Assumptions of Physics
Summer School 2024

Foundational Structures

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Main goal of the project

Identify a handful of physical starting points from which the basic laws can be rigorously derived

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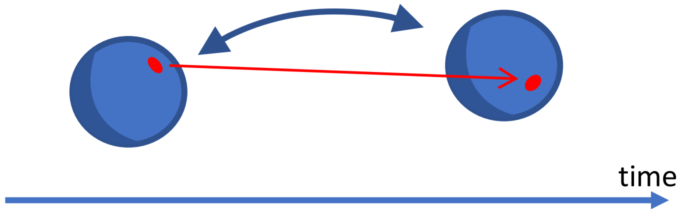
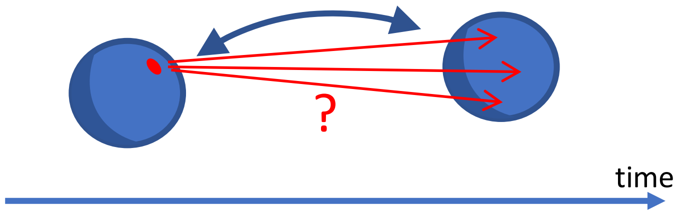
For example:

Infinitesimal reducibility ⇒ Classical state



Irreducibility ⇒ Quantum state



This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based ⇒ scientific theory must be characterized by experimentally verifiable statements ⇒ topology and σ-algebras

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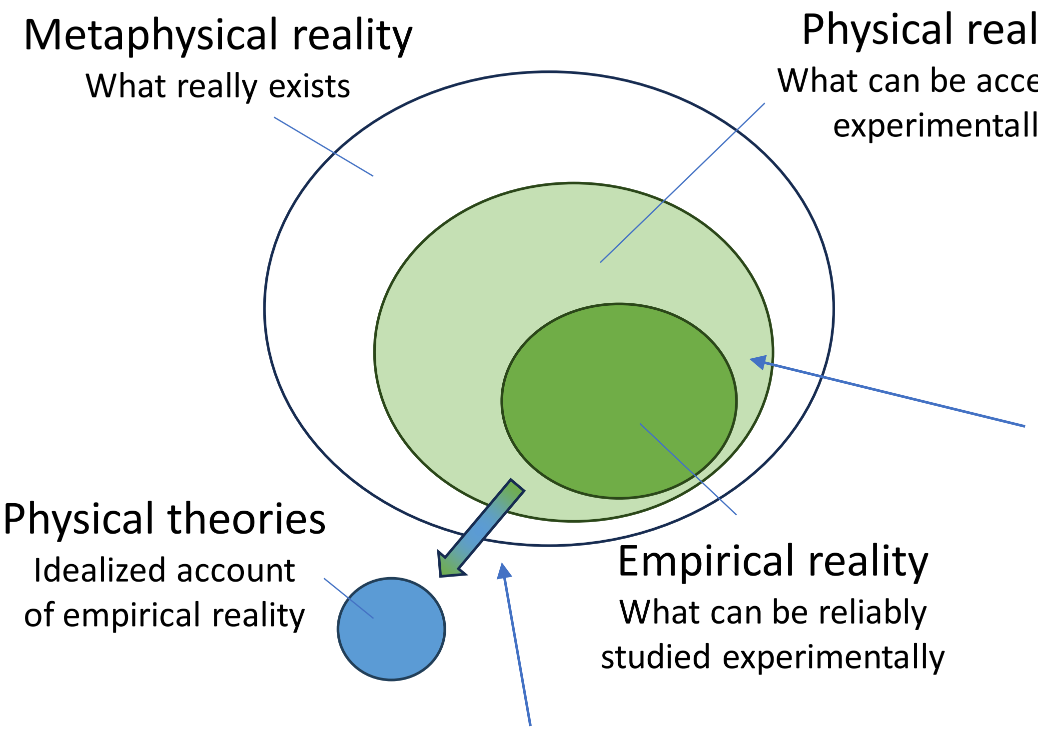
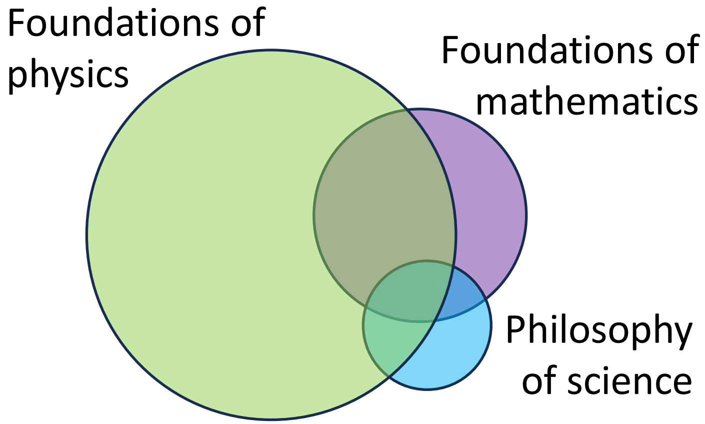
Underlying perspective



What is the boundary?
What are the requirements?

How exactly does the abstraction/idealization process work?



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If physics is about creating models of empirical reality, the foundations of physics should be a theory of models of empirical reality

Requirements of experimental verification, assumptions of each theory, realm of validity of assumptions, …

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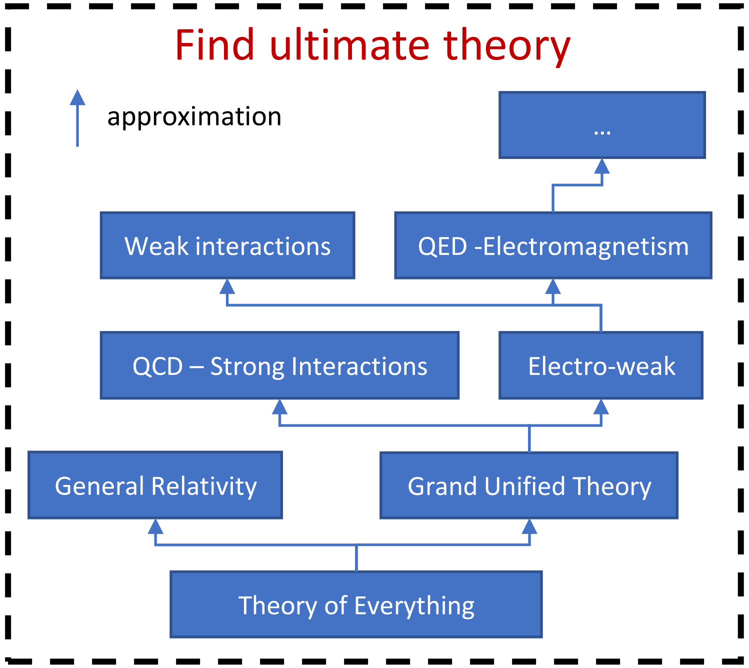
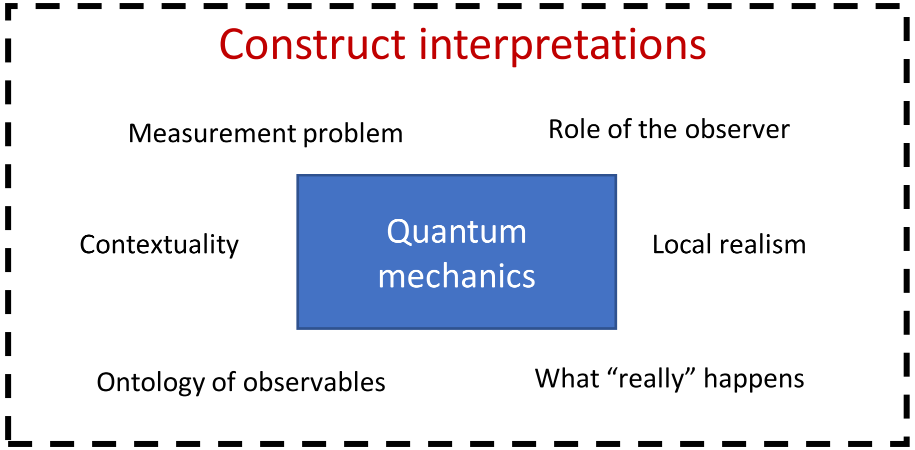
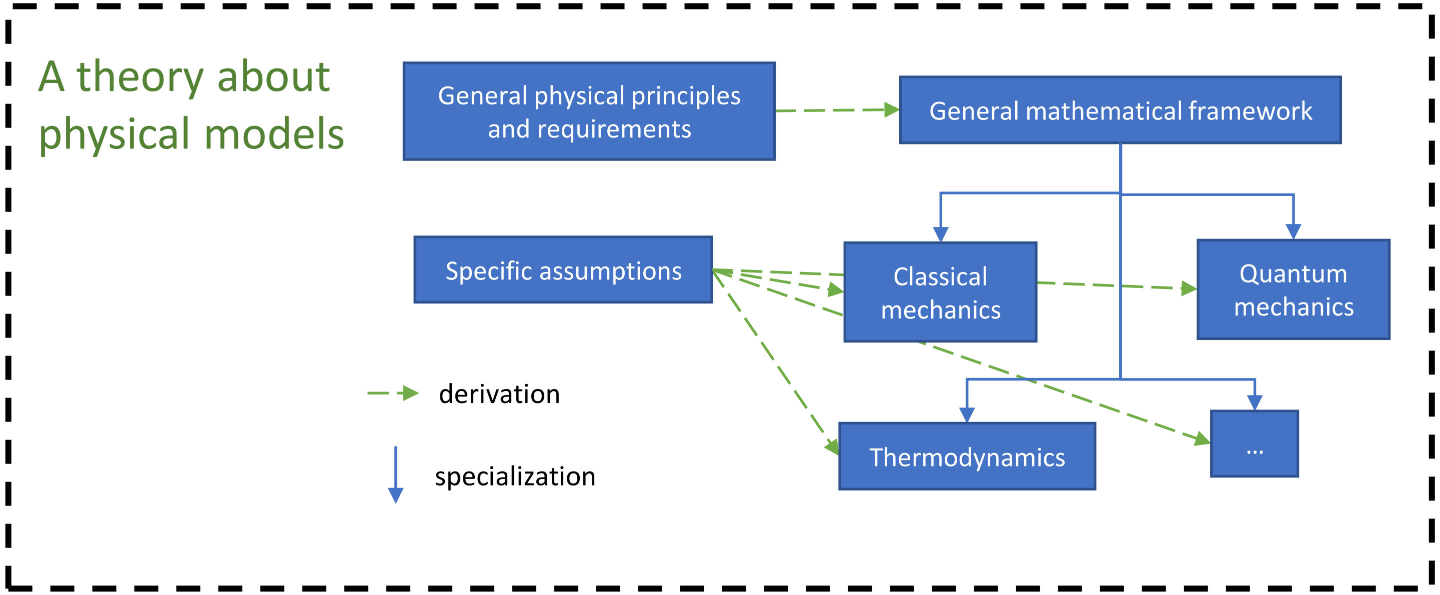
Different approach to the foundations of physics



Typical approaches



Our approach



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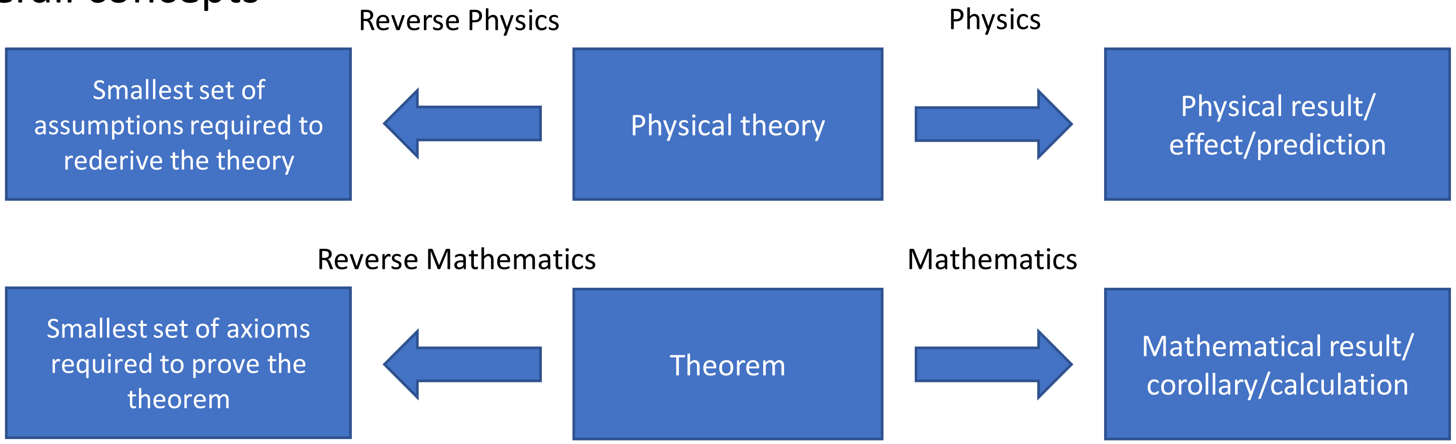
Find the right overall concepts

Reverse physics:

Start with the equations, reverse engineer physical assumptions/principles

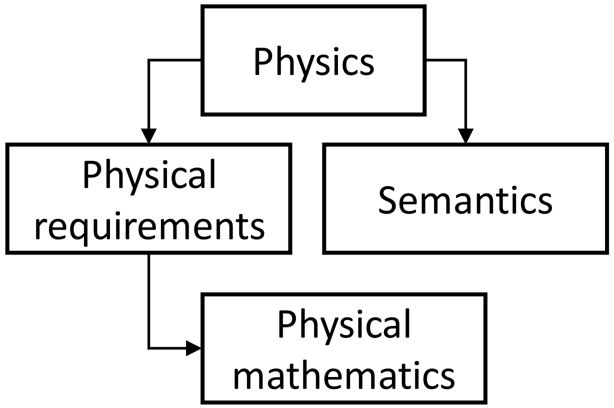
Found Phys **52**, 40 (2022)



Goal: find the right overall physical concepts, “elevate” the discussion from mathematical constructs to physical principles

Physical mathematics:

Start from scratch and rederive all mathematical structures from physical requirements



Goal: get the details right, perfect one-to-one map between mathematical and physical objects

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This session

Physical Mathematics: Foundational Structures

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Michigan Publishing (v2 2023)

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Formal system for physics

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Formal system:

primitive notions

Basic objects that are taken as-is,
without definition in terms of other objects

formal language

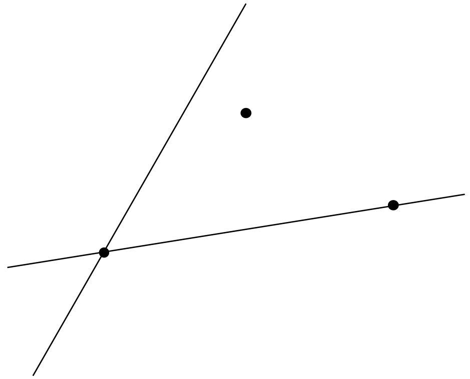
Symbols and rules to write sentences
in the formal system

axioms

Statements about primitive objects that
are to be taken as true

e.g. Euclidean geometry

E.g. Points and lines



E.g. $\mathrm{A}, \mathrm{B}, \mathrm{C}$ for points
$\overline{AB}$ for segment

E.g. Given two points,
there is a line that joins them

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Formal system for all of mathematics:

Sets + first-order logic
+ Zermelo–Fraenkel axioms (+ axiom of choice)

Formal system for all of physics:

???

