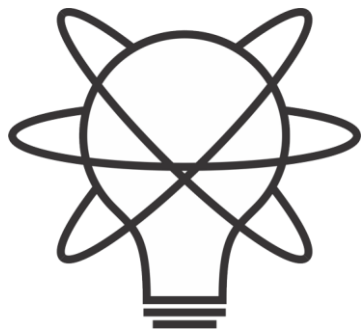


Assumptions of Physics  
Summer School 2024

# Foundational Structures

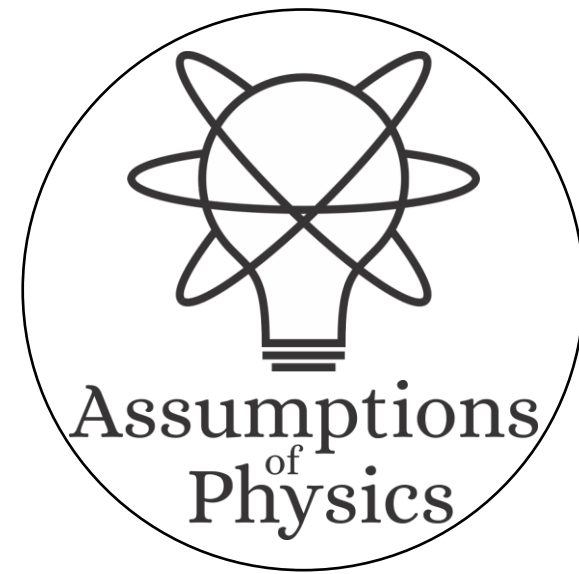
Gabriele Carcassi and Christine A. Aidala

Physics Department  
University of Michigan



Assumptions  
of  
Physics

<https://assumptionsofphysics.org>



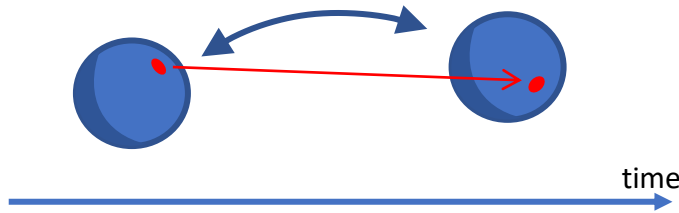
Assumptions  
of  
Physics

# Main goal of the project

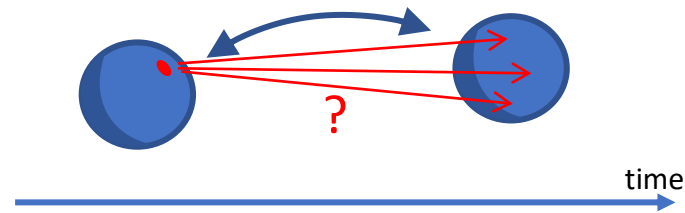
*Identify a handful of physical starting points from which the basic laws can be rigorously derived*

For example:

Infinitesimal reducibility  $\Rightarrow$  Classical state



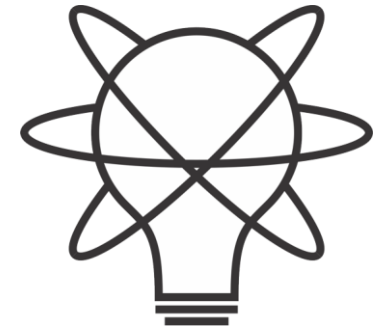
Irreducibility  $\Rightarrow$  Quantum state



This also requires rederiving all mathematical structures from physical requirements

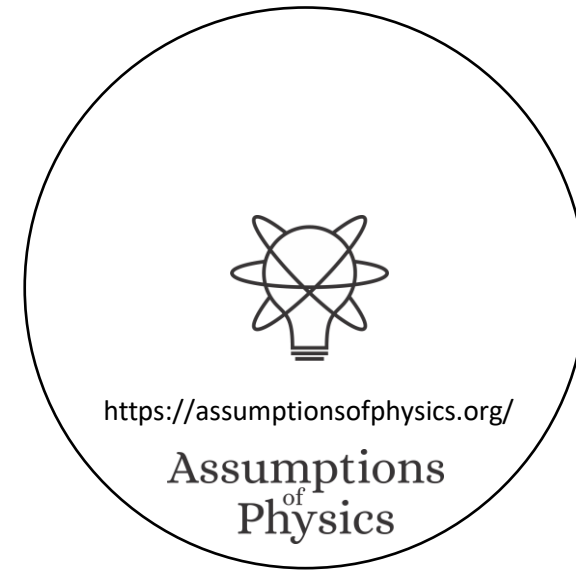
For example:

Science is evidence based  $\Rightarrow$  scientific theory must be characterized by experimentally verifiable statements  $\Rightarrow$  topology and  $\sigma$ -algebras

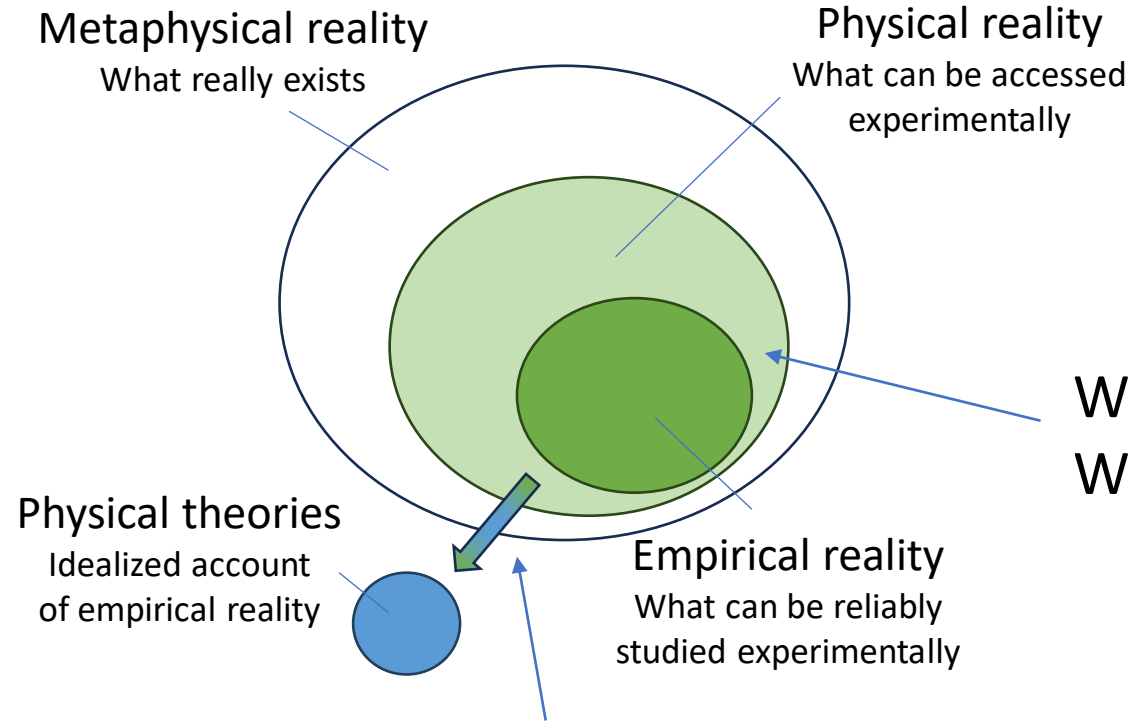


Assumptions  
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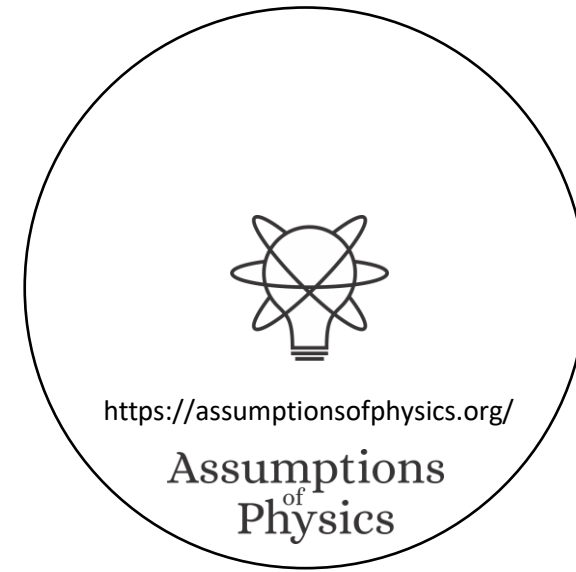
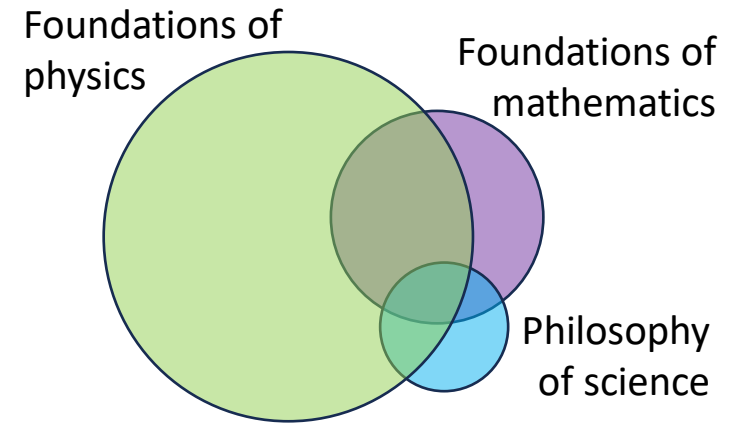


