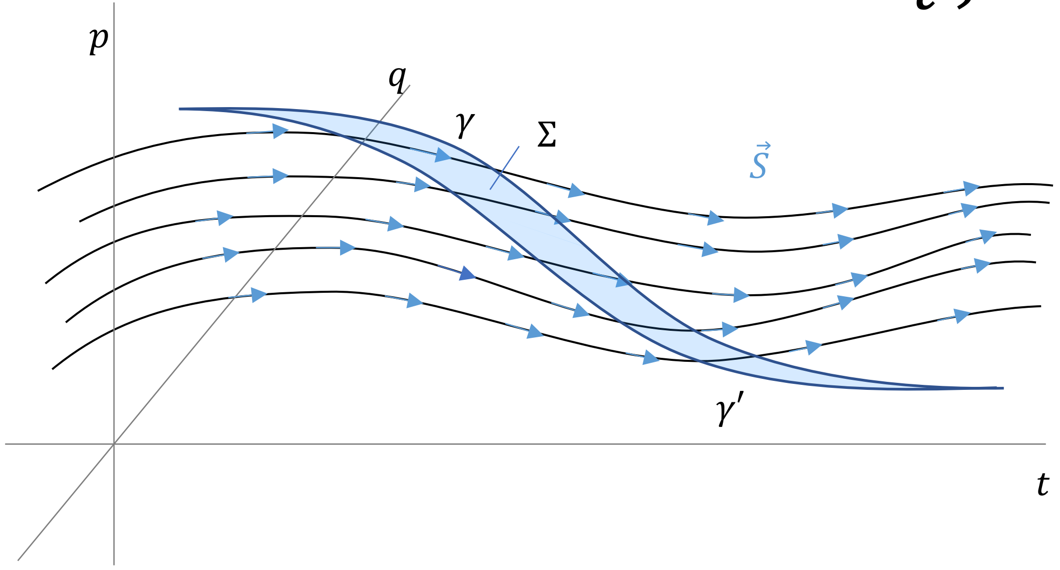
The variation of the action is the flow between path and its variation

$$\delta\mathcal{A}[\gamma] = 0 \iff d\xi^a \omega_{ab} d\eta^b = dt d_t\xi^a \omega_{ab} d\eta^b = 0 \;\; \forall d\eta\hat{}b$$

$$\iff d_t\xi^a \omega_{ab} = 0$$

$$d_t\xi^a = S^a$$

Action is stationary when the path kills the flow



https://assumptionsofphysics.org/

Assumptions of Physics

DOF independence

+

Determinism/Reversibility

+

Kinematic equivalence

$\Updownarrow$

Principle of stationary action

https://assumptionsofphysics.org/

Assumptions of Physics

Relativistic mechanics

https://assumptionsofphysics.org/

Assumptions of Physics

Required background

- Special relativity
  - Space and time coordinates are grouped together: $q^\alpha = \left[ct, q^i\right]$
  - Metric tensor $g_{\alpha\beta}$ defines the geometry of space-time (i.e. lengths and angles)
  - Using $(-,+,+,+)$ signature: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
  - Energy and momentum are grouped together (four-momentum): $p_\alpha = \left[-\frac{E}{c}, p_i\right]$
  - Four-velocity is the velocity with respect to proper time (time in the rest frame) and its norm is $-c^2$: $u^\alpha = d_\tau q^\alpha \quad u^\alpha g_{\alpha\beta} u^\beta = -c^2$
  - For a free particle, relationship between four-momentum and four-velocity: $p_\beta g^{\beta\alpha} = p^\alpha = m u^\alpha$

https://assumptionsofphysics.org/

Assumptions of Physics

Want to extend Hamiltonian mechanics to handle change of coordinates that mix space and time variables

$$q^\alpha = [t, q^i]$$

$$[q^i(t)] \to [t(s), q^i(s)]$$

Separate time-the-variable from time-the-parameter

Hamiltonian mechanics on the extended phase space

Recall

$$\theta_a = [\overset{\theta_{q^i}}{p_i}, 0, \overset{\theta_t}{-H}]$$

Form a covector

$$p_\alpha = [-E, p_i]$$

$\Rightarrow$ only possible choice for conjugates

$$\frac{dq^\alpha}{ds} = \frac{\partial \mathcal{H}}{\partial p_\alpha} \quad \frac{dp_\alpha}{ds} = -\frac{\partial \mathcal{H}}{\partial q^\alpha}$$