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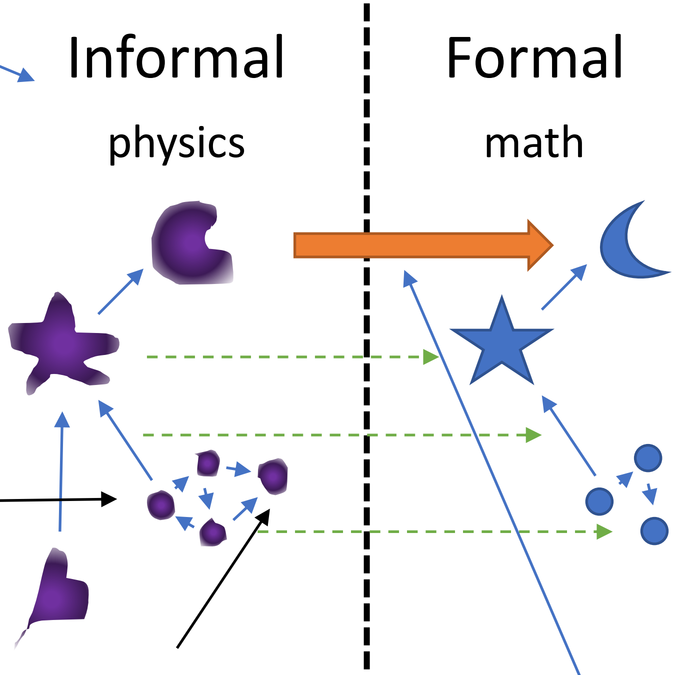
Problems in formalizing physical concepts

Physical objects live in the physical (informal) world

(e.g. connection to experiment is outside of the formal system)

Physical concepts are "fuzzy"

Physical concepts may have circular definitions



Mathematical concepts are "crisper idealizations"

Mathematical concepts cannot have circular definitions

Some concepts will have to remain informal

Choose axioms/primitive notions so that the justification is straightforward

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What should our primitive "informal" notion be?

Guiding principle

Principle of scientific objectivity: science is universal, non-contradictory and evidence based.

Suggest logic as fundamental …

like mathematics!

Universal → same for everybody

Non-contradictory → something is either true or false

Evidence based → truth is determined experimentally

… with some extensions

⇒ Logic of experimentally verifiable statements!

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Not “verifiable statements”

Chocolate tastes good (not universal)

It is immoral to kill one person to save ten (not universal and/or evidence based)

The number 4 is prime (not evidence based)

This statement is false (not non-contradictory)

The mass of the photon is exactly 0 eV (not verifiable due to infinite precision)

“Verifiable statements”

The mass of the photon is less than 10^{-13} eV

If the height of the mercury column is between 24 and 25 millimeters then its temperature is between 24 and 25 Celsius

If I take 2 ± 0.01 Kg of Sodium-24 and wait 15 ± 0.01 hours there will be only 1 ± 0.01 Kg left

A scientific theory needs “at least” the concept of a verifiable statement: good primitive notion

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Takeaways

- A good part of physics must remain informal
- Formal part is “precise” because it represents only an idealized part
- Pragmatic considerations as to what is formalized
- We take verifiable statements as the basic building blocks of our formal system

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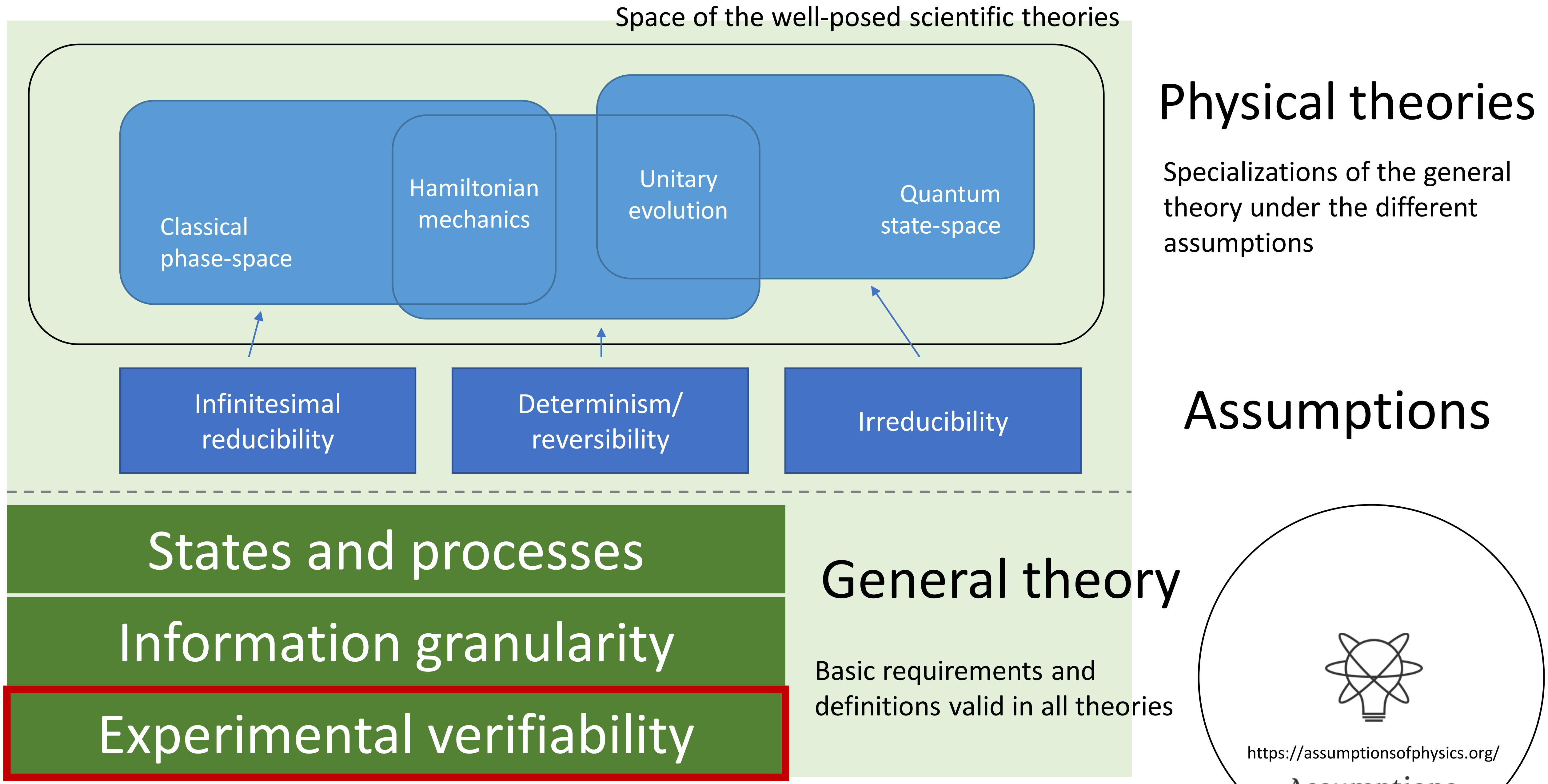

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Axioms of logic

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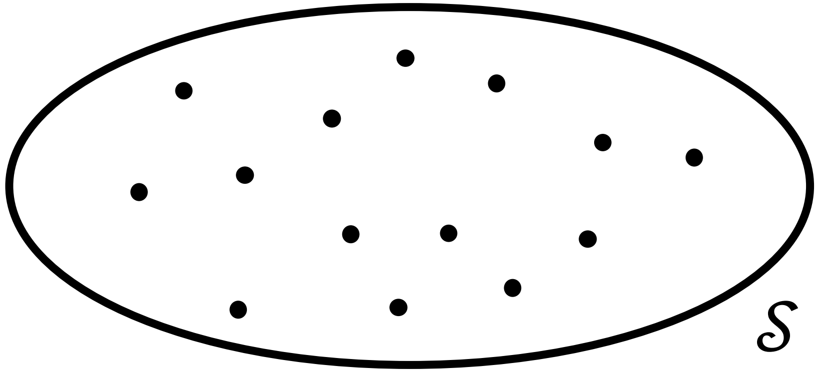
Axiom 1.2 (Axiom of context). *A* ***statement*** s *is an assertion that is either true or false. A* ***logical context*** $\mathcal{S}$ *is a collection of statements with well defined logical relationships. Formally, a logical context* $\mathcal{S}$ *is a collection of elements called statements upon which is defined a function* $\text{truth} : \mathcal{S} \to \mathbb{B}$.



$a \rightarrow$

s_1	s_2	s_3	…
T	T	F	…
T	F	T	…
T	F	F	…

$\mathcal{A}_\mathcal{S}$

Axiom 1.4 (Axiom of possibility). *A* ***possible assignment*** *for a logical context* $\mathcal{S}$ *is a map* $a : \mathcal{S} \to \mathbb{B}$ *that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set* $\mathcal{A}_\mathcal{S} \subseteq \mathbb{B}^\mathcal{S}$ *such that* $\text{truth} \in \mathcal{A}_\mathcal{S}$. *A map* $a : \mathcal{S} \to \mathbb{B}$ *is a possible assignment for* $\mathcal{S}$ *if* $a \in \mathcal{A}_\mathcal{S}$.

Axiom 1.9 (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let* $S \subseteq \mathcal{S}$ *be a set of statements and* $f_\mathbb{B} : \mathbb{B}^S \to \mathbb{B}$ *an arbitrary function from an assignment of* S *to a truth value. Then we can always find a statement* $\bar{s} \in \mathcal{S}$ *that depends on* S *through* $f_\mathbb{B}$.

s_1	s_2	s_3	…
T	T	F	…
T	F	T	…
T	F	F	…

→ $f_\mathbb{B}$ →

$f(s_1, s_2, s_3)$
T
T
F

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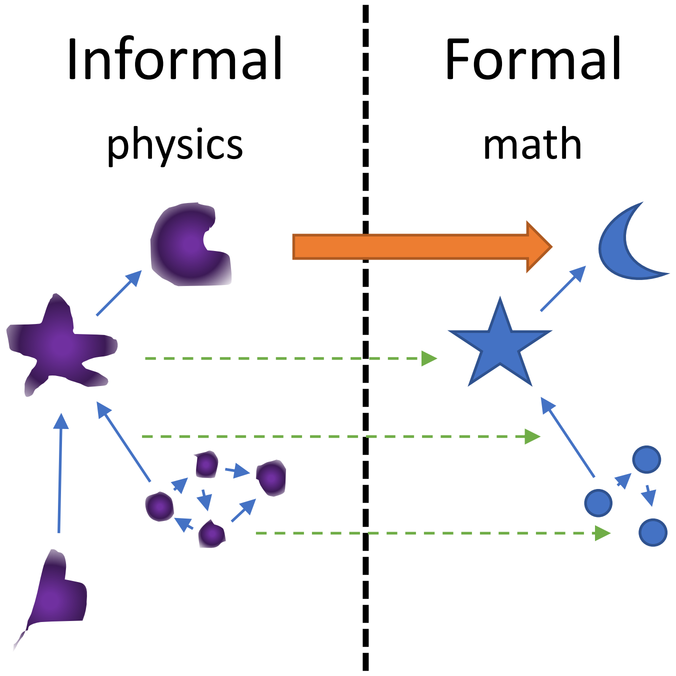
Definition 1.1. *The **Boolean domain** is the set* $\mathbb{B} = \{\text{FALSE}, \text{TRUE}\}$ *of all possible truth values.*

Axiom 1.2 (Axiom of context)**.** *A **statement*** s *is an assertion that is either true or false. A **logical context*** $\mathcal{S}$ *is a collection of statements with well defined logical relationships.* *Formally, a logical context* $\mathcal{S}$ *is a collection of elements called statements upon which is defined a function* $\text{truth} : \mathcal{S} \to \mathbb{B}$.

Informal part

Formal part

Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed



Each axiom/definition has two parts:

- Informal part: tells us what elements in the physical world we are characterizing
- Formal part: how the elements are characterized mathematically

Each axiom/definition has a justification: argues why the mathematical characterization follows from the physical one

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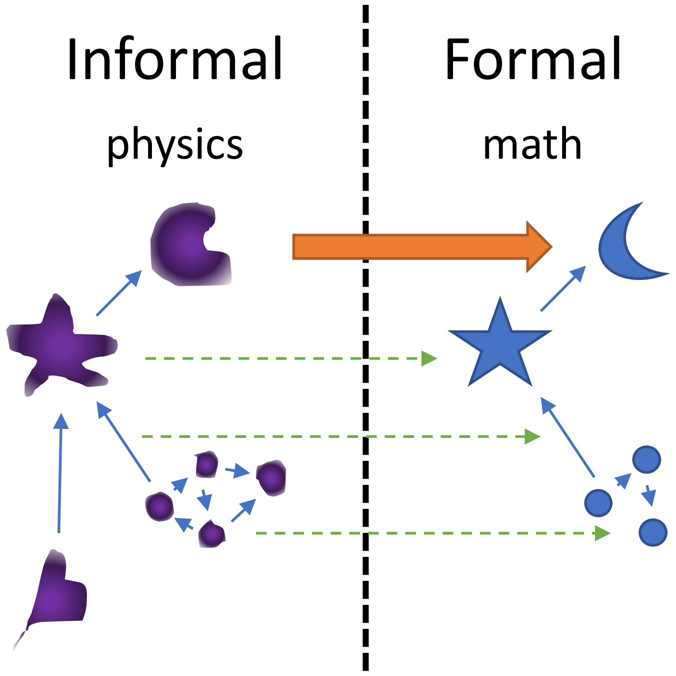

Axiom: brings objects from the informal to the formal

Definition: further specializes formal objects

Axioms/definitions should be formulated so that they are easy to justify…

… not so that they follow trends in mathematics

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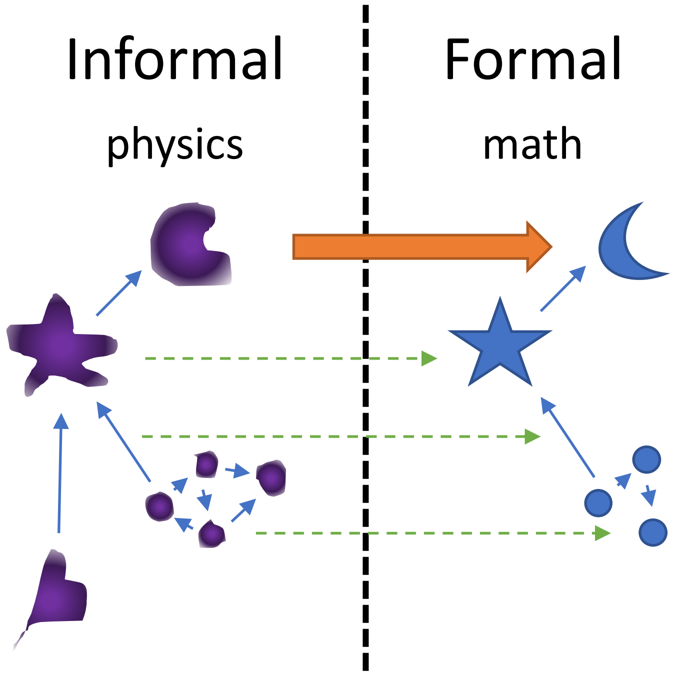

Physical objects are made "mathematically precise" by throwing out everything that can't be made precise

Syntax, grammar, meaning, … can't be made precise, so are not part of the formal system

⇒ Statements are primitive objects

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In mathematics, primitive objects (i.e. those that are left unspecified) must be elements of a set. The logical context, then, has two functions:

1) in the formal system, it is the “container” for the primitive objects (i.e. the statements)

2) in the informal system, consistency/semantics/… are properties of groups of statements (i.e. of the context)

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Assumptions of Physics

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A statement here represents the assertion and not the sentence that declares the assertion. Therefore the translation of a sentence into another language represents the same statement.