# Underlying perspective



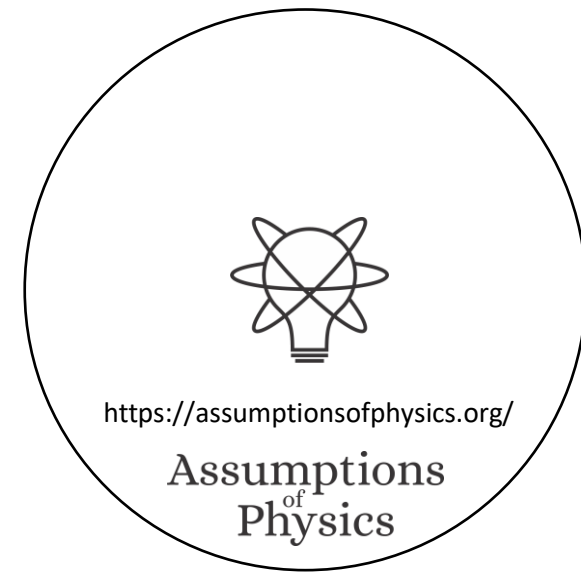
What is the boundary?  
What are the requirements?

How exactly does the abstraction/idealization process work?



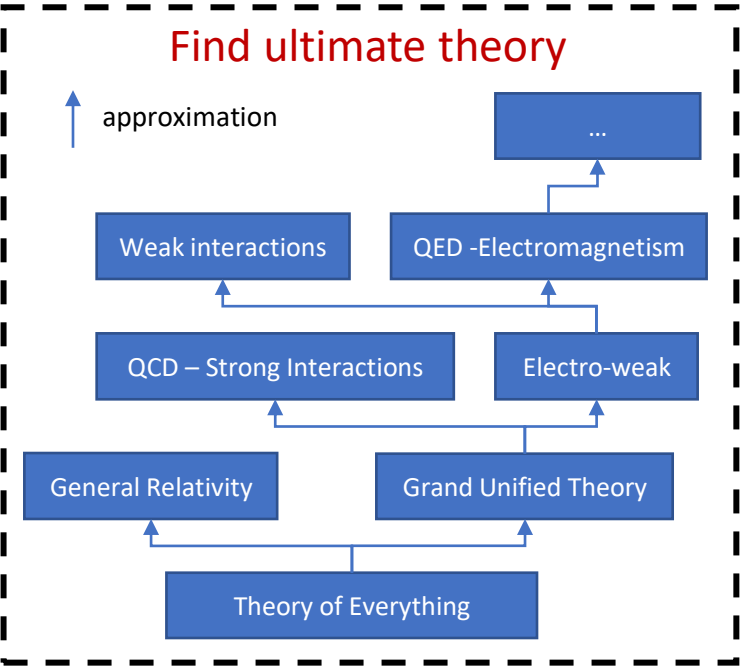
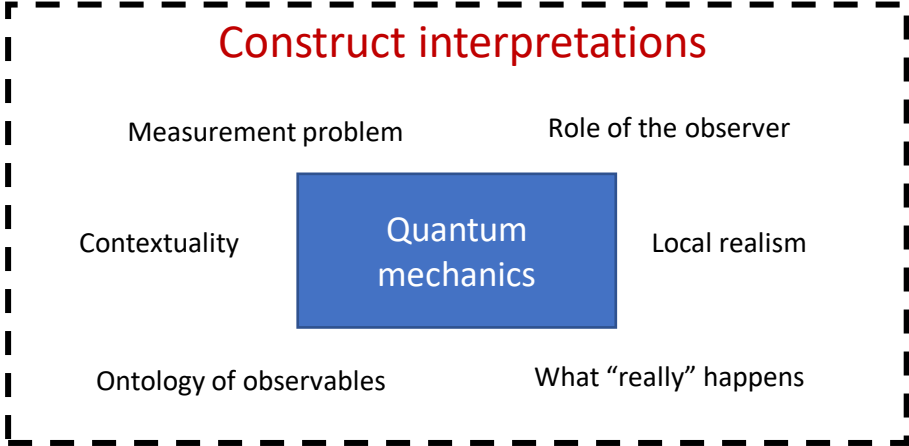
If physics is about creating models of empirical reality, the foundations of physics should be a theory of models of empirical reality

Requirements of experimental verification, assumptions of each theory, realm of validity of assumptions, ...

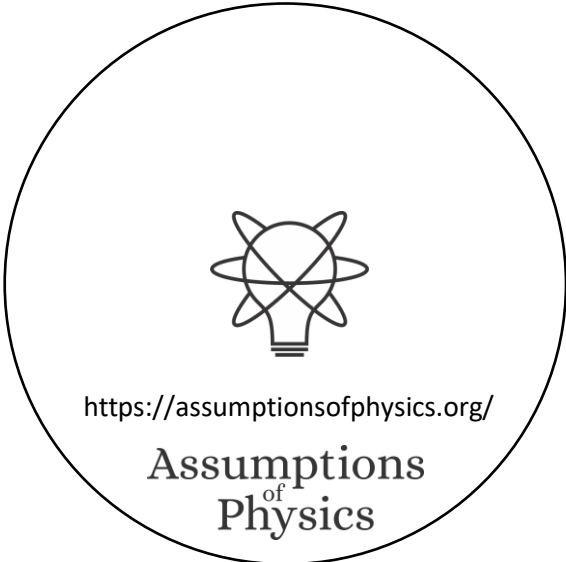
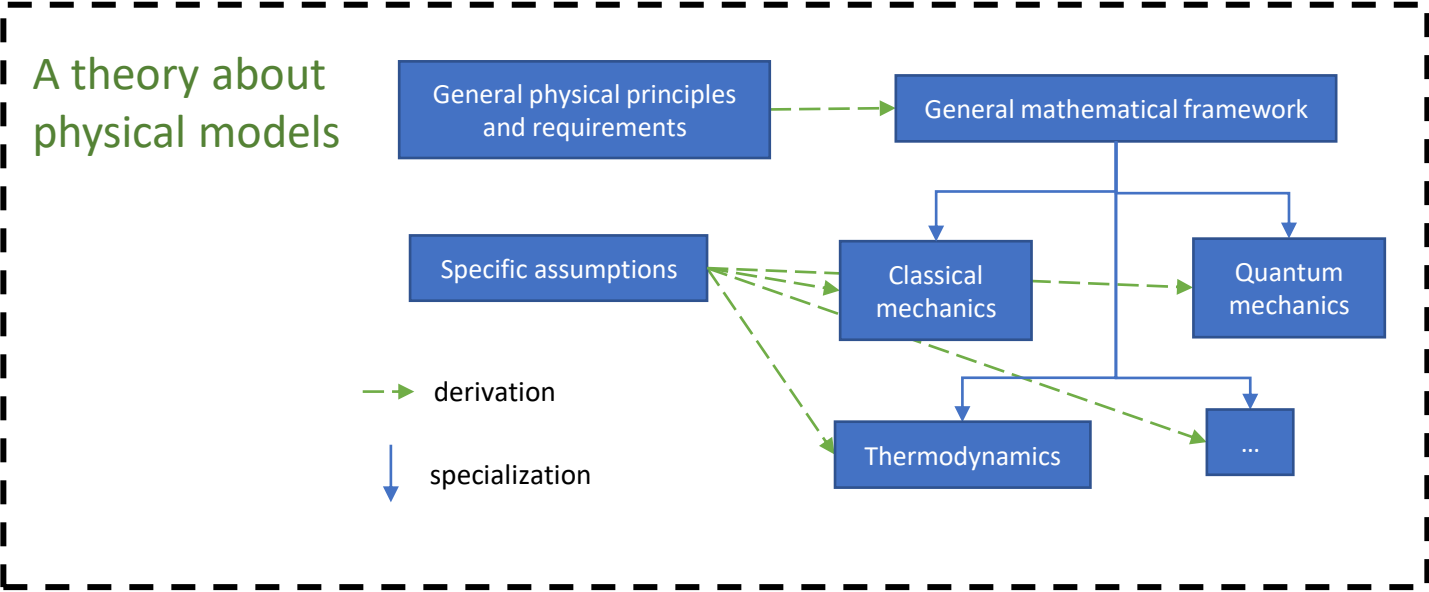