Hamiltonian constraint

$$\mathcal{H}(t, q^i, E, p_i) = 0$$

Not just conserved quantity

https://assumptionsofphysics.org/

Assumptions of Physics

$$q^\alpha = [t, q^i]$$
$$p_a = [-E, p_i]$$

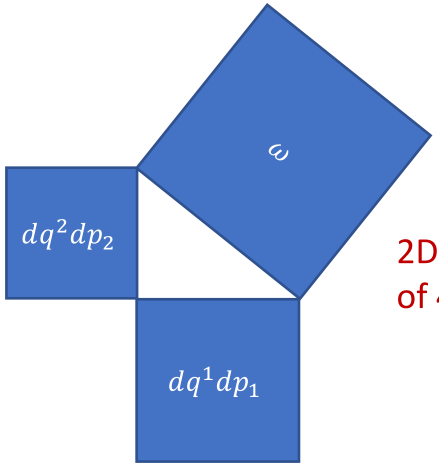
$$\omega = dq^i dp_i - dt dE$$

Energy-momentum covector and negative sign for temporal DOF appear in the extended phase space without knowledge of space-time geometry (or c)

Lorentzian relativity is the only "correct" one

It extends the count of configurations per DOF to the temporal DOF in the correct way

$$\omega = dq^1 dp_1 + dq^2 dp_2$$

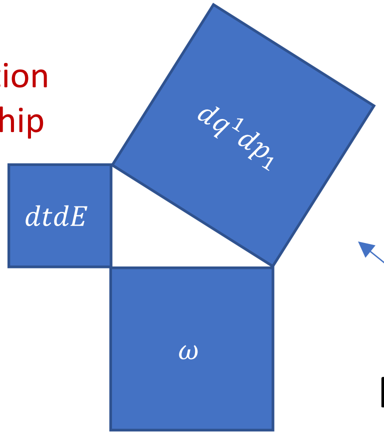


Independent DOFs are orthogonal

2D representation of 4D relationship

$$\omega = dq^1 dp_1 - dt dE$$

$$dq^1 dp_1 = \omega + dt dE$$



States are counted at equal time: temporal DOF orthogonal to ω

Pythagoras's formula still present

https://assumptionsofphysics.org/

Assumptions of Physics

Example: free particle in an inertial frame

$$\mathcal{H} = \frac{1}{2m}\left(p_i \delta^{ij} p_j - \left(\frac{E}{c}\right)^2 + m^2 c^2\right)$$

This Hamiltonian constraint is the generator for proper time (i.e. affine parameter s corresponds to proper time)

$$\frac{dt}{ds} = \frac{\partial \mathcal{H}}{\partial(-E)} = -\frac{1}{2mc^2}\frac{\partial}{\partial(-E)}(-E)^2 = -\frac{1}{2mc^2}(-2E) = \frac{E}{mc^2}$$

$$\frac{dq^i}{ds} = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{1}{2m}\frac{\partial}{\partial p_i}\left(p_i \delta^{ij} p_j\right) = \frac{2\delta^{ij} p_j}{2m} = \frac{p_i}{m}$$

$$\frac{dE}{ds} = -\frac{d(-E)}{ds} = \frac{\partial \mathcal{H}}{\partial t} = 0$$

$$\frac{dp_i}{ds} = -\frac{\partial \mathcal{H}}{\partial q^i} = 0$$

four momentum

$$\frac{E}{c} = m\frac{d(ct)}{ds}$$

$$p_i = m\frac{dq^i}{ds}$$

four velocity

proper time!