Technically, we only assume the existence of valid statements for doing science. Therefore statements are also primitives in the informal system.

But if they exist, they must follow the axioms we are going to specify.

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Formal part

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The existence of a truth function stems from the assumption of non-contradiction and universality.

Every statement must be either true or false for everybody.

$\mathcal{S}$

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	…
truth	T	T	F	T	T	F	T	T	F	…

Context ⇒ big table where statements are columns

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Note: the semantic content constrains the possible combinations of truth values

“that animal is a cat”	*“that animal is a mammal”*	*“that animal is a bird”*	…
~~T~~	~~T~~	~~T~~	~~…~~
T	T	F	…
~~T~~	~~F~~	~~T~~	~~…~~
~~T~~	~~F~~	~~F~~	~~…~~
~~F~~	~~T~~	~~T~~	~~…~~
F	T	F	…
F	F	T	…
F	F	F	…

impossible

The only semantics captured by the formal system is the set of possible combinations of truth values

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Axiom 1.4 (Axiom of possibility). *A **possible assignment** for a logical context $\mathcal{S}$ is a map $a : \mathcal{S} \to \mathbb{B}$ that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set $\mathcal{A}_\mathcal{S} \subseteq \mathbb{B}^\mathcal{S}$ such that* truth $\in \mathcal{A}_\mathcal{S}$. *A map $a : \mathcal{S} \to \mathbb{B}$ is a possible assignment for $\mathcal{S}$ if $a \in \mathcal{A}_\mathcal{S}$.*

$\mathcal{A}_\mathcal{S}$

	s_1	s_2	s_3	…	
a_1	T	T	F	…	truth
a_2	F	F	T	…	
a_3	F	T	F	…	
…	…	…	…	…	

Possible assignments are those assignments consistent with the meaning (semantics) of the statements

Context ⇒ big table where statements are columns and possible assignments are rows

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Assumptions of Physics

Definition 1.6. *Statements are categorized based on their possible assignments.*

- *A certain statement, or* ***certainty****, is a statement* $\top$ *that must be true simply because of its content. Formally,* $a(\top) = \text{TRUE}$ *for all possible assignments* $a \in \mathcal{A}_{\mathcal{S}}$*.*
- *An impossible statement, or* ***impossibility****, is a statement* $\perp$ *that must be false simply because of its content. Formally,* $a(\perp) = \text{FALSE}$ *for all possible assignments* $a \in \mathcal{A}_{\mathcal{S}}$*.*
- *A statement is* ***contingent*** *if it is neither certain nor impossible.*

Corollary 1.7. *A statement* $\mathsf{s} \in \mathcal{S}$ *can only be exactly one of the following: impossible, contingent, certain.*

"that cat is a mammal"	*"that mammal is a cat"*	*"that mammal is a bird"*
T	T	F
T	F	F
certain	contingent	impossible

Certainties and impossibilities have the same truth value in all rows

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- *An impossible statement, or* ***impossibility****, is a statement* $\bot$ *that must be false simply because of its content. Formally,* $a(\bot) = \text{FALSE}$ *for all possible assignments* $a \in \mathcal{A}_\mathcal{S}$.
- *A statement is* ***contingent*** *if it is neither certain nor impossible.*

Corollary 1.7. *A statement* $\mathsf{s} \in \mathcal{S}$ *can only be exactly one of the following: impossible, contingent, certain.*

Whether a statement is certain or contingent depends on context!

the mass of the electron is 510 $\pm$ *0.5 KeV*

Contingent when measuring the mass of the electron

Certain when performing particle identification

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Some statements depend on other statements

That animal is a mammal — incompatible — That animal has feathers

narrower — equivalent

That animal is a cat — That animal is a bird

independent

That animal is black — negation — That animal is not black

⇒ possible assignments determine the logical relationship

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Equivalent

"that animal has feathers"	*"that animal is a bird"*
T	T
~~T~~	~~F~~
~~F~~	~~T~~
F	F

	T	F
T	✓	×
F	×	✓

Independent

"that animal is a cat"	*"that animal is black"*
T	T
T	F
F	T
F	F

	T	F
T	✓	✓
F	✓	✓

Incompatible

"that animal is a mammal"	*"that animal is a bird"*
~~T~~	~~T~~
T	F
F	T
F	F

	T	F
T	×	✓
F	✓	✓

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Definition 1.15. *Two statements* s_1 *and* s_2 *are* ***equivalent*** $\mathsf{s}_1 \equiv \mathsf{s}_2$ *if they must be equally true or false simply because of their content. Formally,* $\mathsf{s}_1 \equiv \mathsf{s}_2$ *if and only if* $a(\mathsf{s}_1) = a(\mathsf{s}_2)$ *for all possible assignments* $a \in \mathcal{A}_{\mathcal{S}}$.

s_1	s_2	s_3	s_4	s_5	s_6	s_7	…
T	T	T	T	F	T	T	…
F	F	F	T	T	T	T	…
F	F	F	F	T	T	T	…
T	F	T	T	F	T	T	…
T	F	T	F	F	T	T	…
…	…	…	…	…	…	…	…

Corollary 1.16. *All certainties are equivalent. All impossibilities are equivalent.*

Corollary 1.18. *Statement equivalence satisfies the following properties:*

- *reflexivity:* $\mathsf{s} \equiv \mathsf{s}$
- *symmetry: if* $\mathsf{s}_1 \equiv \mathsf{s}_2$ *then* $\mathsf{s}_2 \equiv \mathsf{s}_1$
- *transitivity: if* $\mathsf{s}_1 \equiv \mathsf{s}_2$ *and* $\mathsf{s}_2 \equiv \mathsf{s}_3$ *then* $\mathsf{s}_1 \equiv \mathsf{s}_3$

and is therefore an ***equivalence relationship***.

From now on, unless otherwise stated, by statement we mean an equivalence class of statements

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Definition 1.20. *Given two statements* s_1 *and* s_2*, we say that:*

- s_1 ***is narrower than*** s_2 *(noted* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$*) if* s_2 *is true whenever* s_1 *is true simply because of their content. That is, for all* $a \in \mathcal{A}_\mathcal{S}$ *if* $a(\mathsf{s}_1) = \mathrm{TRUE}$ *then* $a(\mathsf{s}_2) = \mathrm{TRUE}$.
- s_1 ***is broader than*** s_2 *(noted* $\mathsf{s}_1 \succcurlyeq \mathsf{s}_2$*) if* $\mathsf{s}_2 \preccurlyeq \mathsf{s}_1$.
- s_1 ***is compatible to*** s_2 *(noted* $\mathsf{s}_1 \overset{\frown}{\frown} \mathsf{s}_2$*) if their content allows them to be true at the same time. That is, there exists* $a \in \mathcal{A}_\mathcal{S}$ *such that* $a(\mathsf{s}_1) = a(\mathsf{s}_2) = \mathrm{TRUE}$.

The negation of these properties will be noted by $\not\preccurlyeq$, $\not\succcurlyeq$, $\not\overset{\frown}{\frown}$ *respectively.*

s_1	s_2	s_3	s_4	s_5	s_6	s_7	…
T	T	T	T	F	T	T	…
F	F	F	T	T	T	T	…
F	F	F	F	T	T	T	…
T	F	F	T	F	T	F	…
T	F	T	F	F	T	T	…
…	…	…	…	…	…	…	…

That animal is a mammal $\overset{\frown}{\frown}$ That animal lays eggs

That animal is a cat $\preccurlyeq$ That animal is a mammal

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Proposition 1.23. *Statement narrowness satisfies the following properties:*

- *reflexivity:* $\mathsf{s} \preccurlyeq \mathsf{s}$
- *antisymmetry: if* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$ *and* $\mathsf{s}_2 \preccurlyeq \mathsf{s}_1$ *then* $\mathsf{s}_1 \equiv \mathsf{s}_2$
- *transitivity: if* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$ *and* $\mathsf{s}_2 \preccurlyeq \mathsf{s}_3$ *then* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_3$

and is therefore a ***partial order****.*

Narrowness $\preccurlyeq$ is related to material implication $\rightarrow$ but:

Material implication is a logical operation that returns a new statement:

$a \rightarrow b = \neg a \vee b$ (i.e. NOT(a) OR b)

Narrowness $\preccurlyeq$ is a binary relationship between statements

The order imposed by narrowness allows us to understand the context as an order theoretic structure

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NOTE: AND, OR and NOT ($\wedge, \vee, \neg$)
are operations within the context

Equivalence, narrowness, compatibility, … ($\equiv, \preccurlyeq, \frown$, …)
are not: they describe the context (i.e. metalanguage)

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Assumptions of Physics

Definition 1.8. *Let* $\bar{\mathsf{s}} \in \mathcal{S}$ *be a statement and* $S \subseteq \mathcal{S}$ *be a set of statements. Then* $\bar{\mathsf{s}}$ ***depends on*** S *(or it is a function of* S*) if we can find an* $f_{\mathbb{B}} : \mathbb{B}^S \to \mathbb{B}$ *such that*

$$a(\bar{\mathsf{s}}) = f_{\mathbb{B}}(\{a(\mathsf{s})\}_{\mathsf{s} \in S})$$

for every possible assignment $a \in \mathcal{A}_S$*. We say* $\bar{\mathsf{s}}$ *depends on* S ***through*** $f_{\mathbb{B}}$*. The relationship is illustrated by the following diagram:*

$\mathcal{A}_S$	s_1	s_2	s_3	…	…	$\bar{s}$	
a_1	T	T	F	…	…	T	$a_1(\bar{s}) = f_{\mathbb{B}}(a_1(s_1), a_1(s_2), a_1(s_3))$
a_2	F	F	T	…	$f_{\mathbb{B}}$ →	F	$a_2(\bar{s}) = f_{\mathbb{B}}(a_2(s_1), a_2(s_2), a_2(s_3))$
a_3	T	F	F	…	s_1 AND (s_2 OR s_3)	F	$a_3(\bar{s}) = f_{\mathbb{B}}(a_3(s_1), a_3(s_2), a_3(s_3))$
…	…	…	…	…	…	…	$f_{\mathbb{B}}(\ldots,\ldots,\ldots)$

S (columns s_1, s_2, s_3, …)

s_1=*"that animal is a cat"*

s_2=*"that animal is a mammal"*

s_3=*"that animal is a bird"*

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Assumptions of Physics

Axiom 1.9 (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let $S \subseteq \mathcal{S}$ be a set of statements and $f_{\mathbb{B}} : \mathbb{B}^S \to \mathbb{B}$ an arbitrary function from an assignment of S to a truth value. Then we can always find a statement $\bar{\mathsf{s}} \in \mathcal{S}$ that depends on S through $f_{\mathbb{B}}$.*

s_1	s_2	s_3		$\bar{s}$	…	$s_{\dots}$	…
T	T	F		T	…	T	…
F	F	T	$f_{\mathbb{B}}$ →	F	…	T	…
F	F	T	s_1 AND (s_2 OR s_3)	F	…	F	…
T	F	F		F	…	T	…
T	F	T		F	…	F	…
…	…	…		…	…	…	…

Not sure whether it is needed as an axiom: the closure may be proven to exist and be unique.

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Assumptions of Physics

Corollary 1.10. *Functions on truth values induce functions on statements. Formally, let I be an index set and $f_{\mathbb{B}} : \mathbb{B}^I \to \mathbb{B}$ be a function. There exists a function $f : \mathcal{S}^I \to \mathcal{S}$ such that*

$$a(f(\{\mathsf{s}_i\}_{i \in I})) = f_{\mathbb{B}}(\{a(\mathsf{s}_i)\}_{i \in I})$$

for every indexed set $\{\mathsf{s}_i\}_{i \in I} \subseteq \mathcal{S}$ and possible assignment $a \in \mathcal{A}_{\mathcal{S}}$.

s_1	s_2	s_3	$f(s_1, s_2, s_3)$ →	$\bar{s}$	...	$s_{...}$	...
T	T	F		T	...	T	...
F	F	T	$f_{\mathbb{B}}$ →	F	...	T	...
F	F	T	s_1 AND (s_2 OR s_3)	F	...	F	...
T	F	F		F	...	T	...
T	F	T		F	...	F	...
...	...	...		...	...	...	...