# Different approach to the foundations of physics

Typical approaches



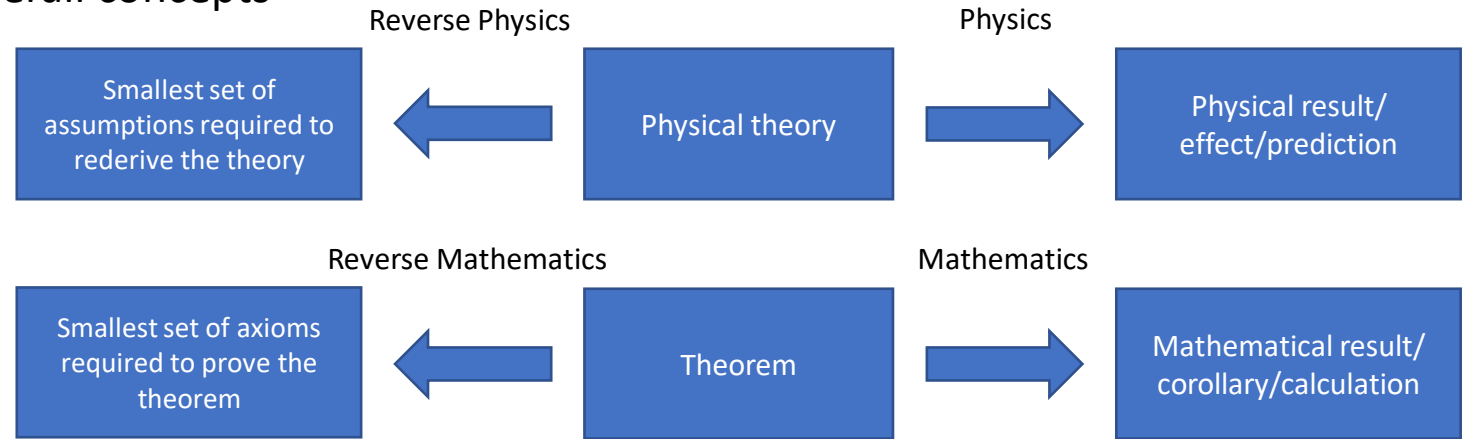
Our approach



Find the right overall concepts

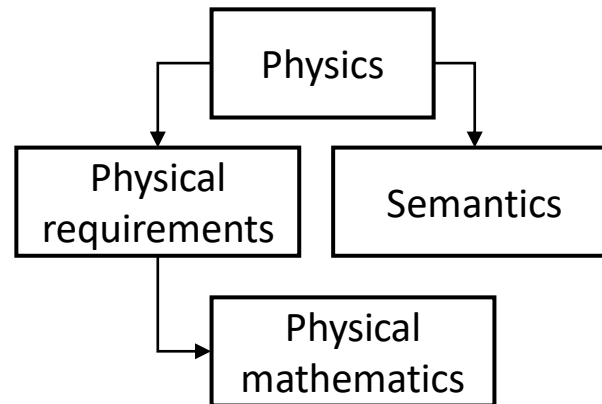
*Reverse physics:*  
Start with the equations,  
reverse engineer physical  
assumptions/principles

*Found Phys* **52**, 40 (2022)

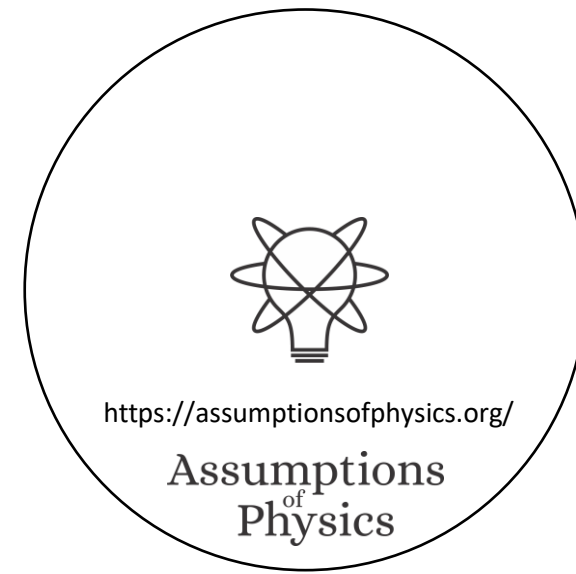


Goal: find the right overall physical concepts, “elevate” the discussion from mathematical constructs to physical principles

*Physical mathematics:*  
Start from scratch and rederive  
all mathematical structures from  
physical requirements



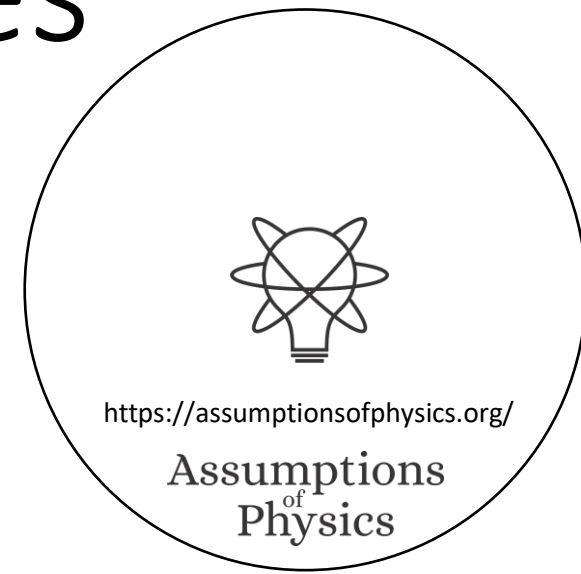
Goal: get the details right, perfect one-to-one map between mathematical and physical objects



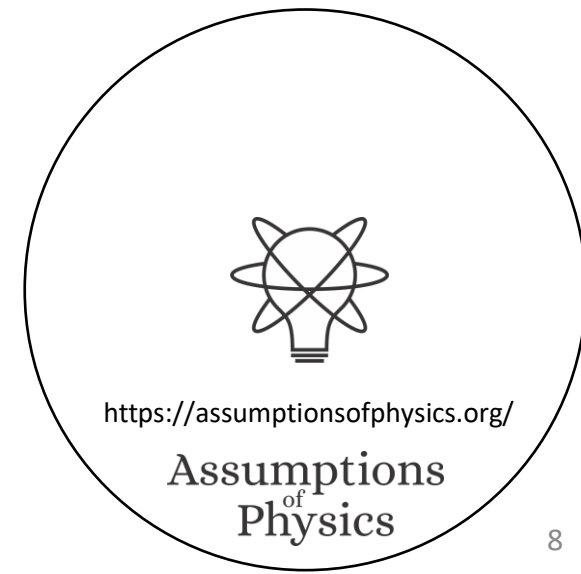
# This session

# Physical Mathematics: Foundational Structures

**Assumptions of Physics,**  
*Michigan Publishing (v2 2023)*



# Formal system for physics





# Formal system:

## primitive notions

Basic objects that are taken as-is,  
without definition in terms of other objects

## formal language

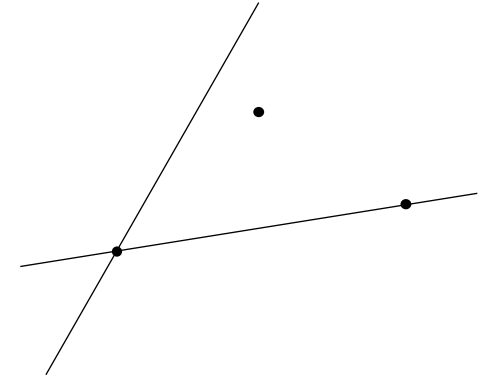
Symbols and rules to write sentences  
in the formal system