Compare to "standard" way

Generator for proper time

Nice and quadratic

$$\mathcal{H} = \frac{1}{2m}\left(p_i\delta^{ij}p_j - \left(\frac{E}{c}\right)^2 + m^2c^2\right)$$

$$H = c\sqrt{m^2c^2 + p_i\delta^{ij}p_j}$$

Hamiltonian is time generator

$$\frac{dq^i}{dt} = \frac{ds}{dt}\frac{dq^i}{ds} = \frac{mc^2}{E}\frac{p_i}{m}$$

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = c\frac{\partial}{\partial p_i}\sqrt{m^2c^2 + p_i\delta^{ij}p_j}$$

$$= \frac{c}{2}\frac{1}{\sqrt{m^2c^2 + p_i\delta^{ij}p_j}}\frac{\partial}{\partial p_i}\left(m^2c^2 + p_i\delta^{ij}p_j\right)$$

$$= \frac{1}{2}\frac{2\delta^{ij}p_j}{\sqrt{m^2 + \frac{p_i\delta^{ij}p_j}{c^2}}} = \frac{p_i}{\sqrt{m^2 + \frac{|p_i|^2}{c^2}}}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} = 0$$

Hamiltonian mechanics on the extended phase-space comes out more naturally and is a nicer formulation

Wait… quadratic?
Can E be negative?

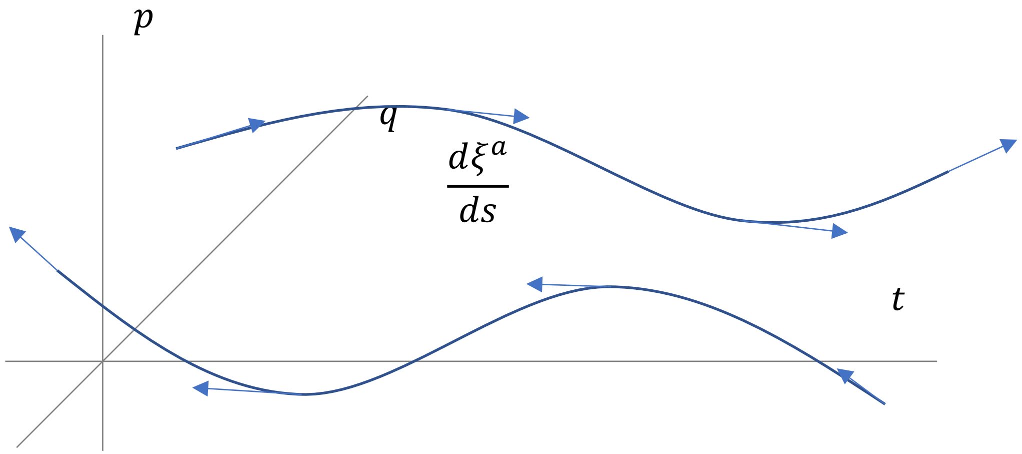
https://assumptionsofphysics.org/

Assumptions of Physics

Classical anti-particles

This can be negative

$$d_s t = \partial_{-E}\mathcal{H} \qquad d_s(-E) = \partial_t \mathcal{H}$$
$$d_s q^i = \partial_{p_i}\mathcal{H} \qquad d_s p_i = \partial_{q^i}\mathcal{H}$$



Particle: affine parameter aligned with time

Anti-particle: affine parameter anti-aligned with time

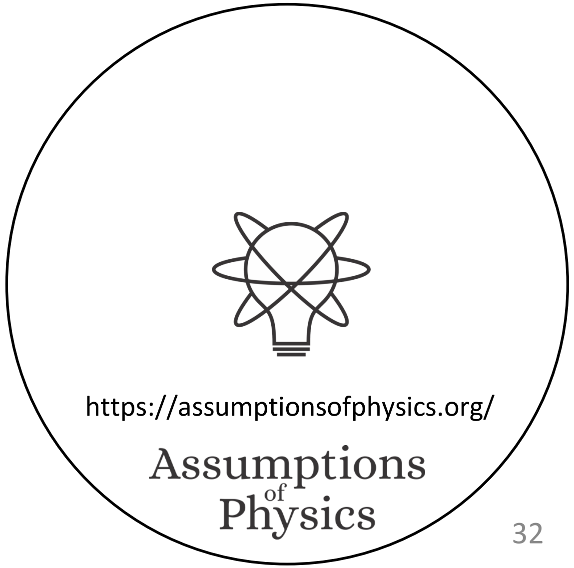
Parametrization flows backwards with respect to time