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Assumptions of Physics

Definition 1.11. *The* ***negation or logical NOT*** *is the function* $\neg : \mathbb{B} \to \mathbb{B}$ *that takes a truth value and returns its opposite. That is:* $\neg\text{TRUE} = \text{FALSE}$ *and* $\neg\text{FALSE} = \text{TRUE}$. *We also call negation* $\neg : \mathcal{S} \to \mathcal{S}$ *the related function on statements.*

t	$\neg t$
T	T
F	F

s_1	s_2	s_3	s_4	s_5	s_6	s_7	…
T	T	F	T	F	T	T	…
F	F	T	T	T	F	F	…
F	F	T	F	T	F	F	…
T	F	F	T	F	F	T	…
T	F	T	F	F	F	T	…
…	…	…	…	…	…	…	…

¬

That animal is a cat

¬

That animal is not a cat

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Assumptions of Physics

Definition 1.12. *The* ***conjunction*** *or* ***logical AND*** *is the function* $\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ *that returns* TRUE *only if all the arguments are* TRUE*. That is:* TRUE $\wedge$ TRUE = TRUE *and* TRUE$\wedge$FALSE = FALSE$\wedge$TRUE = FALSE$\wedge$FALSE = FALSE*. We also call conjunction* $\wedge : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ *the related function on statements.*

t_1	t_2	$t_1 \wedge t_2$
T	T	T
T	F	F
F	T	F
F	F	F

s_1	s_2	s_3	s_4	s_5	s_6	s_7	…
T	T	F	T	F	T	T	…
F	F	T	T	T	F	F	…
F	F	T	F	T	F	F	…
T	F	F	T	F	F	T	…
T	F	T	F	F	F	T	…
…	…	…	…	…	…	…	…

$\wedge$

That animal is a cat

That animal is black

$\wedge$

That animal is a black cat

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Assumptions of Physics

Definition 1.13. *The* ***disjunction or logical OR*** *is the function* $\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ *that returns* FALSE *only if all the arguments are* FALSE. *That is:* FALSE $\vee$ FALSE = FALSE *and* TRUE $\vee$ FALSE = FALSE $\vee$ TRUE = TRUE $\vee$ TRUE = TRUE. *We also call disjunction* $\vee : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ *the related function on statements.*

t_1	t_2	$t_1 \vee t_2$
T	T	T
T	F	T
F	T	T
F	F	F

s_1	s_2	s_3	s_4	s_5	s_6	s_7	…
T	T	F	T	F	T	T	…
F	F	T	T	T	F	F	…
F	F	T	F	T	F	F	…
T	F	F	T	F	F	T	…
T	F	T	F	F	F	T	…
…	…	…	…	…	…	…	…

∨

That animal is a cat

That animal is a dog

∨

That animal is a cat or a dog

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Assumptions of Physics

Corollary 1.19. *A logical context $\mathcal{S}$ satisfies the following properties:*

- *associativity:* $a \vee (b \vee c) \equiv (a \vee b) \vee c$, $a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$
- *commutativity:* $a \vee b \equiv b \vee a$, $a \wedge b \equiv b \wedge a$
- *absorption:* $a \vee (a \wedge b) \equiv a$, $a \wedge (a \vee b) \equiv a$
- *identity:* $a \vee \bot \equiv a$, $a \wedge \top \equiv a$
- *distributivity:* $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$
- *complements:* $a \vee \neg a \equiv \top$, $a \wedge \neg a \equiv \bot$
- *De Morgan:* $\neg a \vee \neg b \equiv \neg(a \wedge b)$, $\neg a \wedge \neg b \equiv \neg(a \vee b)$

Therefore $\mathcal{S}$ is a ***Boolean algebra*** *by definition.*

Recovers the standard structure for classical logic

Note how many properties are part of the definition of a Boolean algebra: if that had been our starting point, we would have had to justify every single one, which is cumbersome and not particularly enlightening

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Assumptions of Physics

Functions in a Boolean algebra have a standard representation important for us

s_1	s_2	s_3	$\bar{s}$	m_1	m_2	m_3	m_4
T	T	F	T	T	F	F	F
F	F	T	T	F	T	F	F
F	T	T	F	F	F	T	F
T	F	F	T	F	F	F	T

m_1, m_2, m_3, m_4 each picks a line of the table, and can be expressed as the conjunction that takes s_1, s_2, s_3 once, negated or not

$$m_1 = s_1 \wedge s_2 \wedge \neg s_3$$
$$m_2 = \neg s_1 \wedge \neg s_2 \wedge s_3$$
$$m_3 = \neg s_1 \wedge s_2 \wedge s_3$$
$$m_4 = s_1 \wedge \neg s_2 \wedge \neg s_3$$

minterms

$\bar{s}$ is a function of s_1, s_2, s_3

$\bar{s}$ is the disjunction of m_1, m_2, m_4

$$\bar{s} = (s_1 \wedge s_2 \wedge \neg s_3) \vee (\neg s_1 \wedge \neg s_2 \wedge s_3) \vee (s_1 \wedge \neg s_2 \wedge \neg s_3)$$

disjunctive normal form

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Assumptions of Physics

Takeaways

- Semantics define which assignments are possible on a given context
- The possible assignments define the logical relationships and operations
- The possible assignments describe “what could happen”, which is inherently tied to the model
  - Certainty, equivalence, narrowness, etc… are all metaconcepts about the theory
- TODOs
  - Statement equivalence should be defined before functions on statements (technically, they should be operations on equivalence classes)

Axioms of verifiability

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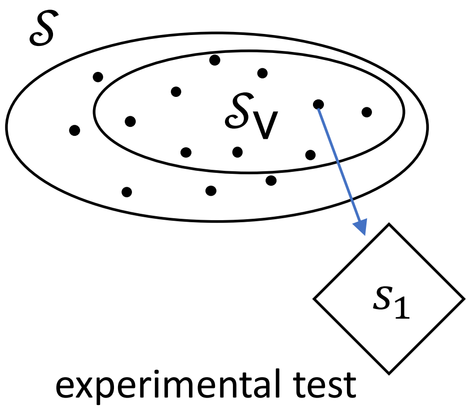
Assumptions of Physics

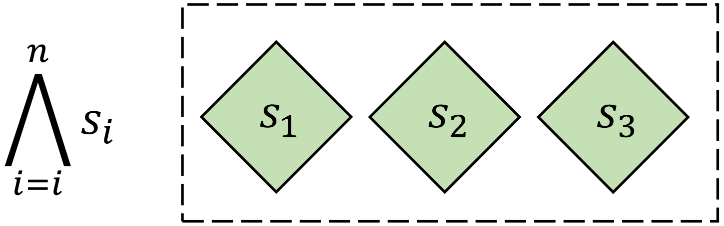
Axiom 1.27 (Axiom of verifiability). *A **verifiable statement** is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_v \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\perp \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.

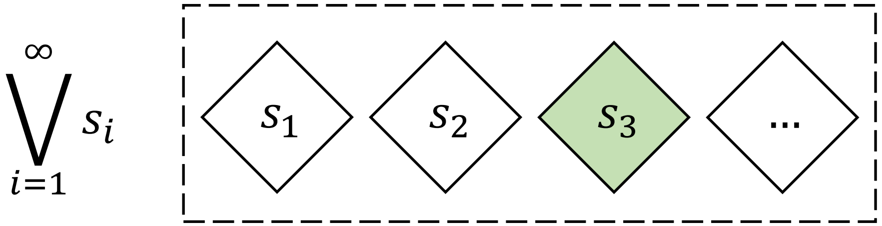


s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED



All tests must succeed

Axiom 1.31 (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^{n} \subseteq \mathcal{S}_v$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge_{i=1}^{n} s_i \in \mathcal{S}_v$ is a verifiable statement.*



One successful test is sufficient

Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_v$ be a countable collection of verifiable statements. Then the disjunction $\bigvee_{i=1}^{\infty} s_i \in \mathcal{S}_v$ is a verifiable statement.*

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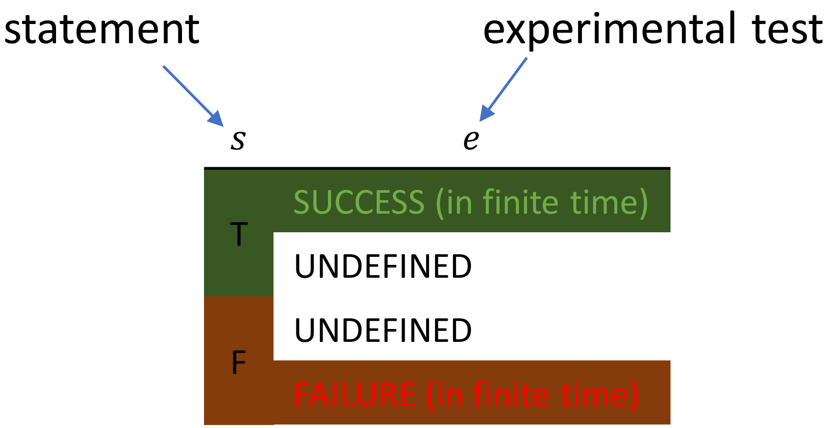
Assumptions of Physics

Axiom 1.27 (Axiom of verifiability). *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_v \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

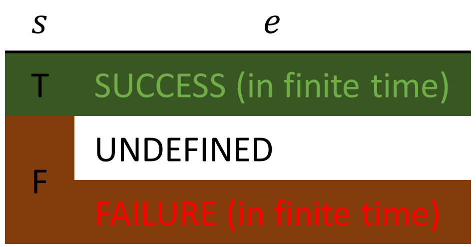
- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

New axiom to bring in the idea that some statements are experimentally verifiable



Tests may or may not terminate

Statements are verifiable if there is a test that always terminates successfully if the statement is true



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Assumptions of Physics

Axiom 1.27 (Axiom of verifiability). *A **verifiable statement** is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_{\mathsf{v}} \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

The tests are not objects in the mathematical framework

Defining tests formally is cumbersome
Capturing which statements are verifiable is enough

Formally we are only "tagging" which statements are verifiable

Only need to tag the verifiable statements:
all other tests can be constructed from those

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Assumptions of Physics

Axiom 1.27 (Axiom of verifiability). *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_v \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Certainties and impossibilities are defined to be true and false, therefore a trivial test that always succeeds or fails is adequate

If two statements are equivalent, the termination conditions on the tests are the same ⇒ we can use the same test

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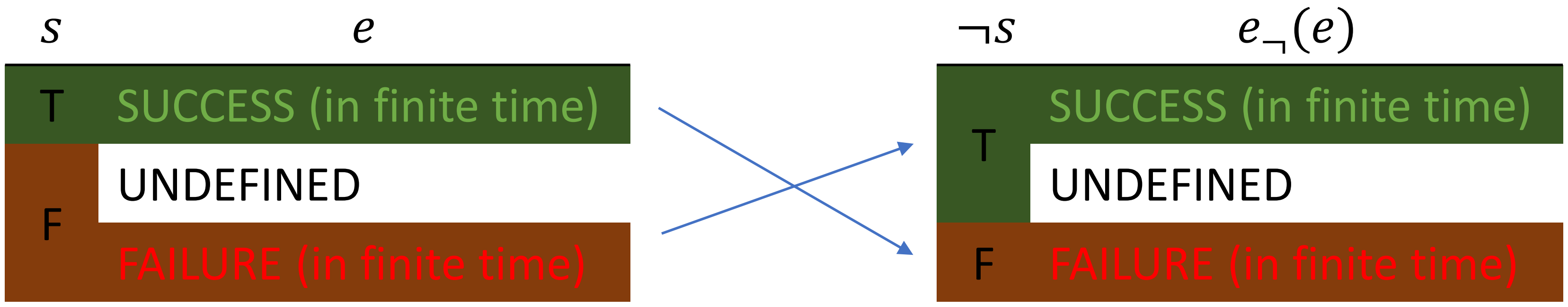
Assumptions of Physics

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



$e_\neg(e)$:

1. Run test e
2. If e fails, return SUCCESS
3. If e succeeds, return FAILURE

From e, we can construct the test $e_\neg(e)$ that switches SUCCESS with FAILURE, but the non-termination remains

$\Rightarrow$ the logic of verifiable statements does not include negation!

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Assumptions of Physics

Definition 1.28. *A **falsifiable statement** is a statement that, if false, can be shown to be so experimentally. Formally, a statement* s *is falsifiable if its negation* $\neg\mathsf{s} \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

s	e
T	SUCCESS (in finite time)
	UNDEFINED
F	FAILURE (in finite time)

Statements are falsifiable if there is a test that always terminates with failure if the statement is true

Note that formally falsifiable is defined to be the negation of verifiable statements

Reduces the number of primitive concepts

⇒ The justification must show these definitions to be equivalent

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Assumptions of Physics

Definition 1.28. *A* ***falsifiable statement*** *is a statement that, if false, can be shown to be so experimentally. Formally, a statement* s *is falsifiable if its negation* $\neg\mathsf{s} \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Suppose $\neg s$ is verifiable. Then we can find a test such that

From e, we can construct the test $e_{\neg}(e)$ that switches SUCCESS with FAILURE

$e_{\neg}(e)$:

1. Run test e
2. If e fails, return SUCCESS
3. If e succeeds, return FAILURE

s	$\neg s$	e	$e_{\neg}(e)$
T	F	FAILURE (in finite time)	SUCCESS (in finite time)
T	F	UNDEFINED	UNDEFINED
F	T	SUCCESS (in finite time)	FAILURE (in finite time)

⇒ If the negation of a statement is verifiable, then the statement is falsifiable

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Assumptions of Physics

Definition 1.29. *A* ***decidable statement*** *is a statement that can be shown to be either true or false experimentally. Formally, a statement* s *is decidable if* $\mathsf{s} \in \mathcal{S}_\mathsf{v}$ *and* $\neg\mathsf{s} \in \mathcal{S}_\mathsf{v}$*. We denote* $\mathcal{S}_\mathsf{d} \subseteq \mathcal{S}_\mathsf{v}$ *the set of all decidable statements.*

s	e
T	SUCCESS (in finite time)
F	FAILURE (in finite time)

Statements are decidable if there is a test that always terminates

Note that formally decidable statements are verifiable statements whose negation is verifiable

⇒ The justification must show these definitions to be equivalent

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Assumptions of Physics

Definition 1.29. *A* ***decidable statement*** *is a statement that can be shown to be either true or false experimentally. Formally, a statement* s *is decidable if* $\mathsf{s} \in \mathcal{S}_\mathsf{v}$ *and* $\neg\mathsf{s} \in \mathcal{S}_\mathsf{v}$*. We denote* $\mathcal{S}_\mathsf{d} \subseteq \mathcal{S}_\mathsf{v}$ *the set of all decidable statements.*

Suppose s is verifiable. Then we can find a test such that

Construct the test $\hat{e}(e, e_\neg)$

s	$\neg s$	e	$e_\neg$	$\hat{e}(e, e_\neg)$
T	F	SUCCESS (in finite time)	FAILURE (in finite time) UNDEFINED	SUCCESS (in finite time)
F	T	UNDEFINED FAILURE (in finite time)	SUCCESS (in finite time)	FAILURE (in finite time)

Suppose $\neg s$ is verifiable. Then we can find a test such that

$\hat{e}(e, e_\neg)$:

1. Initialize n to 1
2. Run e for n seconds
3. If e succeeds, return SUCCESS
4. Run $e_\neg$ for n seconds
5. If $e_\neg$ succeeds, return FAILURE
6. Increment n and go to 2

⇒ If s and $\neg s$ verifiable, then the statement is decidable

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Assumptions of Physics

Definition 1.29. *A* ***decidable statement*** *is a statement that can be shown to be either true or false experimentally. Formally, a statement* s *is decidable if* $\mathsf{s} \in \mathcal{S}_{\mathsf{v}}$ *and* $\neg\mathsf{s} \in \mathcal{S}_{\mathsf{v}}$. *We denote* $\mathcal{S}_{\mathsf{d}} \subseteq \mathcal{S}_{\mathsf{v}}$ *the set of all decidable statements.*

Corollary 1.30. *Certainties and impossibilities are decidable statements.*

Certainties and impossibilities are true and false by definition.
Yet, we can make trivial tests for them.