## axioms

Statements about primitive objects that  
are to be taken as true

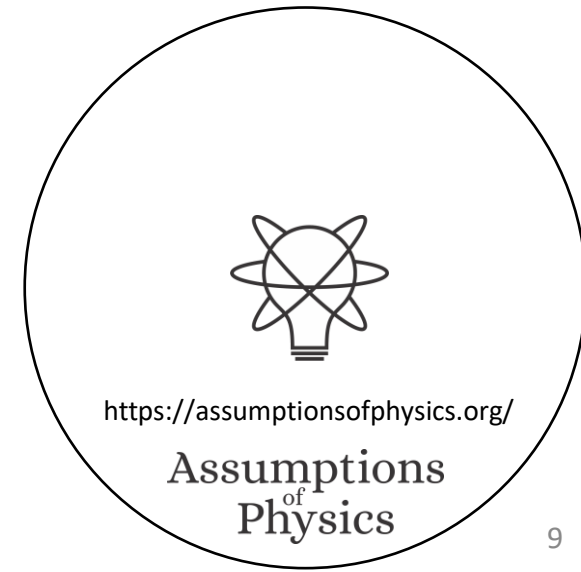
## e.g. Euclidean geometry

E.g. Points and lines



E.g.  $A, B, C$  for points  
 $\overline{AB}$  for segment

E.g. Given two points,  
there is a line that joins them



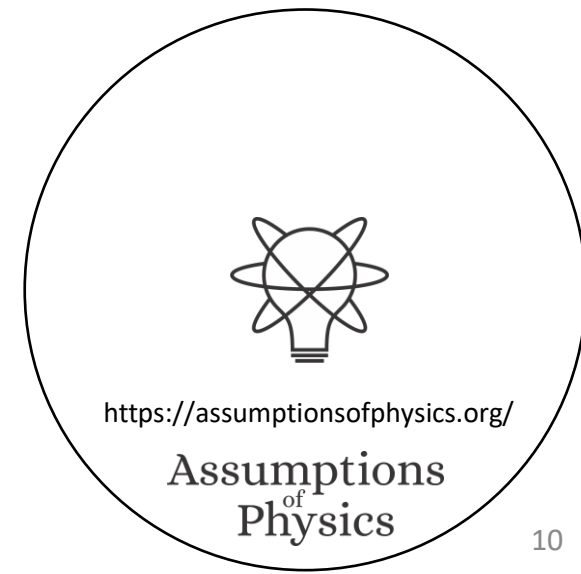
# Formal system for all of mathematics:

Sets + first-order logic

+ Zermelo–Fraenkel axioms (+ axiom of choice)

# Formal system for all of physics:

???



# Problems in formalizing physical concepts

Physical objects live in the physical (informal) world

(e.g. connection to experiment is outside of the formal system)

Informal physics

Formal math

Mathematical concepts are “crisper idealizations”

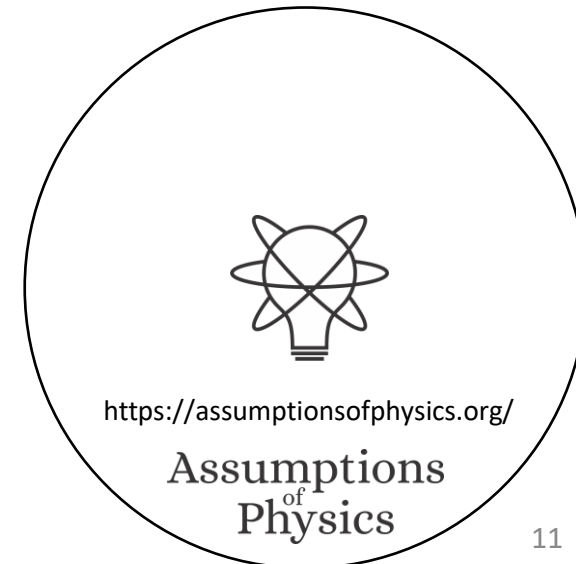
Physical concepts are “fuzzy”

Mathematical concepts cannot have circular definitions

Physical concepts may have circular definitions

Some concepts will have to remain informal

Choose axioms/primitive notions so that the justification is straightforward



Guiding principle

What should our primitive “informal” notion be?

**Principle of scientific objectivity:** science is universal, non-contradictory and evidence based.

Universal → same for everybody

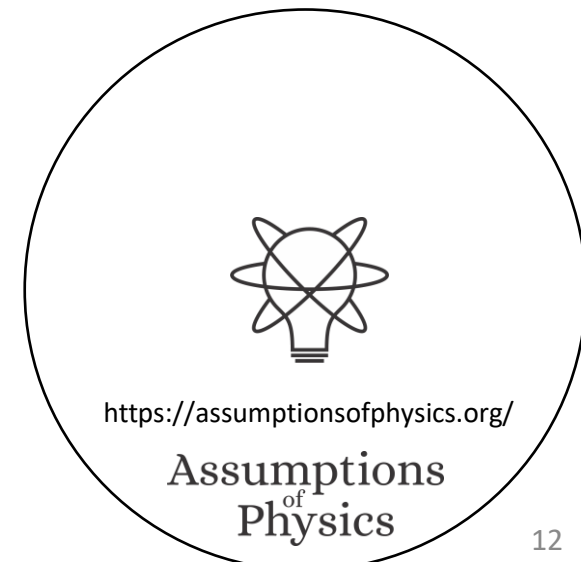
Non-contradictory → something is either true or false

Evidence based → truth is determined experimentally

Suggest logic as fundamental ...  
like mathematics!

... with some extensions

⇒ Logic of experimentally verifiable statements!



## Not “verifiable statements”

Chocolate tastes good (not universal)

It is immoral to kill one person to save ten (not universal and/or evidence based)

The number 4 is prime (not evidence based)

This statement is false (not non-contradictory)

The mass of the photon is exactly 0 eV (not verifiable due to infinite precision)

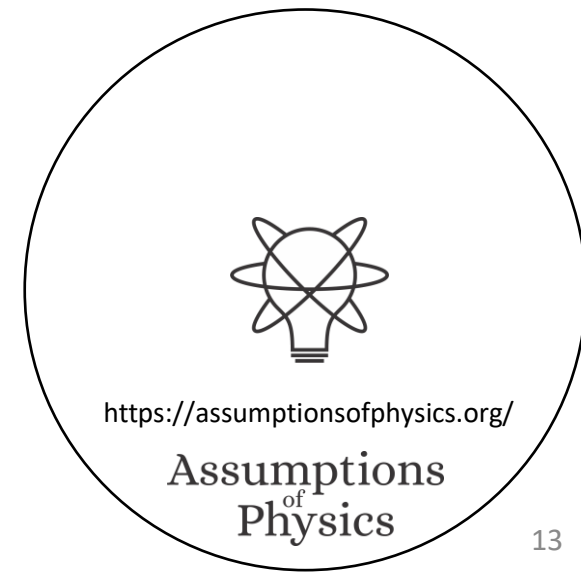
## “Verifiable statements”

The mass of the photon is less than  $10^{-13}$  eV

If the height of the mercury column is between 24 and 25 millimeters then its temperature is between 24 and 25 Celsius

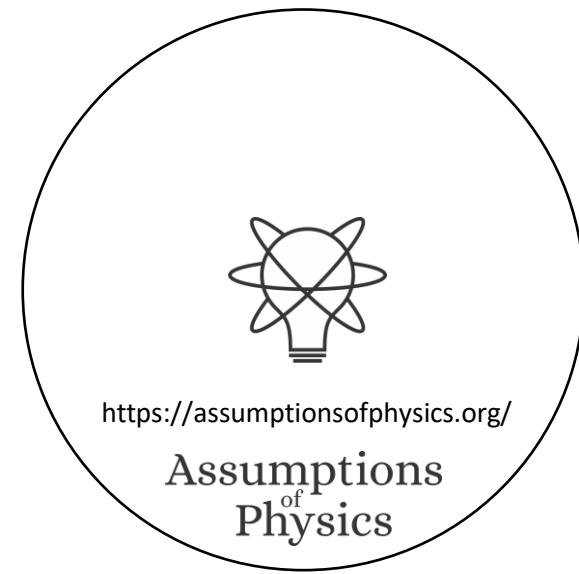
If I take  $2 \pm 0.01$  Kg of Sodium-24 and wait  $15 \pm 0.01$  hours there will be only  $1 \pm 0.01$  Kg left

A scientific theory needs “at least” the concept of a verifiable statement: good primitive notion

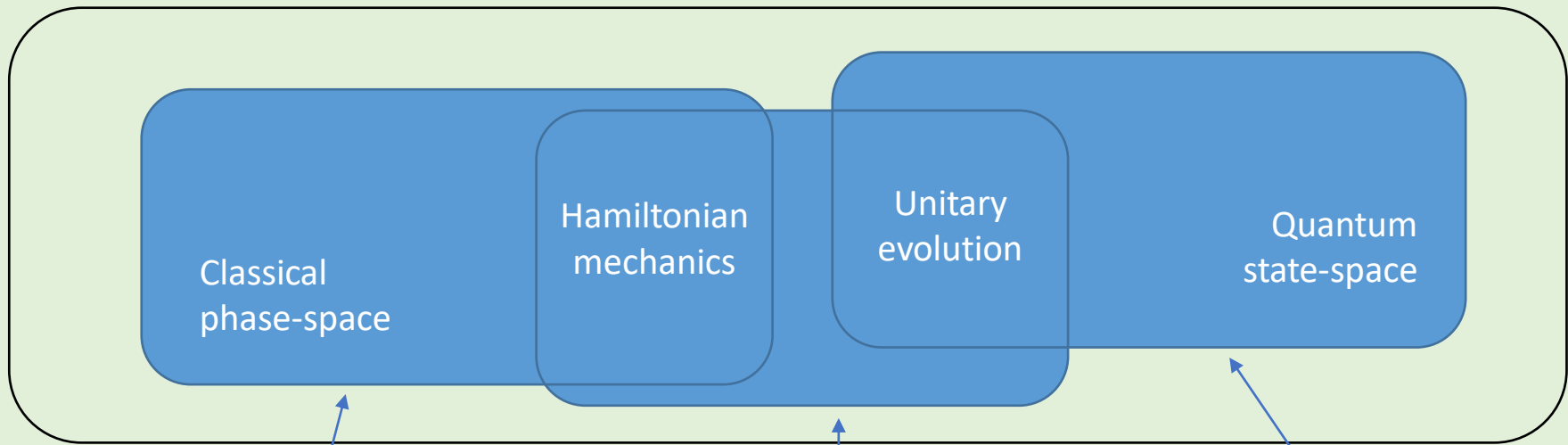


# Takeaways

- A good part of physics must remain informal
- Formal part is “precise” because it represents only an idealized part
- Pragmatic considerations as to what is formalized
- We take verifiable statements as the basic building blocks of our formal system