An evolution cannot change time alignment

Free particle

$$d_s t = \frac{E}{mc^2}$$

Anti-particle = negative energy



Recover standard Hamiltonian

Over valid states $\mathcal{H} = 0$ and $E = H(q^i, p_i, t)$

$$\mathcal{H} = \lambda(H - E)$$

Assume no E dependence

$$E = H$$

$$d_t t = 1 = d_t s \, d_s t = d_t s \, \partial_{-E} \mathcal{H} = d_t s \, \lambda$$

$$d_t s = \frac{1}{\lambda}$$

$= 0$

$$d_t q^i = d_t s \, d_s q^i = d_t s \, \partial_{p_i} \mathcal{H} = \frac{1}{\lambda} \left(\partial_{p_i} \lambda (H - E) + \lambda \partial_{p_i} H \right) = \partial_{p_i} H$$

$$d_t p_i = d_t s \, d_s p_i = -d_t s \, \partial_{q^i} \mathcal{H} = -\frac{1}{\lambda} \left(\partial_{q^i} \lambda (H - E) + \lambda \partial_{q^i} H \right) = -\partial_{q^i} H$$

$$d_t E = d_t s \, d_s E = d_t s \, \partial_t \mathcal{H} = \frac{1}{\lambda} \left(\partial_t \lambda (H - E) + \lambda \partial_t H \right) = \partial_t H$$

https://assumptionsofphysics.org/

Assumptions of Physics

Hamiltonian constraint can "hide" multiple Hs

Free particle Hamiltonian constraint

$$\mathcal{H} = \frac{1}{2mc^2}\left(\sqrt{c^2|p_i|^2 + (mc^2)^2} + E\right)\left(\sqrt{c^2|p_i|^2 + (mc^2)^2} - E\right)$$

$$\mathcal{H} = (H_1 - E)(H_2 - E)\,\frac{-1}{2mc^2}$$

$H_1 = -H_2$

Negative energy

Positive energy

$$d_s t = \frac{1}{2mc^2}(E + E) = \frac{E}{mc^2}$$

https://assumptionsofphysics.org/

Assumptions of Physics

Classical-quantum parallel

Free particle Hamiltonian constraint

$$\mathcal{H} = \frac{1}{2m}\left(p_\alpha \eta^{\alpha\beta} p_\beta + m^2 c^2\right) = \frac{1}{2m}\left(p_i \delta^{ij} p_j - (E/c)^2 + m^2 c^2\right)$$

Hamiltonian constraint becomes Klein-Gordon equation in QM

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2}\right)\psi(t, \mathbf{x}) = 0$$

minus sign from $\imath^2$

Also note: $$\mathcal{H}\rho = 0$$

Density can be non-zero only where $\mathcal{H} = 0$

https://assumptionsofphysics.org/

Assumptions of Physics

Hamiltonian constraint for charged particles

pure four-velocity kinetic term

$$\mathcal{H} = \frac{1}{2m}(p_\alpha - \mathfrak{q}A_\alpha)g^{\alpha\beta}(p_\beta - \mathfrak{q}A_\beta) + \frac{1}{2}mc^2 = \frac{1}{2}mu^\alpha g_{\alpha\beta}u^\beta + \frac{1}{2}mc^2$$

$$mu_\alpha = mu^\beta g_{\beta\alpha} = p_\alpha - \mathfrak{q}A_\alpha$$

Kinetic momentum

Momentum gauge and EM gauge must cancel out

Compare to EM Klein-Gordon equation

$$\left[c^2(\partial_\alpha - \imath\mathfrak{q}A_\alpha)\eta^{\alpha\beta}(\partial_\alpha - \imath\mathfrak{q}A_\alpha) + m^2c^4\right]\psi = \left[c^2 D_\alpha \eta^{\alpha\beta} D_\beta + m^2c^4\right]\psi = 0$$

$$D_\alpha = \partial_\alpha - \imath\mathfrak{q}A_\alpha$$

Gauge-covariant derivative is kinetic momentum

https://assumptionsofphysics.org/

Assumptions of Physics

Takeaways

- Relativistic mechanics is recovered without additional assumptions
  - Relativity is needed to make densities/entropy invariant over time transformations
- Hamiltonian mechanics on the extended phase space is a more relativistic formulation
  - Hamiltonian constraint is the generator of the affine parameter, and can be taken to generate proper time for particles and minus proper time for anti-particles
- Multiple connections to quantum mechanics
  - Anti-particles already present in classical Hamiltonian particle mechanics
  - Hamiltonian constraint related to Klein-Gordon (and Dirac) operators
  - Gauge considerations are already present in classical particle mechanics