$e_{\top}$:
1. return SUCCESS

$e_{\bot}$:
1. return FAILURE

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Assumptions of Physics

Axiom 1.31 (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a finite collection of verifiable statements. Then the conjunction* $\bigwedge_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Conjunction (AND) of verifiable statements: check that all tests terminate successfully

$\wedge (e_i)$:

1. Run all e_i
2. If all succeed, return SUCCESS
3. Return FAILURE

⇒ Only finite conjunction is guaranteed to terminate

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Assumptions of Physics

Axiom 1.32 (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a countable collection of verifiable statements. Then the disjunction* $\bigvee_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Disjunction (OR) of verifiable statements:
check that ONE test terminates successfully

watch out for non-termination!

$\vee\,(e_i)$:

1. Initialize n to 1
2. For each $i = 1 \dots n$
   a) Run e_i for n seconds
   b) If e_i succeeds, return SUCCESS
3. Increment n and go to 2

⇒ Only countable disjunction can reach all tests

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Assumptions of Physics

Proposition 1.33. *The conjunction and disjunction of a finite collection of decidable statements are decidable. Formally, let $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_d$ be a finite collection of decidable statements. Then the conjunction $\bigwedge\limits_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_d$ and the disjuction $\bigvee\limits_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_d$ are decidable statements.*

For decidable statements, we need both the statement and its negation to be verifiable

$$\neg \bigwedge e_i = \bigvee \neg e_i \qquad \neg \bigvee e_i = \bigwedge \neg e_i$$

Using De Morgan properties, we can construct tests using test for negation

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Assumptions of Physics

Takeaways

- Adding the notion of verifiability only requires tagging which statements are verifiable
- We are essentially modeling procedures that output success/failure (i.e. one bit) and may not terminate
- These are the only axioms at this point
  - Everything else is a construction on top of this

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Assumptions of Physics

Topology and the logic of experimental verifiability

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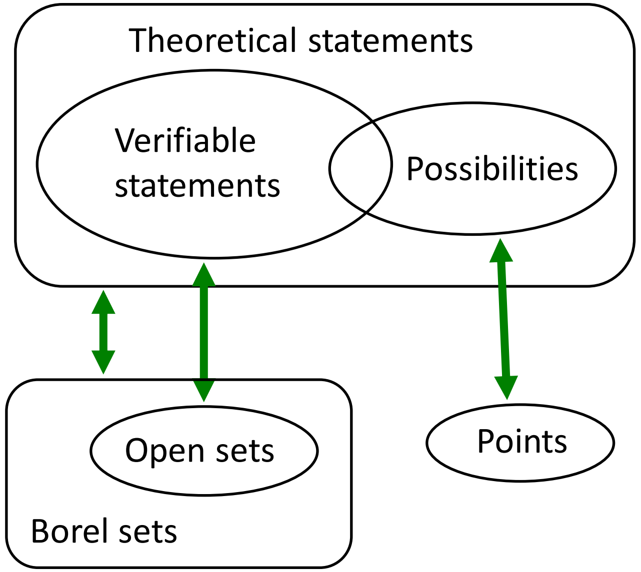
Topology and σ-algebra



s_1	Test Result
T	SUCCESS (in finite time)
T	UNDEFINED
F	UNDEFINED
F	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

Experimental verifiability ⇒ topology and σ-algebras (foundation of geometry, probability, …)

Perfect map between math and physics

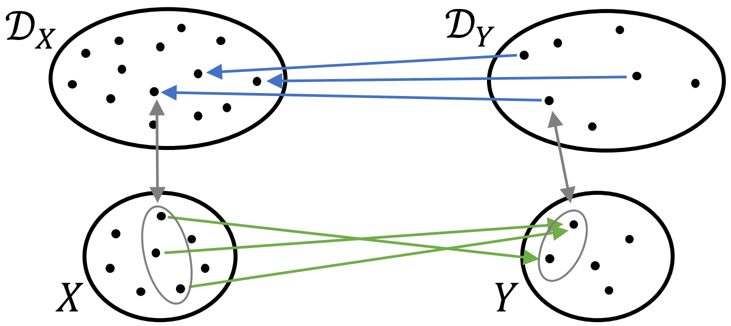
NB: in physics, topology and σ-algebra are parts of the **same** logic structure

Open set $(509.5, 510.5)$ ⟺ Verifiable "the mass of the electron is 510 ± 0.5 KeV"

Closed set $[510]$ ⟺ Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set $\mathbb{Q}$ $(int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset)$ ⟺ Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

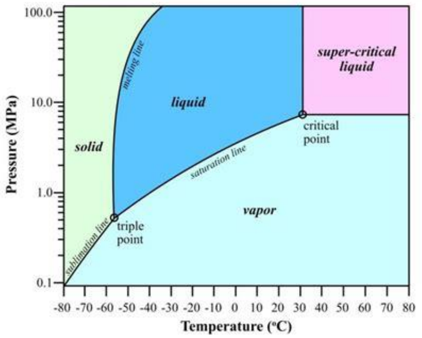


Causal relationship $f: X \to Y$ such that $x \preccurlyeq f(x)$

Inference relationship
⟺
Causal relationship

Relationships must be topologically continuous

Topologically continuous consistent with analytic discontinuity on isolated points



Phase transition ⟺ Topologically isolated regions

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Assumptions of Physics

What is the largest set of verifiable statements it makes sense to consider?

$s_1, s_2, s_3, \dots, s_n, \dots$

Note: even assuming an indeterminate amount of time, we can only run up to countably many tests

$s_1 \vee s_2, s_1 \wedge s_2, \dots$

However, testing those statements implicitly tests all other statements that depend on those

$\Rightarrow$ Set of verifiable statements whose truth can be verified by running countably many tests

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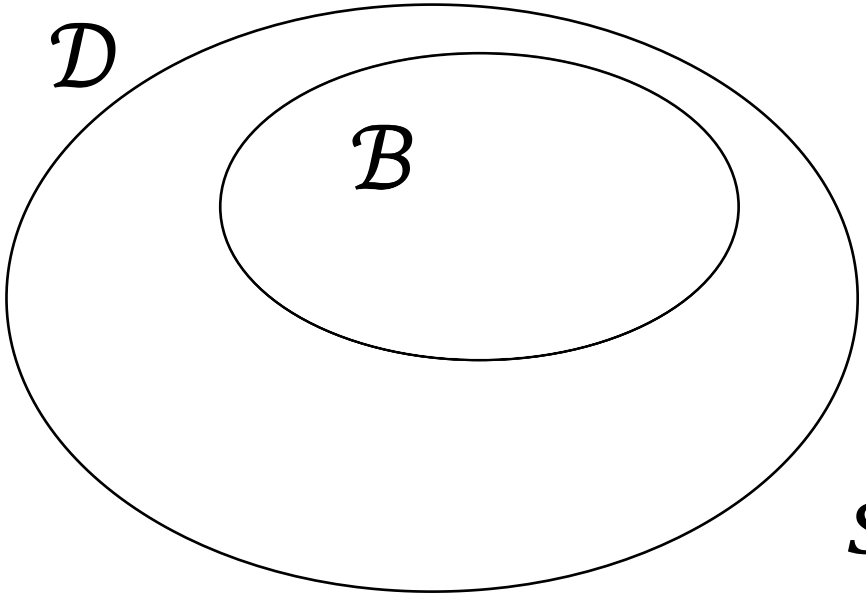
Assumptions of Physics

Definition 1.34. *Given a set $\mathcal{D}$ of verifiable statements, $\mathcal{B} \subseteq \mathcal{D}$ is a* ***basis*** *if the truth values of $\mathcal{B}$ are enough to deduce the truth values of the set. Formally, all elements of $\mathcal{D}$ can be generated from $\mathcal{B}$ using finite conjunction and countable disjunction.*

Definition 1.35. *An* ***experimental domain*** *$\mathcal{D}$ represents a set of verifiable statements that can be tested and possibly verified in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the certainty, the impossibility, and a set of verifiable statements that can be generated from a countable basis.*



$$\mathcal{B} = \{e_1, e_2, e_3, \ldots\}$$

Countable basis

Only finite conjunction and countable disjunction

$$s_1 = (e_1 \vee e_3) \wedge e_2 \ldots$$

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Assumptions of Physics

Definition 1.34. *Given a set $\mathcal{D}$ of verifiable statements, $\mathcal{B} \subseteq \mathcal{D}$ is a* **basis** *if the truth values of $\mathcal{B}$ are enough to deduce the truth values of the set. Formally, all elements of $\mathcal{D}$ can be generated from $\mathcal{B}$ using finite conjunction and countable disjunction.*

Definition 1.35. *An* ***experimental domain*** *$\mathcal{D}$ represents a set of verifiable statements that can be tested and possibly verified in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the certainty, the impossibility, and a set of verifiable statements that can be generated from a countable basis.*

Every physical theory must be fully characterized by an experimental domain

All its content must be expressible in terms of verifiable statements

The theory must be fully explorable with a countable set of tests

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Assumptions of Physics

Definition 1.36. *The* ***theoretical domain*** $\bar{\mathcal{D}}$ *of an experimental domain* $\mathcal{D}$ *is the set of statements constructed from* $\mathcal{D}$ *to which we can associate a test regardless of termination. We call* ***theoretical statement*** *a statement that is part of a theoretical domain. More formally,* $\bar{\mathcal{D}}$ *is the set of all statements generated from* $\mathcal{D}$ *using negation, finite conjunction and countable disjunction.*

Extend the domain to include all statements that are associated with a test, regardless of termination.

All statements depend on the verifiable statements
(which depend on the basis)

No new information is captured

Definition 1.39. *Let* $\bar{\mathsf{s}} \in \bar{\mathcal{D}}$ *be a theoretical statement. We call the* ***verifiable part*** $\mathrm{ver}(\bar{\mathsf{s}}) = \bigvee_{\mathsf{s}\in\mathcal{D} \,|\, \mathsf{s}\preccurlyeq\bar{\mathsf{s}}} \mathsf{s}$ *the broadest verifiable statement that is narrower than* $\bar{\mathsf{s}}$*. We call the* ***falsifiable part*** $\mathrm{fal}(\bar{\mathsf{s}}) = \bigvee_{\mathsf{s}\in\mathcal{D} \,|\, \mathsf{s}\not\asymp\bar{\mathsf{s}}} \mathsf{s}$ *the broadest verifiable statement that is incompatible with* $\bar{\mathsf{s}}$*. We call the* ***undecidable part*** $\mathrm{und}(\bar{\mathsf{s}}) = \neg\,\mathrm{ver}(\bar{\mathsf{s}}) \wedge \neg\,\mathrm{fal}(\bar{\mathsf{s}})$ *the broadest statement incompatible with both the verifiable and the falsifiable part.*

Formalizing successful termination is indeed enough to characterize termination

s	Test Result
T	SUCCESS (in finite time)
	UNDEFINED
F	UNDEFINED
	FAILURE (in finite time)

$\mathrm{ver}(s)$ corresponds to successful termination

$\mathrm{und}(s)$ corresponds to non-termination

$\mathrm{fal}(s)$ corresponds to failure

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Assumptions of Physics

Definition 1.47. *A* ***possibility*** *for an experimental domain* $\mathcal{D}$ *is a statement* $x \in \bar{\mathcal{D}}$ *that, when true, determines the truth value for all statements in the theoretical domain. Formally,* $x \not\equiv \bot$ *and for each* $\mathsf{s} \in \bar{\mathcal{D}}$*, either* $x \preccurlyeq \mathsf{s}$ *or* $x \not\preccurlyeq \mathsf{s}$*. The* ***full possibilities****, or simply the* ***possibilities****,* X *for* $\mathcal{D}$ *are the collection of all possibilities.*

A possibility of a domain is a statement that picks one assignment

Possibilities: experimentally defined alternative cases defined by the verifiable cases

s_1	s_2	s_3	…	x_1	x_2	x_3	x_4
T	T	F	T	T	F	F	F
F	F	T	T	F	T	F	F
F	T	T	F	F	F	T	F
T	F	F	T	F	F	F	T

Proposition 1.48. *Let* $\mathcal{D}$ *be an experimental domain. A possibility for* $\mathcal{D}$ *is any minterm of a basis that is not impossible.*

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Assumptions of Physics

Start with a countable set of verifiable statements

Add all dependent verifiable statements (close under finite AND countable OR)

Add all statements with tests (close under negation as well)

The points of the space (the possibilities, the distinguishable cases) are not given a priori but are constructed from the chosen verifiable statements

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each possible assignment we have a theoretical statement that is true only in that case (minterm). We call these statements possibilities of the domain.

Fill in all possible assignments

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Assumptions of Physics

Each column (statement) is also a set of possibilities $s = \bigvee_{x \in U} x$

Finite AND and countable OR become finite intersection and countable union

Negation and countable AND become complement and countable union

Basis $\mathcal{B}$				Experimental domain $\mathcal{D}_X$			Theoretical domain $\bar{\mathcal{D}}_X$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities $X \subset \bar{\mathcal{D}}_X$

Topologies (needed for manifold/geometric spaces) and σ-algebras (needed for integration and probability spaces) naturally arise from requiring experimental verifiability

The experimental domain $\mathcal{D}_X$ induces a topology on the possibilities X.