# Space of the well-posed scientific theories



## Physical theories

Specializations of the general theory under the different assumptions

## Assumptions

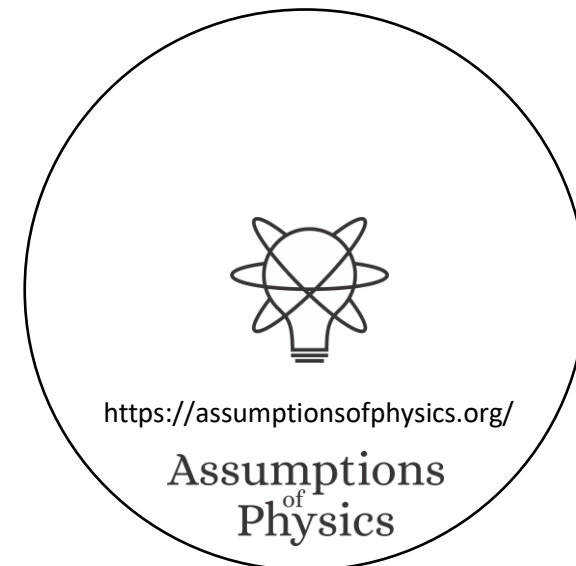
States and processes

Information granularity

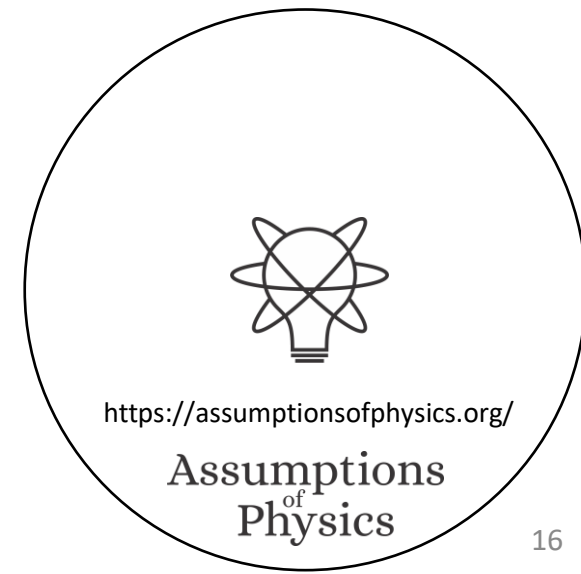
Experimental verifiability

## General theory

Basic requirements and definitions valid in all theories

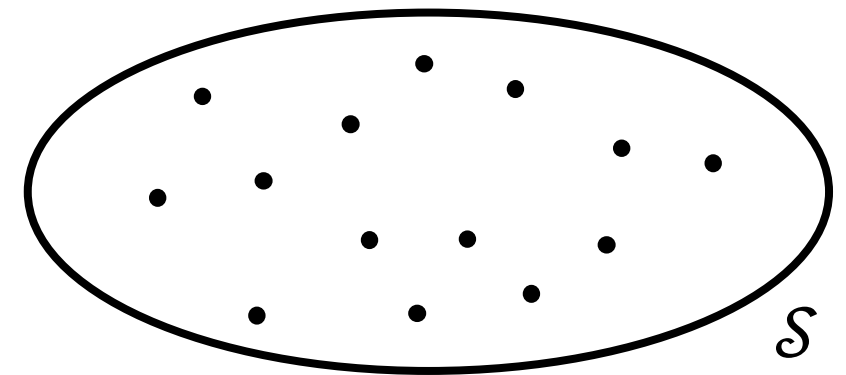


# Axioms of logic





**Axiom 1.2** (Axiom of context). A *statement*  $s$  is an assertion that is either true or false. A *logical context*  $\mathcal{S}$  is a collection of statements with well defined logical relationships. Formally, a logical context  $\mathcal{S}$  is a collection of elements called statements upon which is defined a function  $\text{truth} : \mathcal{S} \rightarrow \mathbb{B}$ .



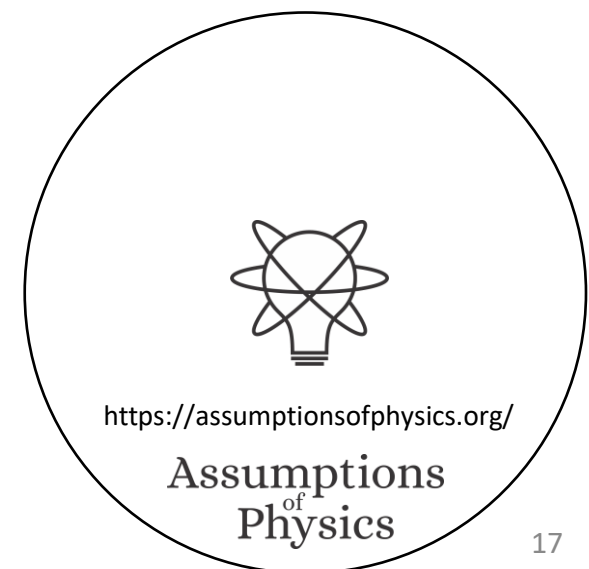
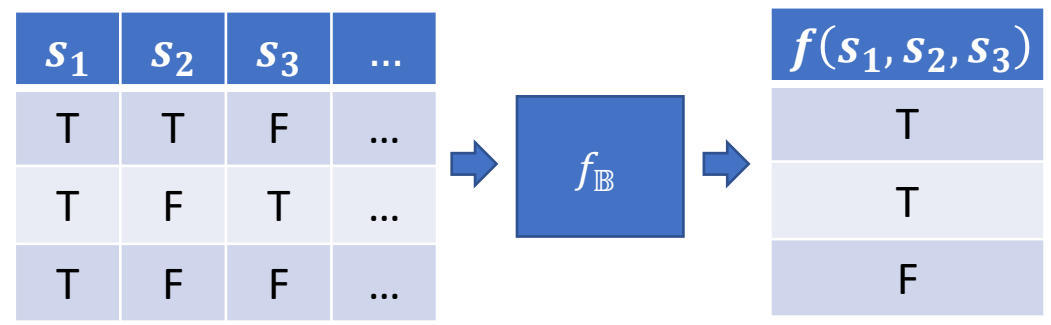
$a \rightarrow$

| $s_1$ | $s_2$ | $s_3$ | ... |
|-------|-------|-------|-----|
| T     | T     | F     | ... |
| T     | F     | T     | ... |
| T     | F     | F     | ... |

}  $\mathcal{A}_{\mathcal{S}}$

**Axiom 1.4** (Axiom of possibility). A *possible assignment* for a logical context  $\mathcal{S}$  is a map  $a : \mathcal{S} \rightarrow \mathbb{B}$  that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set  $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$  such that  $\text{truth} \in \mathcal{A}_{\mathcal{S}}$ . A map  $a : \mathcal{S} \rightarrow \mathbb{B}$  is a possible assignment for  $\mathcal{S}$  if  $a \in \mathcal{A}_{\mathcal{S}}$ .

**Axiom 1.9** (Axiom of closure). We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let  $S \subseteq \mathcal{S}$  be a set of statements and  $f_{\mathbb{B}} : \mathbb{B}^S \rightarrow \mathbb{B}$  an arbitrary function from an assignment of  $S$  to a truth value. Then we can always find a statement  $\bar{s} \in \mathcal{S}$  that depends on  $S$  through  $f_{\mathbb{B}}$ .