https://assumptionsofphysics.org/

Assumptions of Physics

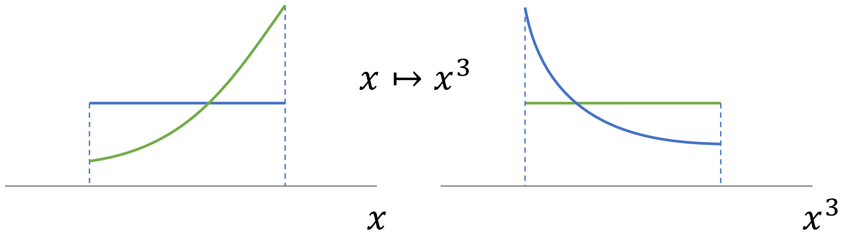
Directional DOF

https://assumptionsofphysics.org/

Assumptions of Physics

Why DOFs are charted by two conjugate variables

Density, entropy, uniform distributions
NOT in general coordinate-invariant



The principle of indifference fails over continuous quantities

Phase space (symplectic) structure is the only one that supports coordinate-invariant density, entropy, state count

$\hat{q} = 100\ cm/m\ q$

Each unit variable (i.e. coordinate) paired with a conjugate of inverse units: number of states $\Delta q \Delta k$ is invariant

$\Delta k = 1\ m^{-1}$ 1

1 $\Delta \hat{k} = 0.01\ cm^{-1}$

$\Delta \hat{q} = 100\ cm$

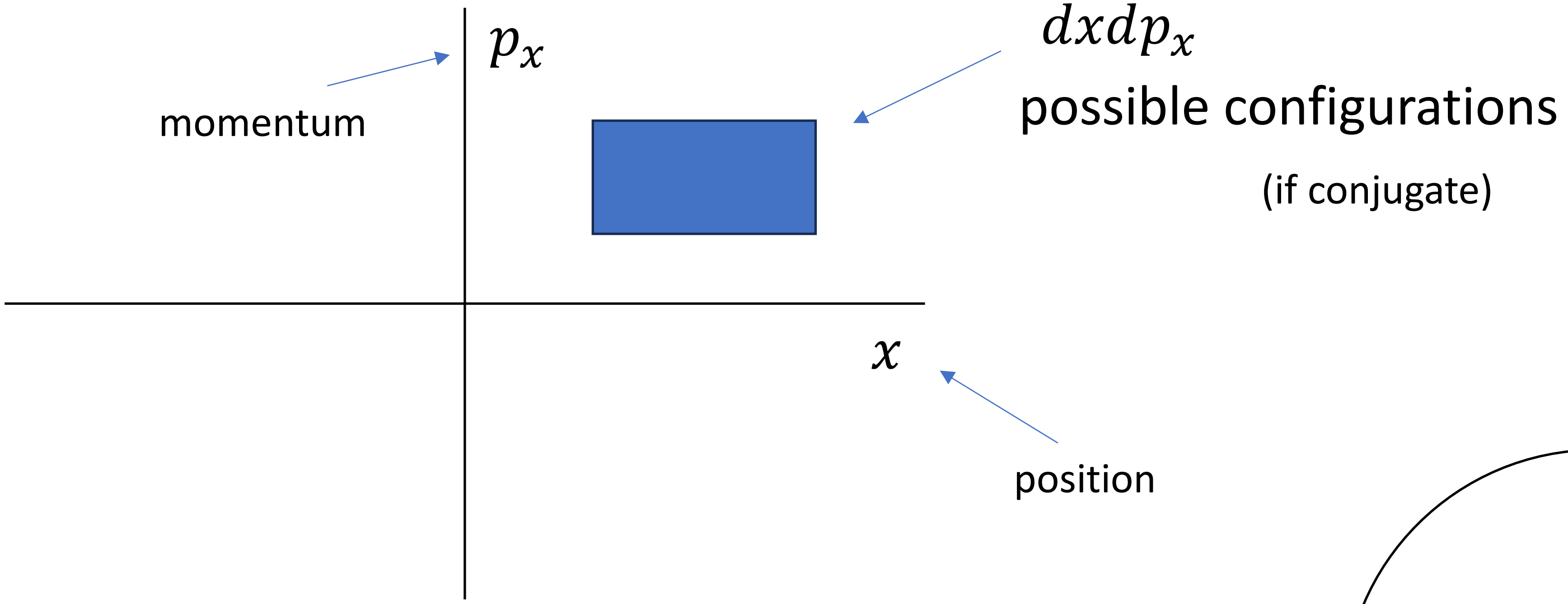
$\Delta q = 1\ m$

Conjugate quantities recover frame independence of states

https://assumptionsofphysics.org/

Assumptions of Physics

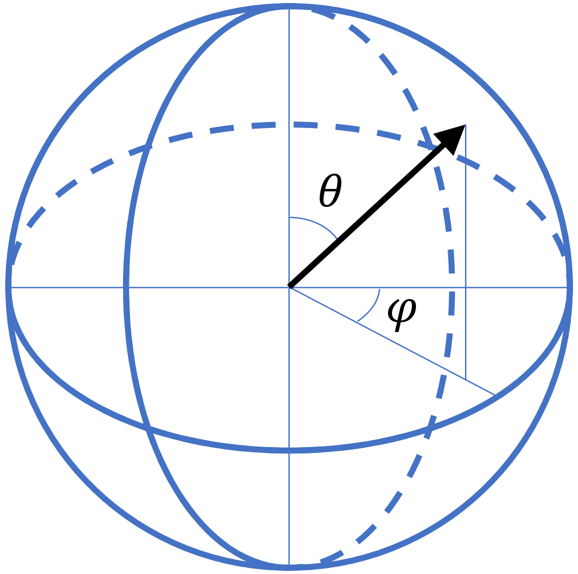
Each DOF is charted by two variables



https://assumptionsofphysics.org/

Assumptions of Physics

Angular momentum



$$\vec{L} = \left[L_x, L_y, L_z\right]$$

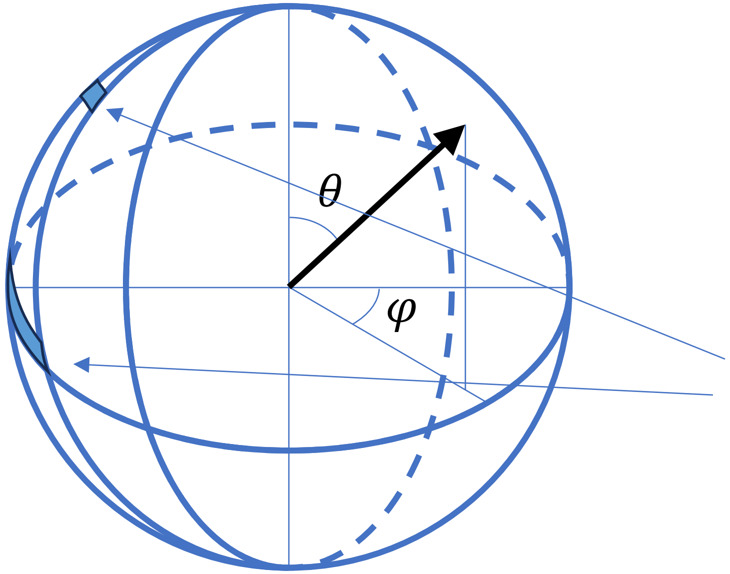
$$= [L \sin\theta \cos\varphi , L \sin\theta \sin\varphi , L \cos\theta]$$