The theoretical domain $\bar{\mathcal{D}}_X$ induces a (Borel) σ-algebra

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Assumptions of Physics

Topologies and σ-algebras

All definitions and all proofs about these structures have precise physical meaning in this context

s_1	Test Result
T	SUCCESS (in finite time)
T	UNDEFINED
F	UNDEFINED
F	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

If $U \subseteq X$ is an open set then "x is in U" is a verifiable statement (e.g. "the mass of the electron is 511 ± 0.5 KeV")

If $V \subseteq X$ is a closed set then "x is in V" is a falsifiable statement (e.g. "the mass of the electron is exactly 511 KeV")

If $A \subseteq X$ is a Borel set then "x is in A" is a theoretical statement: a test can be created, though we have no guarantee of termination (e.g. "the mass of the electron in KeV is a rational number" is undecidable, the test will never terminate)

Topologies and σ-algebras each capture part of the formal structure

For us, they are part of a single unified structure

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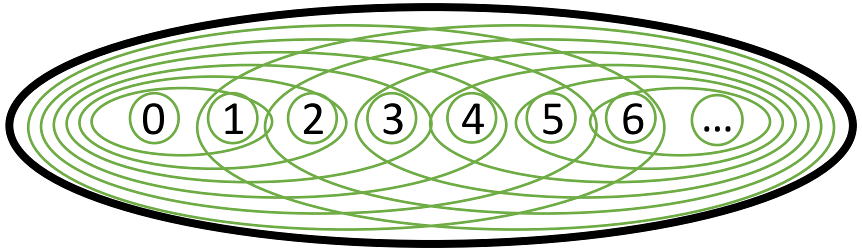
Assumptions of Physics

Examples



Standard topology on integers

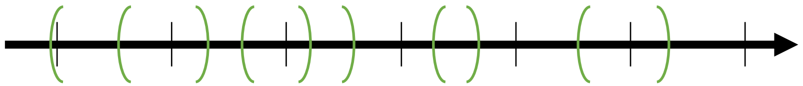
Decidable domain (all statements are decidable)

Discrete topology (every set is clopen); topology and σ-algebra both coincide with the power set

Standard topology on the reals

Finite precision measurements (open intervals are verifiable)

Topology generated by open intervals (coincides with order and metric topology); separable, complete, connected (no clopen sets except full and empty set); σ-algebra is the Borel algebra (strict subset of power set)



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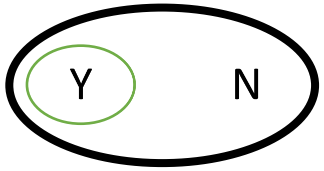
Assumptions of Physics

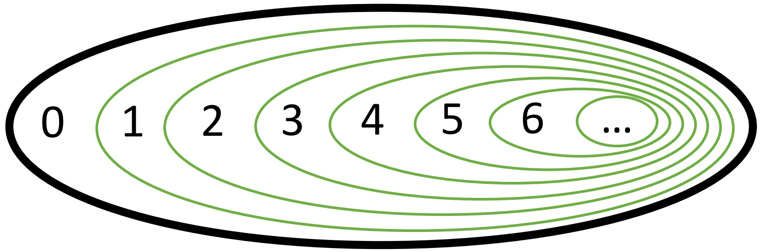
Examples



Does extra-terrestrial life exist?

Semi-decidable question

Topology $\{\emptyset, \{Y\}, \{Y, N\}\}$ is strictly T_0; σ-algebra is the power set



How many leptons (electron-like particles) are there? (through direct observation)

Can only measure lower bound (e.g. "there are at least i")

Topology contains empty set and $\{i, i+1, i+2, \ldots\}$ for all i; strictly T_0; σ-algebra is the power set

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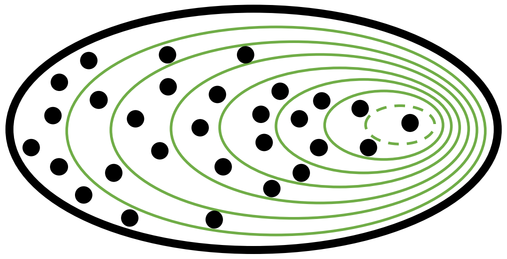
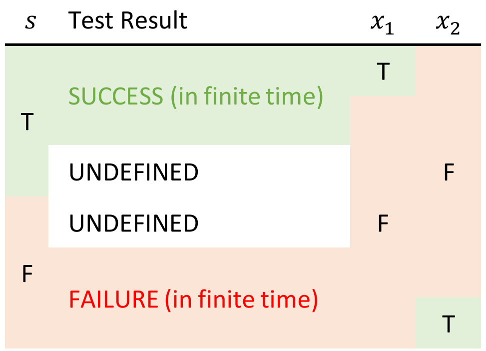
Assumptions of Physics

Physical meaning of separation axioms

- All topologies are Kolmogorov (i.e. T_0)
  - Possibilities are experimentally well-defined
    i.e. possibilities constructible from a base by countable AND/OR and NOT (singletons in the σ-algebra)
- The topology is T_1 if all possibilities are approximately verifiable
  - Possibilities are the limit of a sequence of verifiable statements
    i.e. possibilities are the countable conjunction of verifiable statements



- The topology is Hausdorff (i.e. T_2) if all possibilities are pairwise experimentally distinguishable
  - Given two possibilities, we can find a test that confirms one and excludes the other
  - i.e. for any $x_1, x_2 \in X$ there is a statement $s \in \bar{\mathcal{D}}_X$ such that $x_1 \preccurlyeq ver(s)$ and $x_2 \preccurlyeq fal(s)$



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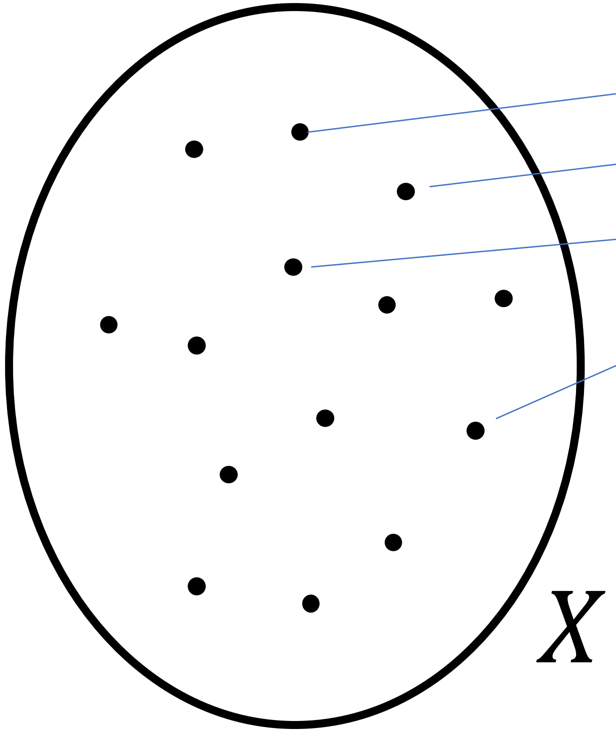
Assumptions of Physics

Maximum cardinality of distinguishable cases

Set of distinguishable cases

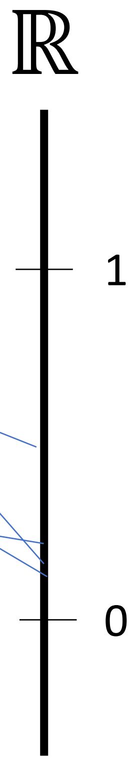


Test results for countable basis

FTFFFFTTTFTFTT…
TFFTTFTTFFFTF…
FTFFFFTTFTFFTF…
FTTFTFTTFTFFT…

Correspond to binary expansion

0.0100011101011…
0.1001101100010…
0.0100011010010…
0.0110101101001…



Correspondence to binary sequence

0100011101011…
1001101100010…
0100011010010…
0110101101001…

$|X| \leq |\mathbb{R}|$

- Sets with greater cardinality (e.g. the set of all discontinuous functions from $\mathbb{R}$ to $\mathbb{R}$) cannot represent physical objects
- Issues about higher infinities (e.g. large cardinals) are not relevant, but those surrounding the continuum hypothesis may be

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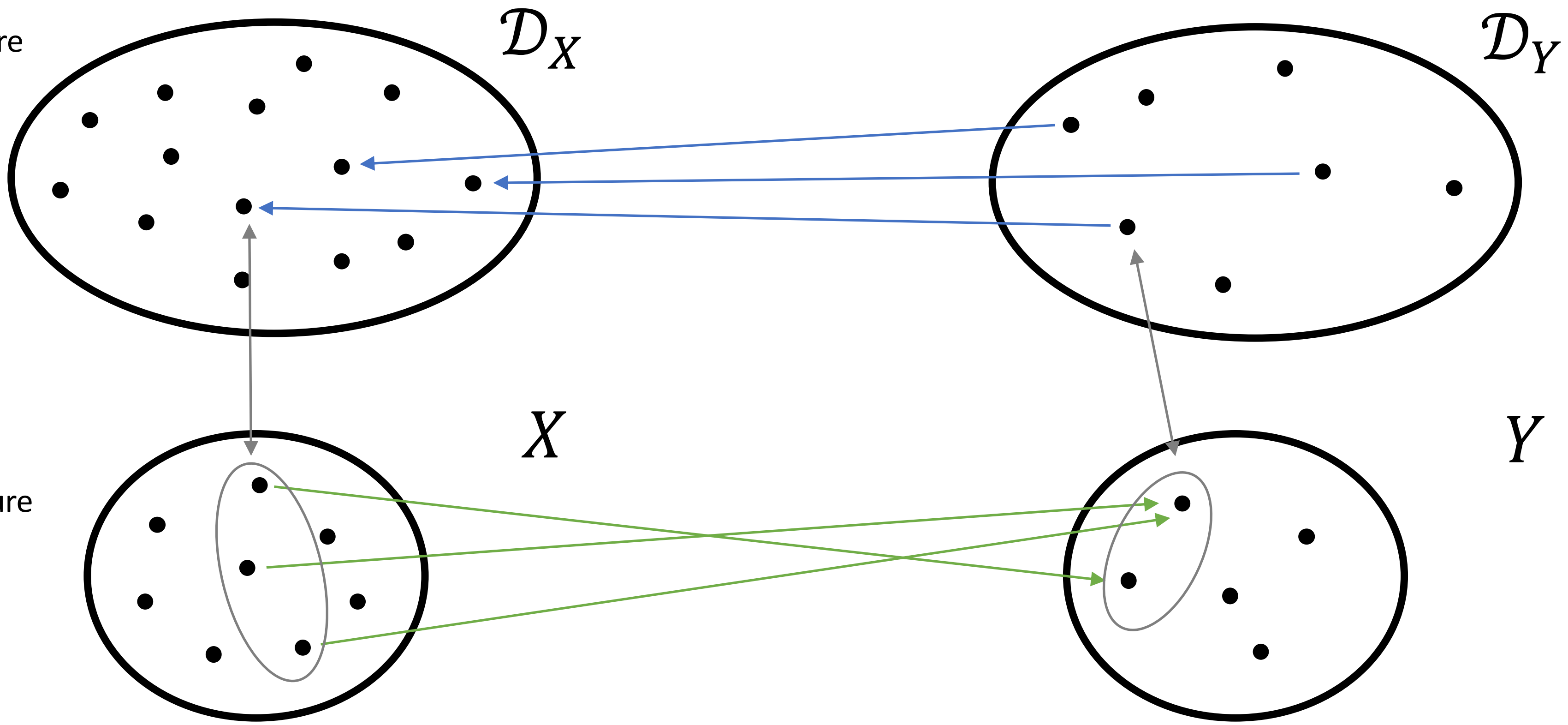
Assumptions of Physics

An inference relationship is a map $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

e.g. the water temperature is between 0 and 0.52 Celsius or between 7.6 and 9.12 Celsius

e.g. the water density is between 999.8 and 999.9 kg/m^3



e.g. the water temperature is exactly 4 Celsius

e.g. the water density is exactly 1 kg/m^3

A causal relationship is a map $f: X \to Y$ such that $x \preccurlyeq f(x)$

Two general and important results:

1) Two domains admit an inference relationship if and only if they admit a causal relationship

2) The causal relationship must be a continuous map in the natural topology

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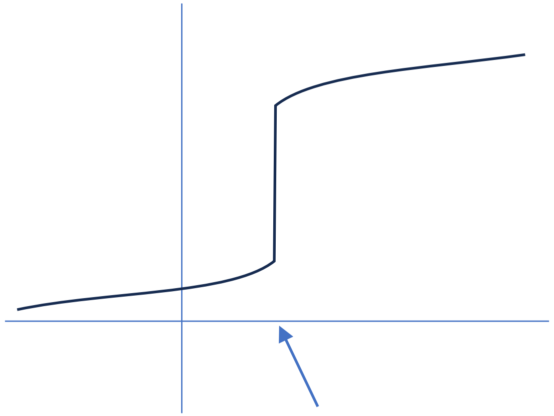
Assumptions of Physics

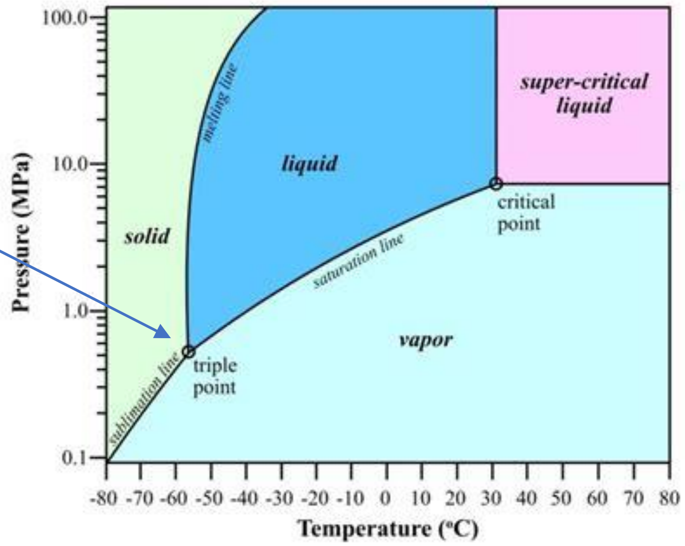
Functions in physics must be "well-behaved"



Topologically continuous function can be analytically discontinuous at a topologically isolated point

Second countable space
⇒ up to countably many isolated points
⇒ up to countably many discontinuity
⇒ "well-behaved"

We can verify we are in the triple point ⇒ topologically isolated point



Phase transition ⇔ Topologically isolated regions

Internal energy can change discontinuously through phase transitions

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Assumptions of Physics

Takeaway

- The most fundamental mathematical structures (topology and σ-algebra) are there to capture the logic of experimental verifiability
    - Precise science/math dictionary
    - “Well-behaved” mathematical objects are really “well-defined” physical objects
- Experimental verifiability is the basis for scientifically well-defined objects
- TODOs:
    - Space of possible composite experimental domains
    - Approximations of domains
    - Projections to domains

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Assumptions of Physics

Quantities and ordering

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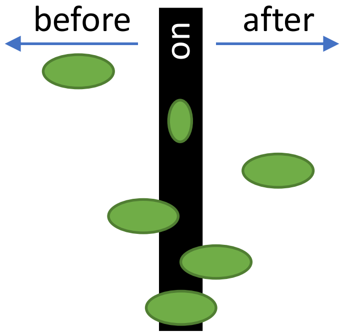
Assumptions of Physics

Quantities and ordering

Phys. Scr. **95** 084003 (2020)

Goal: deriving the notion of quantities and numbers (i.e. integers, reals, …) from an operational (metrological) model

A **reference** (i.e. a tick of a clock, notch on a ruler, sample weight with a scale) is something that allows us to distinguish between a before and an after



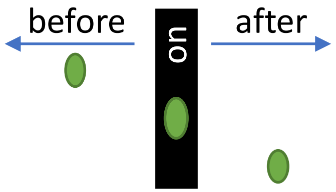
Mathematically, it is a triple (b, o, a) such that:

- b and a are verifiable
- The reference has an extent ($o \not\equiv \bot$)
- If it's not before or after, it is on ($\neg b \wedge \neg a \preccurlyeq o$)
- If it's before and after, it is on ($b \wedge a \preccurlyeq o$)

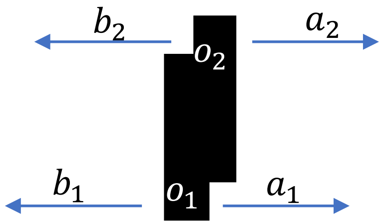
Numbers defined by metrological assumptions, NOT by ontological assumptions

To define an **ordered** sequence of possibilities, the references must be (nec/suff conditions):

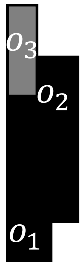
Strict



Aligned



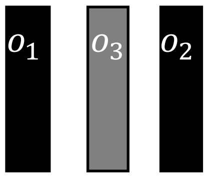
Refinable



$\Rightarrow (X, \leq)$

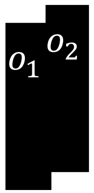
+

Dense



$\Rightarrow (X, \leq) \cong (\mathbb{R}, \leq)$

Sparse



$\Rightarrow (X, \leq) \cong (\mathbb{Z}, \leq)$

The hard part is to recover ordering. After that, recovering reals and integers is simple.