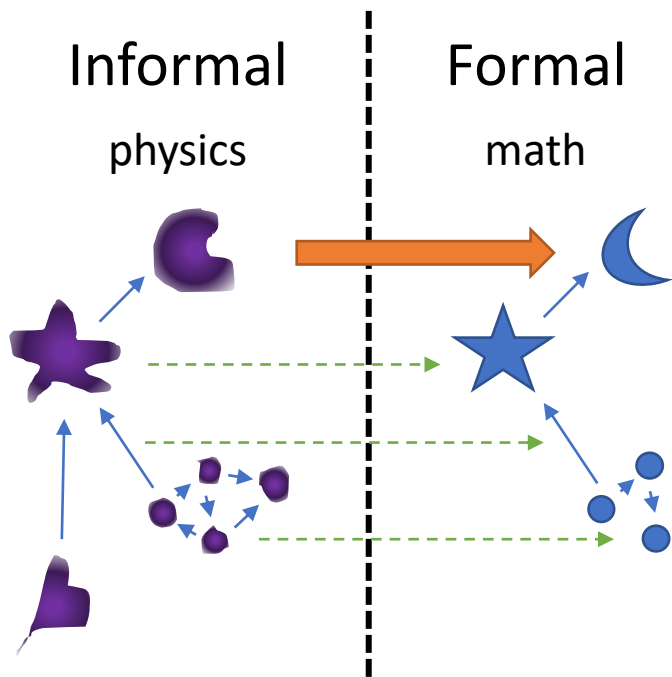
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Informal part

Formal part

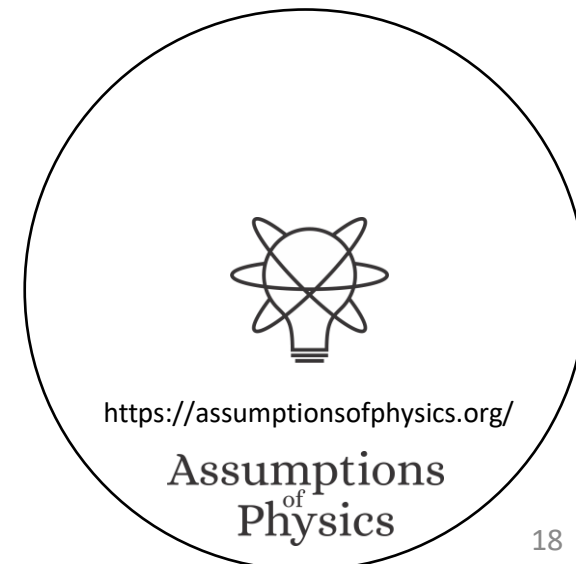
*Justification.* As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed



Each axiom/definition has two parts:

- Informal part: tells us what elements in the physical world we are characterizing
- Formal part: how the elements are characterized mathematically

Each axiom/definition has a justification: argues why the mathematical characterization follows from the physical one



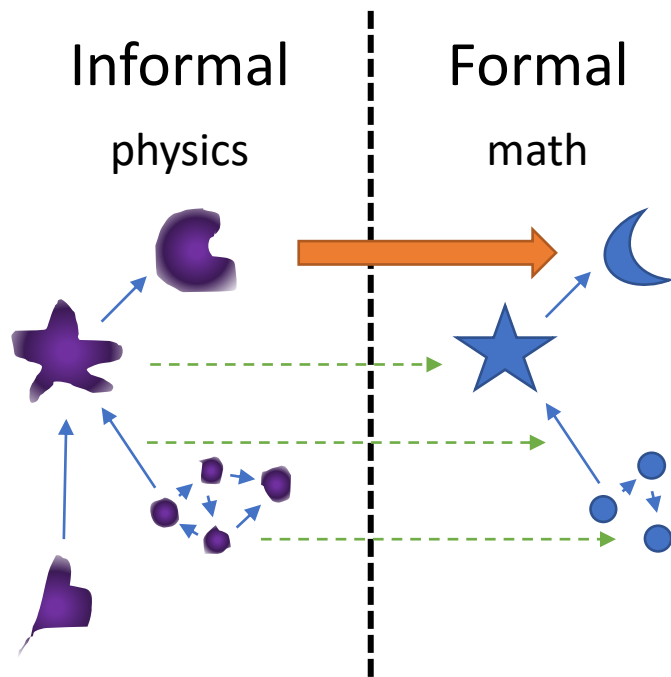
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Formal part

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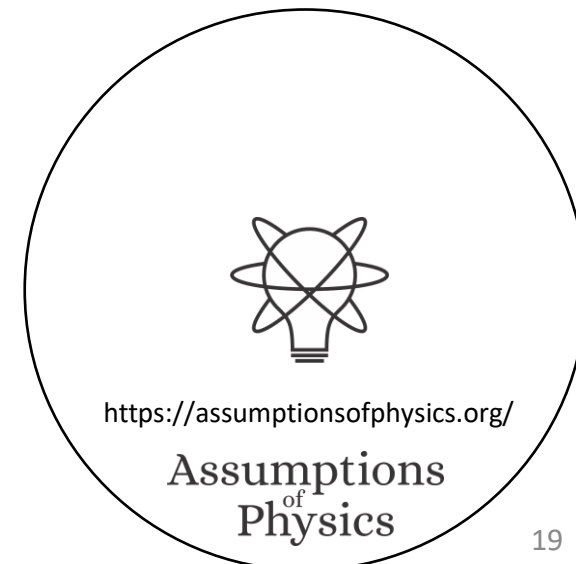


Axiom: brings objects from the informal to the formal

Definition: further specializes formal objects

Axioms/definitions should be formulated so that they are easy to justify...

... not so that they follow trends in mathematics



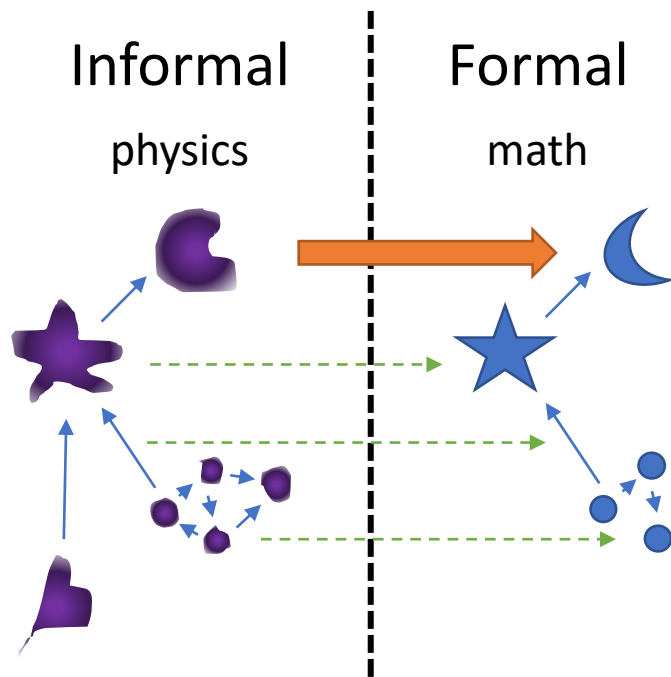
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Informal part

Formal part

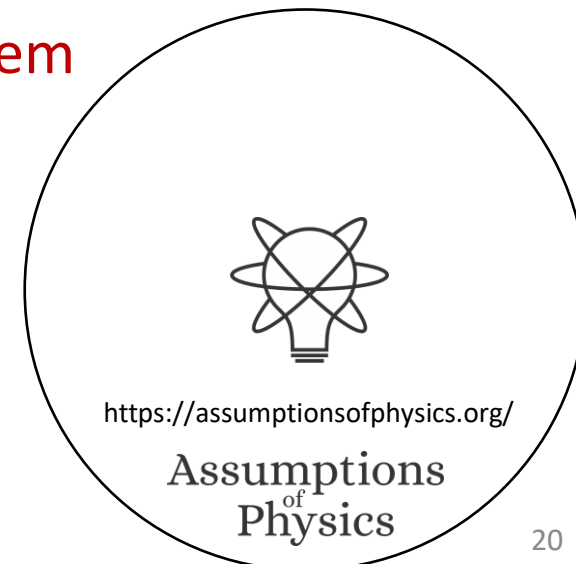
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Physical objects are made “mathematically precise” by throwing out everything that can’t be made precise

Syntax, grammar, meaning, ... can’t be made precise, so are not part of the formal system

⇒ Statements are primitive objects



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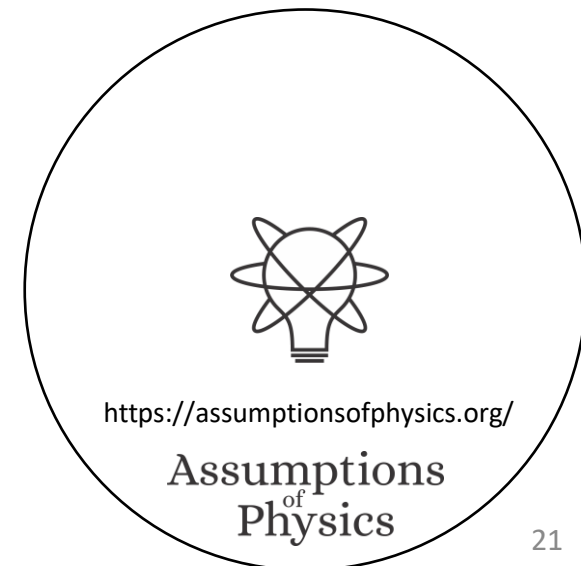
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Informal part