Direction independent from the magnitude

https://assumptionsofphysics.org/

Assumptions of Physics

Angular momentum direction



Direction is an independent DOF

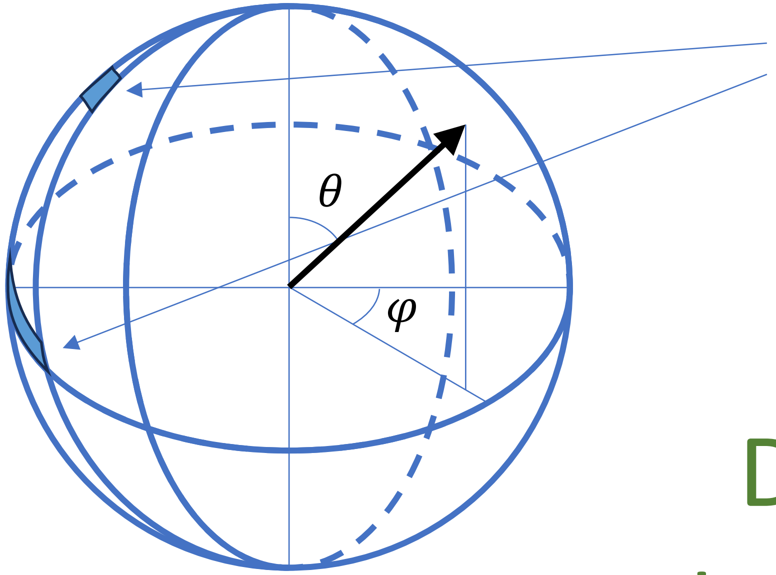
However, θ and φ are not conjugate

$d\theta d\varphi \neq$ possible configurations

https://assumptionsofphysics.org/

Assumptions of Physics

Angular momentum direction



$$Ld\Omega = L\sin\theta\, d\theta d\varphi = d(L\cos\theta)d\varphi$$
$$= dL_z d\varphi_{xy}$$

φ_{xy} and L_z are conjugate

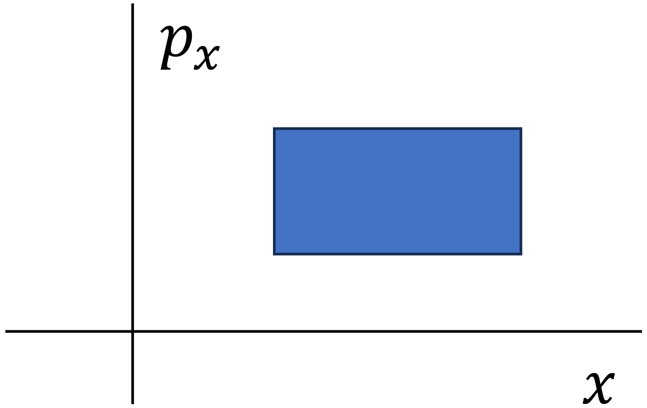
Direction is charted by two conjugate quantities

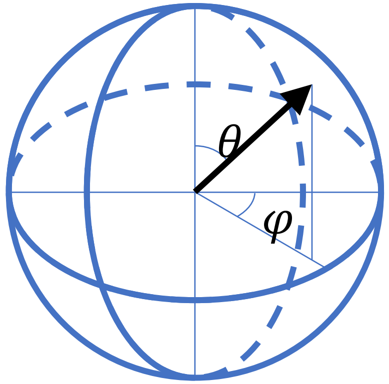
$\Longleftrightarrow$ Space is three dimensional

https://assumptionsofphysics.org/

Assumptions of Physics

Independent DOF —— Two dimensional



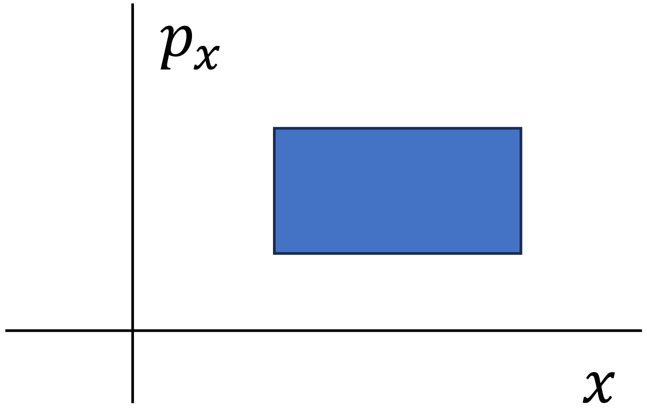


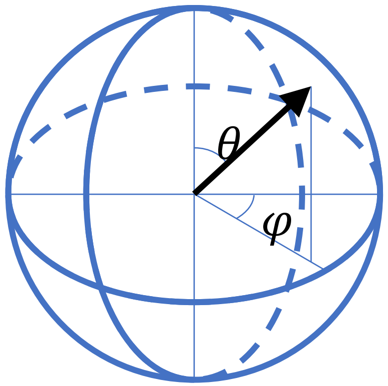
Independent directional DOF —— Space three dimensional