Assumptions untenable at Planck scale:
no consistent **ordering**: no "objective" "before" and "after"

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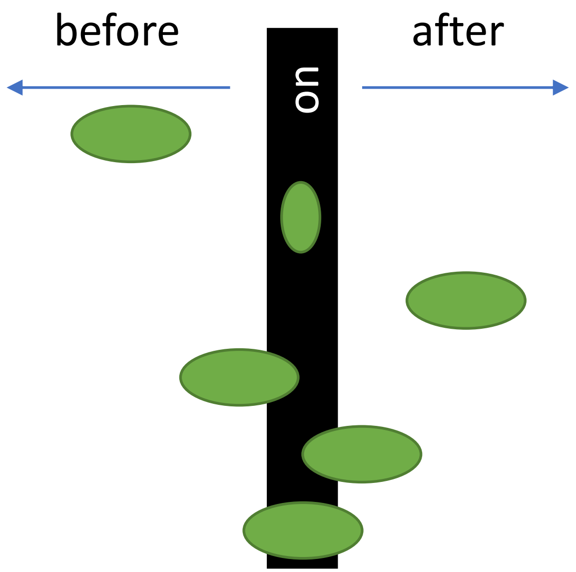
Assumptions of Physics

How do we formally model a quantity?

A **reference** (e.g. a tick of a clock) is something that allows us to distinguish between a before and an after



Mathematically, it is a triple (b, o, a) such that:

- b and a are verifiable
- The reference has an extent ($o \not\equiv \bot$)
- If it's not before or after, it is on ($\neg b \wedge \neg a \preccurlyeq o$)
- If it's before and after, it is on ($b \wedge a \preccurlyeq o$)

Before	On	After
T	F	F
F	T	F
F	F	T
T	T	F
F	T	T
T	T	T

The experimental domain for a quantity is a collection of references

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Assumptions of Physics

Imagine collecting the references of all possible clocks into a single logical structure. What are the necessary and sufficient conditions such that they identify a point on the real line?

Intuitively, we would need clocks at higher and higher resolutions, all perfectly synchronized, …

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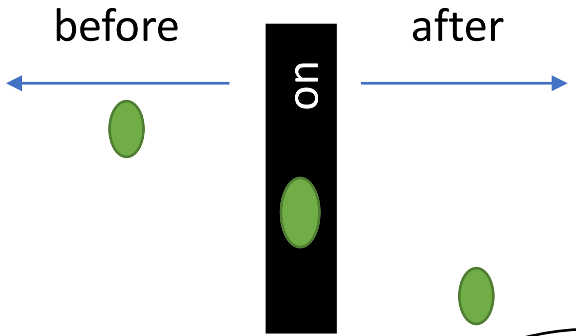
Assumptions of Physics

1. Strict references

A reference is strict if before/on/after are mutually exclusive

Before	On	After
T	F	F
F	T	F
F	F	T



Physically, the extent of what we measure is assumed to be smaller than the extent of our reference

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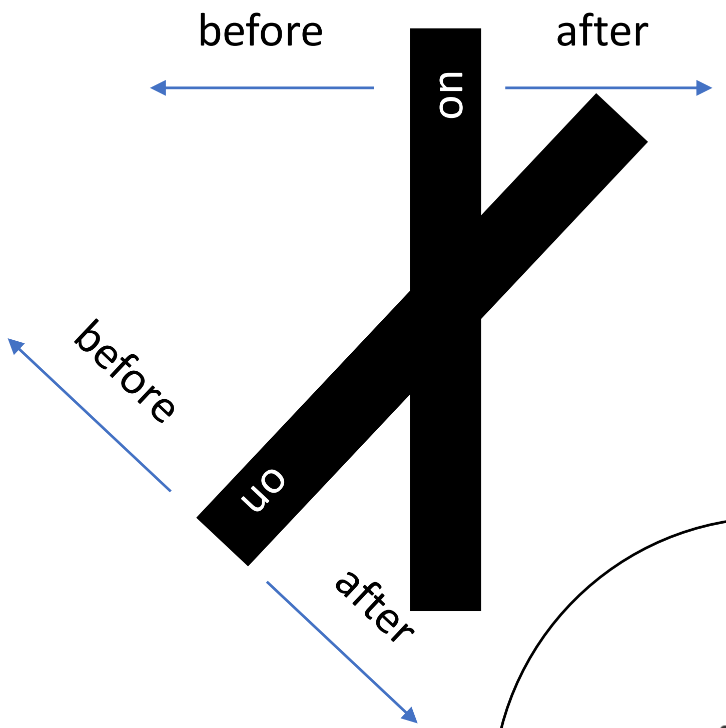
Assumptions of Physics

Multiple references

Without further constraints, references would not lead to a linear order

	b_2	o_2	a_2
b_1	✓	✓	✓
o_1	✓	✓	✓
a_1	✓	✓	✓



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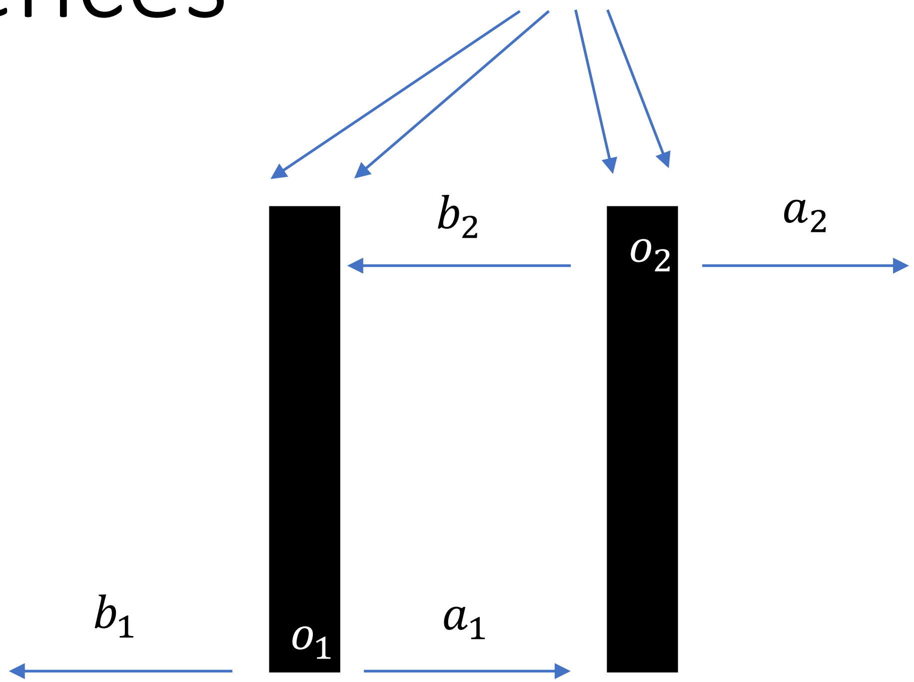
Assumptions of Physics

Multiple references

Note: the “boundaries” are ordered



The fact that a reference is “before” or “after” another is captured by the statements’ logical relationship

	b_2	o_2	a_2
b_1	✓	×	×
o_1	✓	×	×
a_1	✓	✓	✓

Order relationship between references is too restrictive

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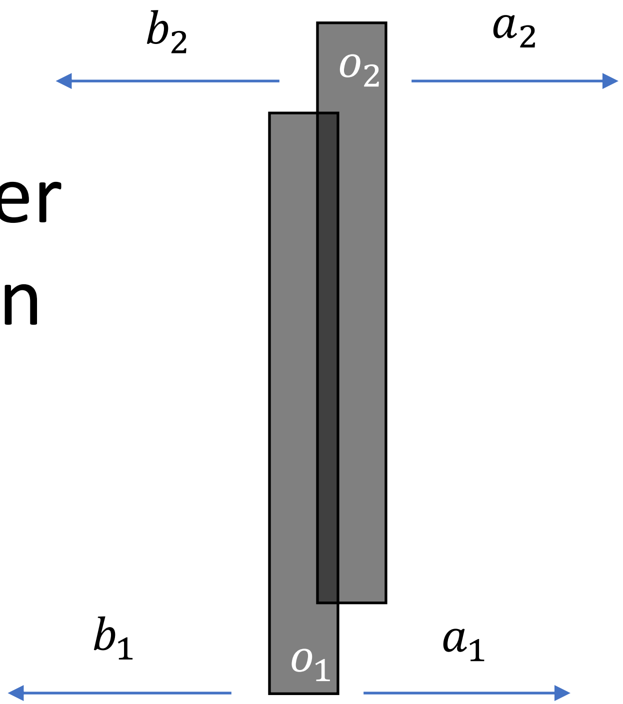
Assumptions of Physics

2. Aligned references



Two references are aligned if the before and not-after statement can be ordered by narrowness/implication

For example, $b_1 \preccurlyeq b_2 \preccurlyeq \neg a_1 \preccurlyeq \neg a_2$

$\preccurlyeq$ Means that if the first statement is true then the second statement will be true as well

That is, the first statement is narrower, more specific

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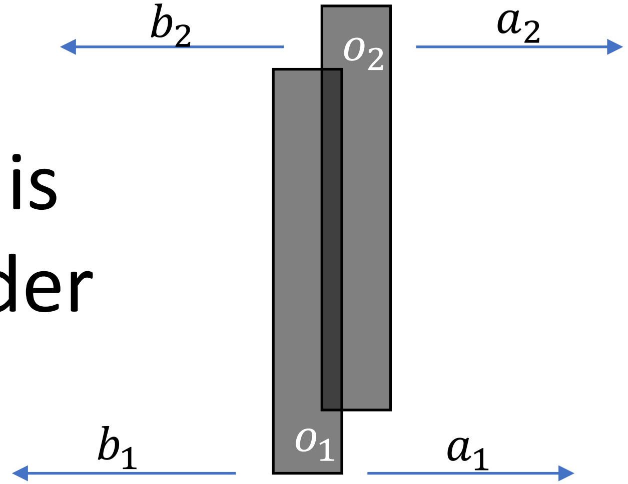
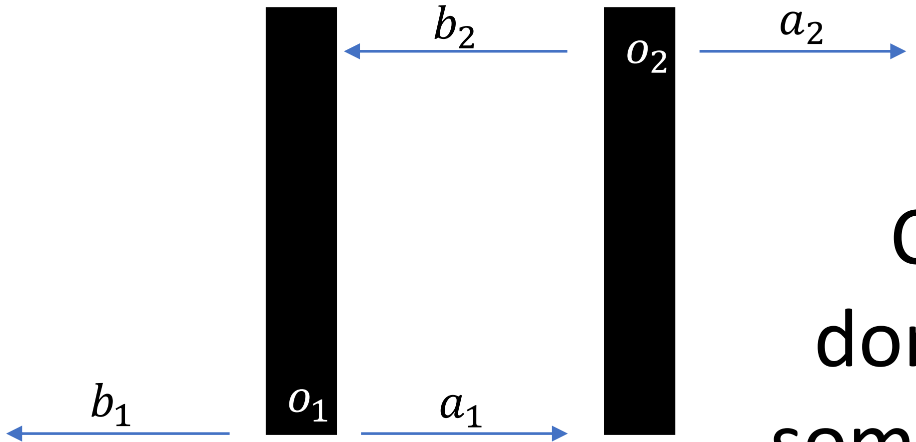
Assumptions of Physics

Filling the whole region



If two different references overlap, we can't say one is before the other: we can't fully resolve the linear order



Conversely, if two references don't overlap and there can be something in between, we must be able to put a reference there

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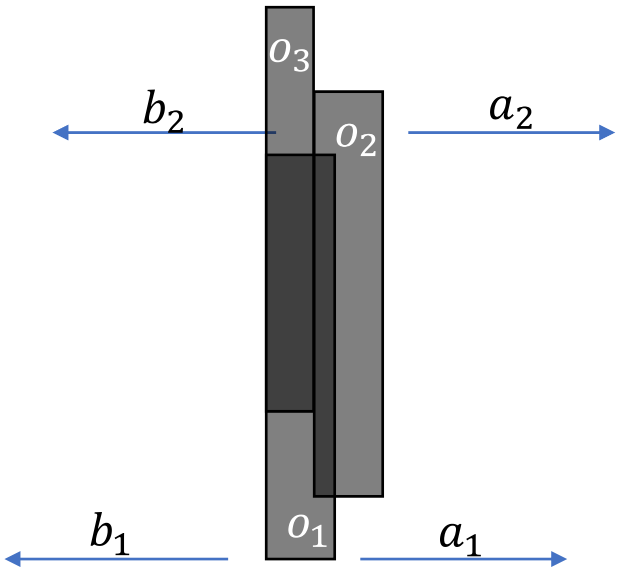
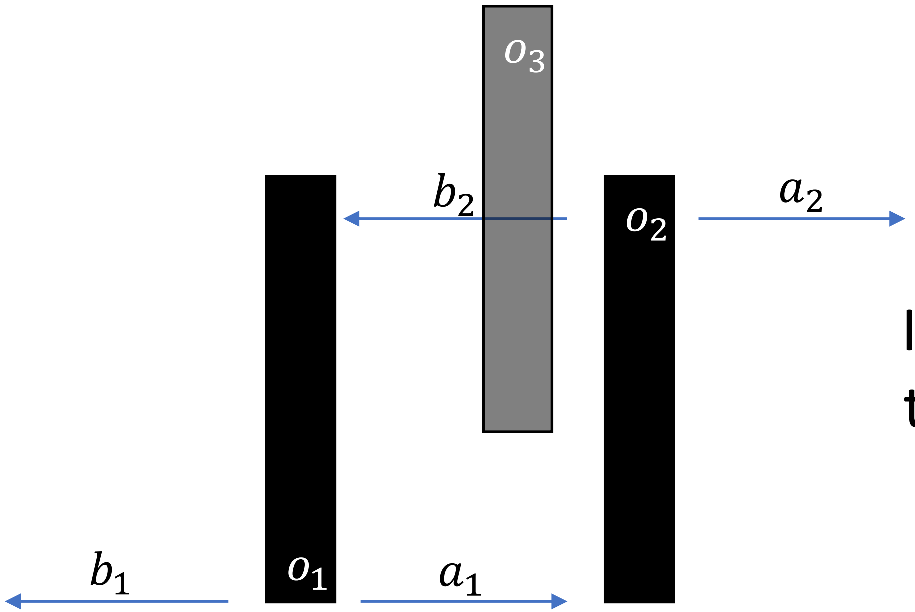
3. Refinable references