Formal part

In mathematics, primitive objects (i.e. those that are left unspecified) must be elements of a set. The logical context, then, has two functions:

- 1) in the formal system, it is the “container” for the primitive objects (i.e. the statements)
- 2) in the informal system, consistency/semantics/... are properties of groups of statements (i.e. of the context)



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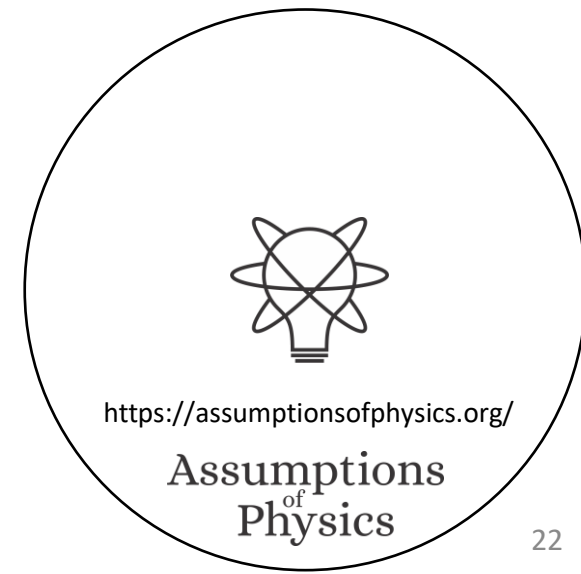
Informal part

Formal part

A statement here represents the assertion and not the sentence that declares the assertion. Therefore the translation of a sentence into another language represents the same statement.

Technically, we only assume the existence of valid statements for doing science. Therefore statements are also primitives in the informal system.

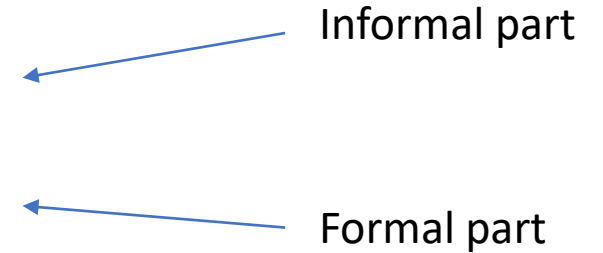
But if they exist, they must follow the axioms we are going to specify.



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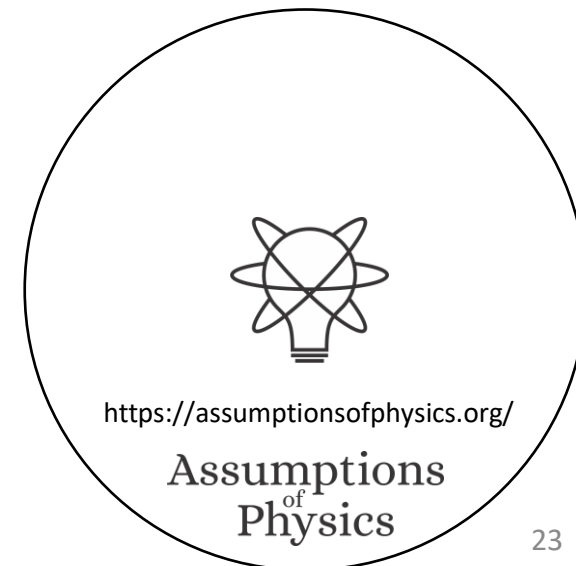
The existence of a truth function stems from the assumption of non-contradiction and universality.

Every statement must be either true or false for everybody.

$\mathcal{S}$

|       | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ | ... |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----|
| truth | T     | T     | F     | T     | T     | F     | T     | T     | F     | ... |

Context  $\Rightarrow$  big table where statements are columns

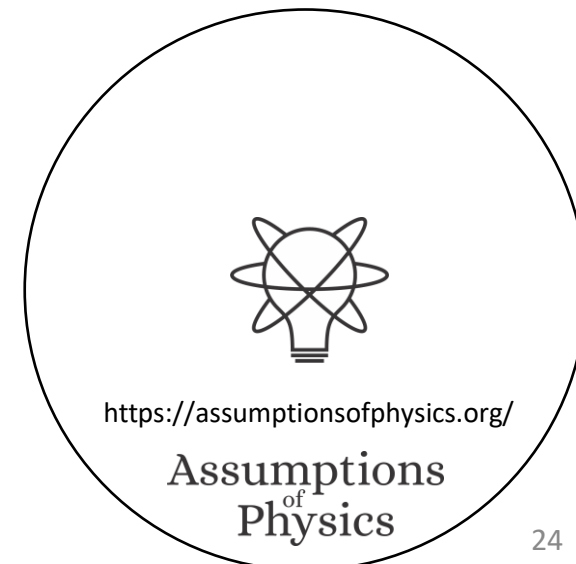


# Note: the semantic content constrains the possible combinations of truth values

| <i>"that animal is a cat"</i> | <i>"that animal is a mammal"</i> | <i>"that animal is a bird"</i> | ... |
|-------------------------------|----------------------------------|--------------------------------|-----|
| <del>T</del>                  | <del>T</del>                     | <del>T</del>                   | ... |
| T                             | T                                | F                              | ... |
| <del>T</del>                  | <del>F</del>                     | <del>T</del>                   | ... |
| <del>T</del>                  | <del>F</del>                     | <del>F</del>                   | ... |
| <del>F</del>                  | <del>T</del>                     | <del>T</del>                   | ... |
| F                             | T                                | F                              | ... |
| F                             | F                                | T                              | ... |
| F                             | F                                | F                              | ... |

impossible

The only semantics captured by the formal system is the set of possible combinations of truth values



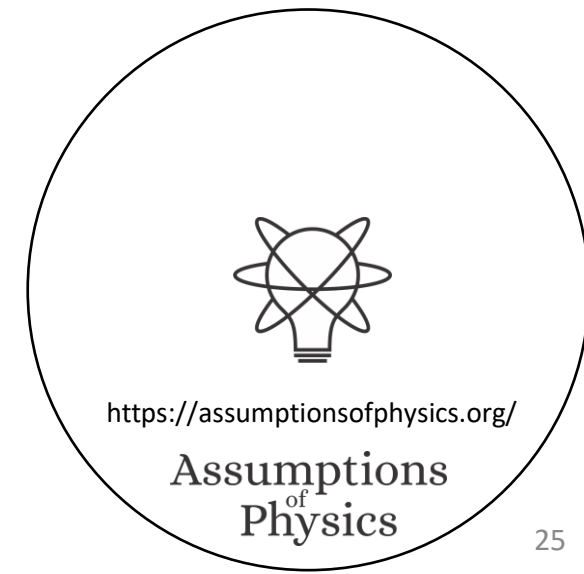


**Axiom 1.4** (Axiom of possibility). A *possible assignment* for a logical context  $\mathcal{S}$  is a map  $a : \mathcal{S} \rightarrow \mathbb{B}$  that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set  $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$  such that  $\text{truth} \in \mathcal{A}_{\mathcal{S}}$ . A map  $a : \mathcal{S} \rightarrow \mathbb{B}$  is a possible assignment for  $\mathcal{S}$  if  $a \in \mathcal{A}_{\mathcal{S}}$ .

|       | $s_1$ | $s_2$ | $s_3$ | ... |       |
|-------|-------|-------|-------|-----|-------|
| $a_1$ | T     | T     | F     | ... | truth |
| $a_2$ | F     | F     | T     | ... |       |
| $a_3$ | F     | T     | F     | ... |       |
| ...   | ...   | ...   | ...   | ... |       |

Possible assignments are those assignments consistent with the meaning (semantics) of the statements

Context  $\Rightarrow$  big table where statements are columns and possible assignments are rows



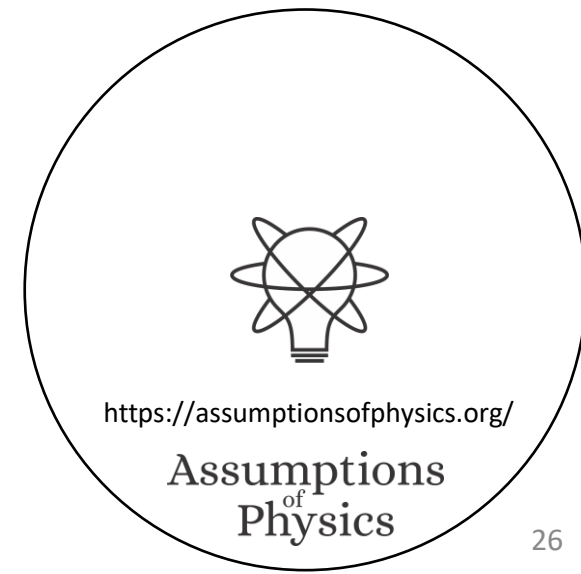
**Definition 1.6.** *Statements are categorized based on their possible assignments.*

- A certain statement, or **certainty**, is a statement  $\top$  that must be true simply because of its content. Formally,  $a(\top) = \text{TRUE}$  for all possible assignments  $a \in \mathcal{A}_S$ .
- An impossible statement, or **impossibility**, is a statement  $\perp$  that must be false simply because of its content. Formally,  $a(\perp) = \text{FALSE}$  for all possible assignments  $a \in \mathcal{A}_S$ .
- A statement is **contingent** if it is neither certain nor impossible.