https://assumptionsofphysics.org/

Assumptions of Physics

Independent DOF $\overset{\text{Suppose}}{\Longrightarrow}$ Two dimensional





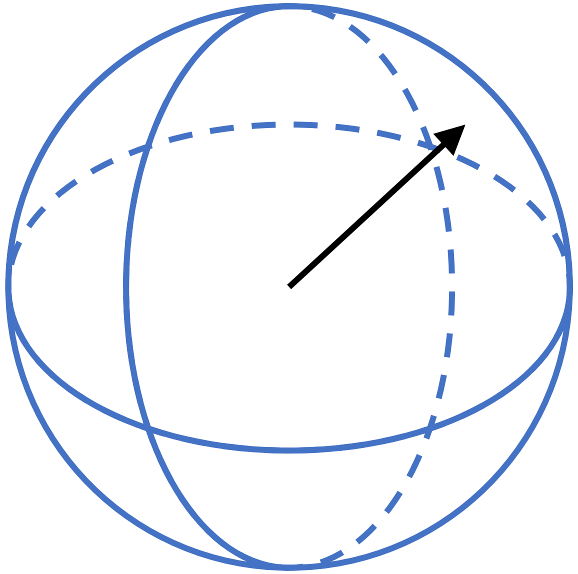
Independent directional DOF $\overset{???}{\Longrightarrow}$ Space three dimensional

A single directional DOF could be made
of multiple independent DOF

https://assumptionsofphysics.org/

Assumptions of Physics

Direction in multiple dimensions



Direction is a point on a hypersphere

$$\hat{L} = [\hat{L}_1, \hat{L}_2, \hat{L}_3, \dots, \hat{L}_n] \qquad \sum_i \hat{L}_i = 1$$

normalized vector

2-sphere is the only symplectic manifold

Direction cannot be described
by multiple independent DOFs

https://assumptionsofphysics.org/

Assumptions of Physics

Takeaways

- Invariant densities, entropy and count of states cannot be defined in general over the continuum
- The additional structure required is exactly the structure of phase space (symplectic structure)
- If a direction in space forms an independent DOF, then it must be a sphere that has the structure of phase space
- But the two-sphere is the only one possible
- ⇒ Space has three dimensions

https://assumptionsofphysics.org/

Assumptions of Physics

Wrapping it up

- All the core elements of classical mechanics can be rederived and understood using four physically meaningful assumptions
  - Infinitesimal Reducibility (IR), DOF independence (IND), Determinism and Reversibility (DR) and Kinematic Equivalence (KE)
- Many physical ideas are a consequence of those assumptions
  - Principle of stationary action, massive particles under scalar/vector potential forces, relativistic mechanics, phase space (conjugate quantities), …
- A great number of better insights from a theory that is generally considered as “understood”
  - Probably even better insights to be found

https://assumptionsofphysics.org/

Assumptions of Physics

Reading group?

- If there is interest, we can set up an online reading group, livestreamed on the research channel
  - Go through the book chapter on classical mechanics together, section by section, discuss and think of potential diagrams/figures that can help
- I'd need one person to volunteer to organize
  - Help get list of interested people, find a time, send reminders, prepare thumbnail, etc…

https://assumptionsofphysics.org/

Assumptions of Physics

To learn more

- Project website
  - https://assumptionsofphysics.org for papers, presentations, …
  - https://assumptionsofphysics.org/book for our open access book (updated every few years with new results)
- YouTube channels
  - https://www.youtube.com/@gcarcassi
    Videos with results and insights from the research
  - https://www.youtube.com/@AssumptionsofPhysicsResearch
    Research channel, with open questions and livestreamed work sessions
- GitHub
  - https://github.com/assumptionsofphysics
    Book, research papers, slides for videos...

https://assumptionsofphysics.org/

Assumptions of Physics

https://assumptionsofphysics.org/

Assumptions of Physics