A set of references is refinable if we can address the previous two problems and resolve the full space



If two references overlap, we can find a reference that refines the overlap



If something can be found between two references, then there must be another reference in between

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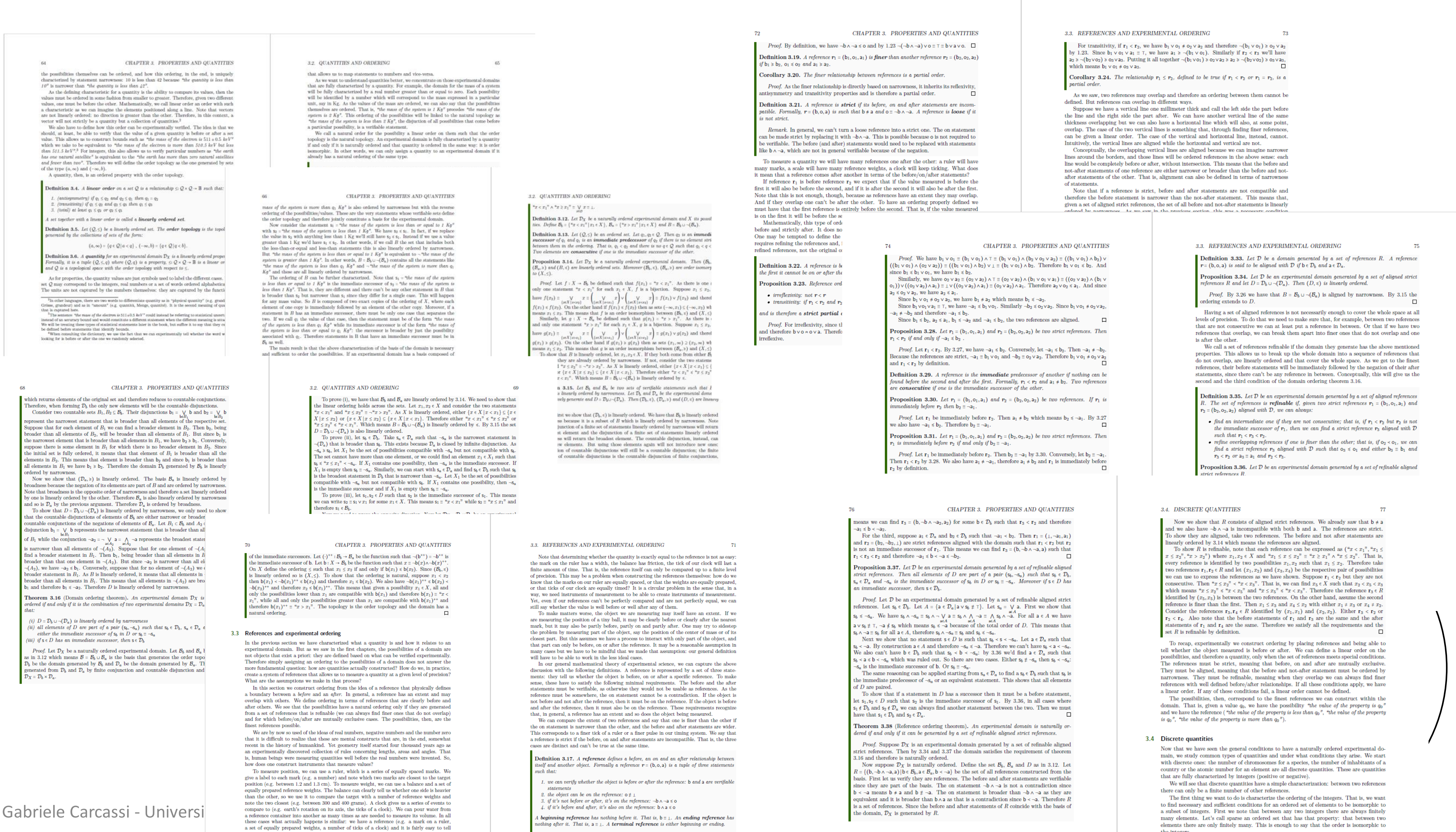

Reference ordering theorem

To define an **ordered** sequence (e.g. of “instants”), the references must be (nec/suff conditions):

- Strict – an event is strictly before/on/after the reference (doesn’t extend over the tick)
- Aligned – shared notion of before and after (logical relationship between statements)
- Refinable – overlaps can always be resolved

Additionally:

Between any two references we can always have another reference ⇒ **real numbers**

Only finitely many references between any two references ⇒ **integers**

For time/space, these conditions are idealizations

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Assumptions of Physics

How does this model break down?

The ticks of a clock have an extent and so do the events (references not strict)

If clocks have jitter, they cannot achieve perfect synchronization (references not aligned)

We cannot make clock ticks as narrow as we want (references not refinable)

No consistent **ordering**: no “objective” “before” and “after”

In relativity, different observers measure time differently, but the order is the same. We should expect this to fail at “small” scales.

A better understanding of space-time means creating a more realistic formal model that accounts for those failures

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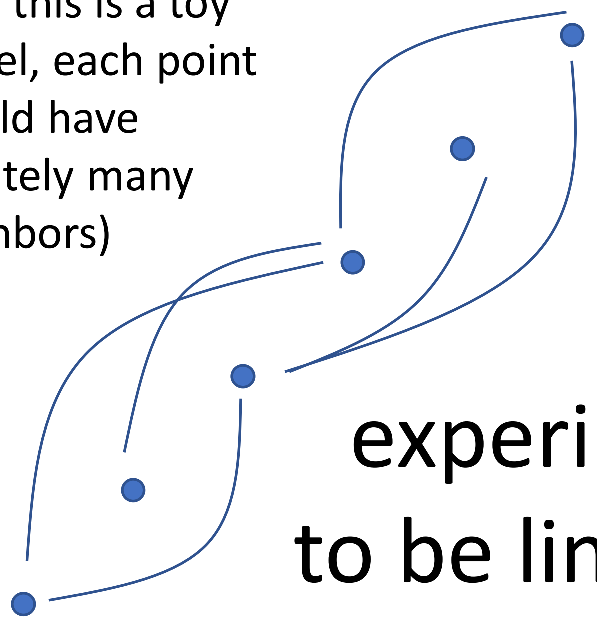
Assumptions of Physics

What type of models should we use?

Hard to say, but we can argue from necessity

(N.B. this is a toy model, each point should have infinitely many neighbors)



Lack of order at small scales, order at large enough scale

What we can distinguish experimentally (i.e. topology) seems to be linked to how precisely we want to distinguish (i.e. geometry)

Current mathematical tools have a hard division between topology and geometry

Likely need new math

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Assumptions of Physics

Our reasoning contradicts the expectations of many that time is simply "discrete" at the smallest scale

This intuition is based on the idea that the continuum is like the discrete but "with more points"

This idea (though extremely common in physics) is flawed

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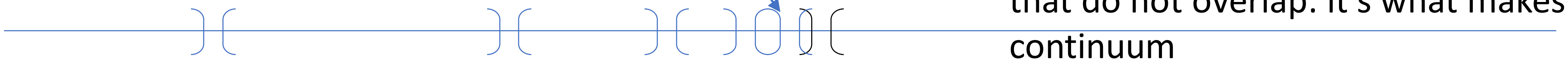
Topology of the reals

Connected: cannot be divided into two disjoint open sets

Open set – open interval



Always space between two references that do not overlap: it's what makes it a continuum

Reference

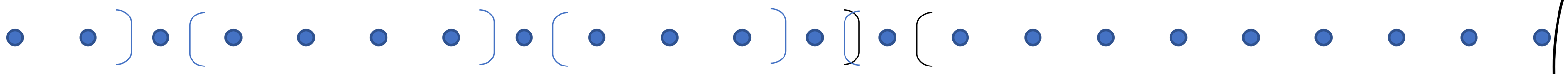
No contingent decidable statements

Topology of the integers

Disconnected: can be divided into two disjoint open sets

No space between two consecutive references

Reference



All contingent statements are decidable

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Takeaway

- Ordering, the defining features of quantities, is a logical structure
  - $3 \leq 5$ precisely because “there are less than 3 items” $\preccurlyeq$ “there are less than 5 items”
- TODOs:
  - Find whether one can construct topological spaces that are not locally metrizable but are “sort of metrizable” on long “distances”

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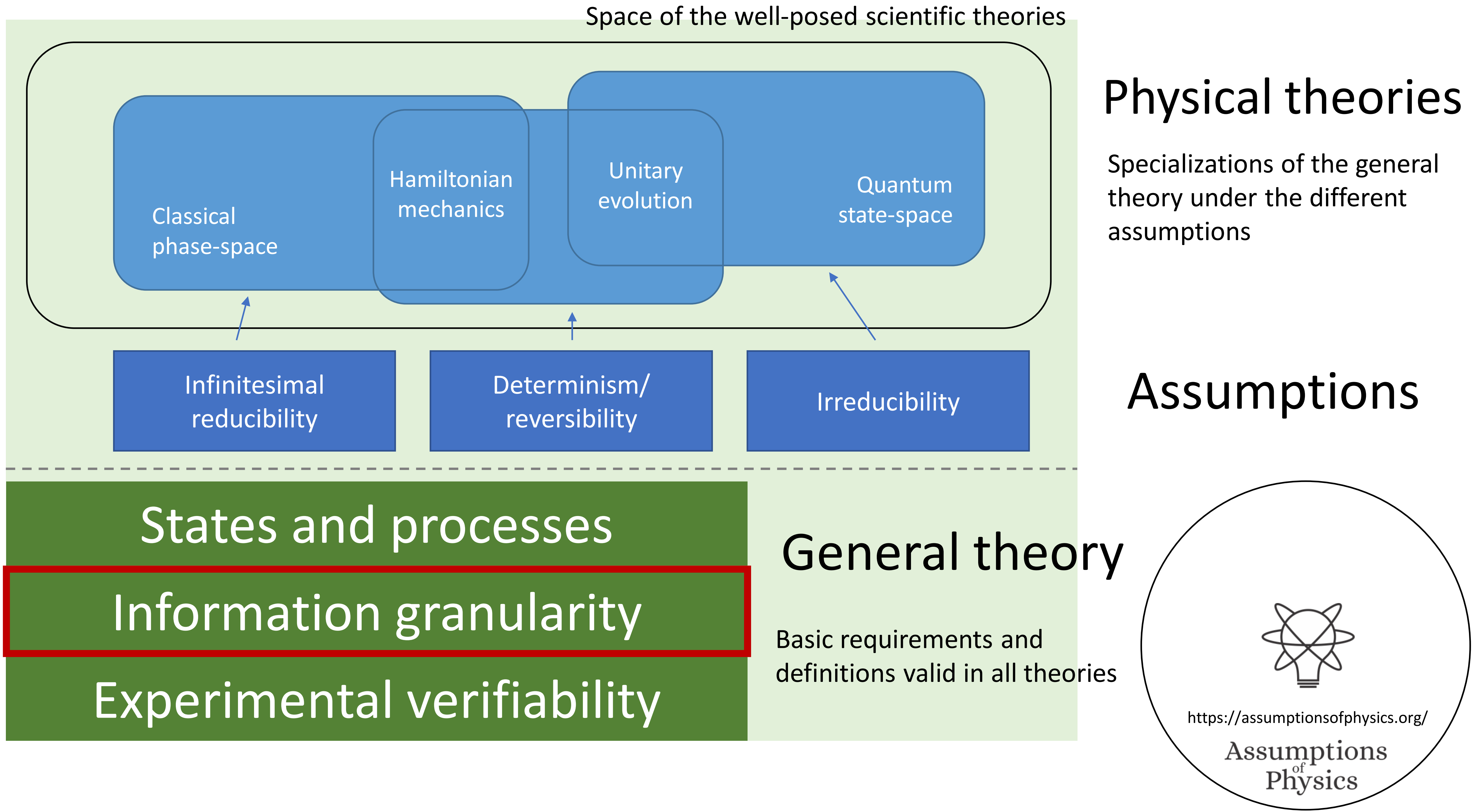

Space of the well-posed scientific theories
Classical phase-space
Hamiltonian mechanics
Unitary evolution
Quantum state-space
Infinitesimal reducibility
Determinism/ reversibility
Irreducibility
States and processes
Information granularity
Experimental verifiability
Physical theories
Specializations of the general theory under the different assumptions
Assumptions
General theory
Basic requirements and definitions valid in all theories
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Information granularity

Logical relationships $\Leftrightarrow$ Topology/σ-algebra

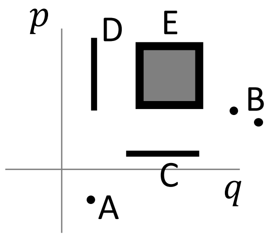
- "The position of the object is between 0 and 1 meters" $\preccurlyeq$ "The position of the object is between 0 and 1 kilometers"
- "The fair die landed on 1" $\preccurlyeq$ "The fair die landed on 1 or 2"
- "The first bit is 0 and the second bit is 1" $\preccurlyeq$ "The first bit is 0"

Granularity relationships $\Leftrightarrow$ Geometry/Probability/Information

- "The position of the object is between 0 and 1 meters" $\lesssim$ "The position of the object is between 2 and 3 kilometers"
- "The fair die landed on 1" $\lesssim$ "The fair die landed on 3 or 4"
- "The first bit is 0 and the second bit is 1" $\lesssim$ "The third bit is 0"

$\Rightarrow$ Measure theory, geometry, probability theory, information theory, … all quantify the level of granularity of different statements

A partially ordered set allows us to compare size at different level of infinity and to keep track of incommensurable quantities (i.e. physical dimensions)



A $\lesssim$ B $\lesssim$ C $\lesssim$ E

C $\not\lesssim$ D

D $\not\lesssim$ C

Once a "unit" is chosen, a measure quantifies the granularity of another statement with respect to the unit

$$\mu_u : \overline{\mathcal{D}} \to \mathbb{R}$$

$$\mu_u(u) = 1$$

$$s_1 \lesssim s_2 \Rightarrow \mu_u(s_1) \le \mu_u(s_2)$$

$\mu_u(s_1 \vee s_2) = \mu_u(s_1) + \mu_u(s_2)$ if s_1 and s_2 are incompatible

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Takeaway

- Only rough ideas at this point
- TODOs:
  - Find “right” basic axioms by reverse engineering measure theory

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Wrapping it up

- We have a good foundational layer done that recovers topological structures from requiring experimental verifiability
  - Though some elements can still be developed and better understood
- The layer to describe more quantitative elements (geometry, probability, …) is still to be understood

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