**Corollary 1.7.** *A statement  $s \in \mathcal{S}$  can only be exactly one of the following: impossible, contingent, certain.*

| "that cat is a mammal" | "that mammal is a cat" | "that mammal is a bird" |
|------------------------|------------------------|-------------------------|
| T                      | T                      | F                       |
| T                      | F                      | F                       |
| certain                | contingent             | impossible              |

Certainties and impossibilities have the same truth value in all rows



**Definition 1.6.** *Statements are categorized based on their possible assignments.*

- A certain statement, or **certainty**, is a statement  $\top$  that must be true simply because of its content. Formally,  $a(\top) = \text{TRUE}$  for all possible assignments  $a \in \mathcal{A}_S$ .
- An impossible statement, or **impossibility**, is a statement  $\perp$  that must be false simply because of its content. Formally,  $a(\perp) = \text{FALSE}$  for all possible assignments  $a \in \mathcal{A}_S$ .
- A statement is **contingent** if it is neither certain nor impossible.

**Corollary 1.7.** *A statement  $s \in \mathcal{S}$  can only be exactly one of the following: impossible, contingent, certain.*

Whether a statement is certain or contingent depends on context!

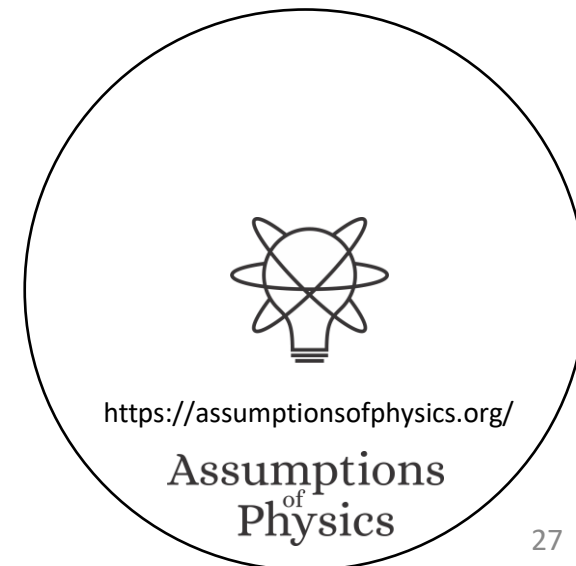
*the mass of the electron is  $510 \pm 0.5$  KeV*



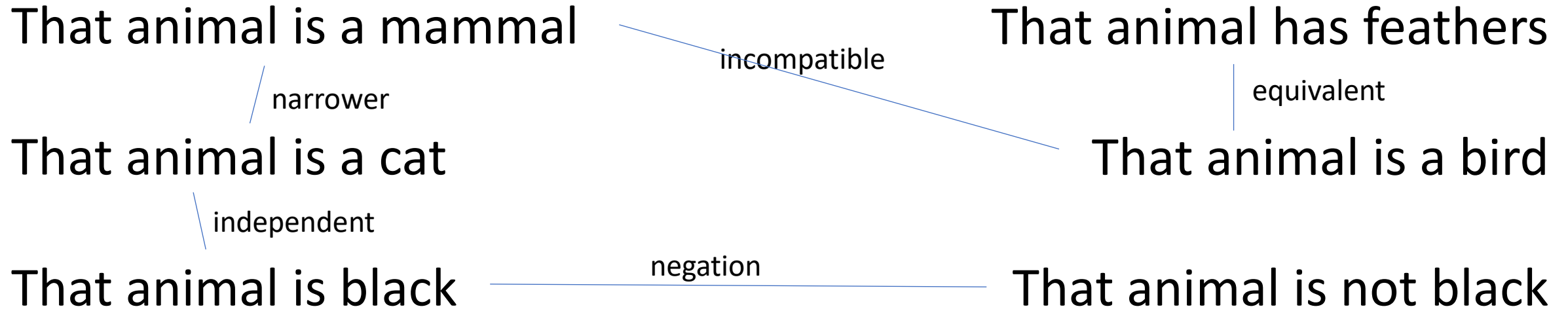
Contingent when measuring  
the mass of the electron



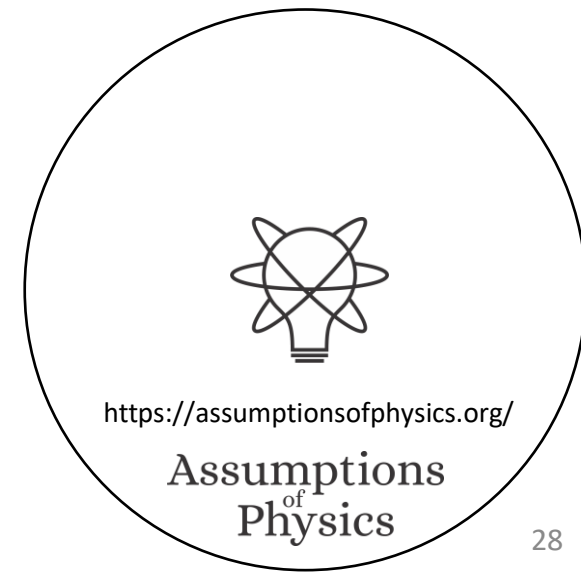
Certain when performing  
particle identification



# Some statements depend on other statements



⇒ possible assignments determine the logical relationship



# Equivalent

| <i>"that animal has feathers"</i> | <i>"that animal is a bird"</i> |
|-----------------------------------|--------------------------------|
| T                                 | T                              |
| <del>T</del>                      | <del>F</del>                   |
| <del>F</del>                      | <del>T</del>                   |
| F                                 | F                              |

|   | T | F |
|---|---|---|
| T | ✓ | ✗ |
| F | ✗ | ✓ |

# Independent

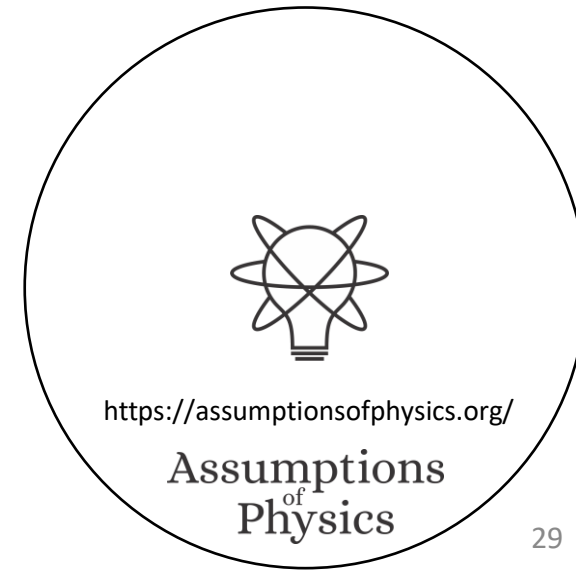
| <i>"that animal is a cat"</i> | <i>"that animal is black"</i> |
|-------------------------------|-------------------------------|
| T                             | T                             |
| T                             | F                             |
| F                             | T                             |
| F                             | F                             |

|   | T | F |
|---|---|---|
| T | ✓ | ✓ |
| F | ✓ | ✓ |

# Incompatible

| <i>"that animal is a mammal"</i> | <i>"that animal is a bird"</i> |
|----------------------------------|--------------------------------|
| <del>T</del>                     | <del>T</del>                   |
| T                                | F                              |
| F                                | T                              |
| F                                | F                              |

|   | T | F |
|---|---|---|
| T | ✗ | ✓ |
| F | ✓ | ✓ |



**Definition 1.15.** Two statements  $s_1$  and  $s_2$  are **equivalent**  $s_1 \equiv s_2$  if they must be equally true or false simply because of their content. Formally,  $s_1 \equiv s_2$  if and only if  $a(s_1) = a(s_2)$  for all possible assignments  $a \in \mathcal{A}_S$ .

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| T     | T     | T     | T     | F     | T     | T     | ... |
| F     | F     | F     | T     | T     | T     | T     | ... |
| F     | F     | F     | F     | T     | T     | T     | ... |
| T     | F     | T     | T     | F     | T     | T     | ... |
| T     | F     | T     | F     | F     | T     | T     | ... |
| ...   | ...   | ...   | ...   | ...   | ...   | ...   | ... |



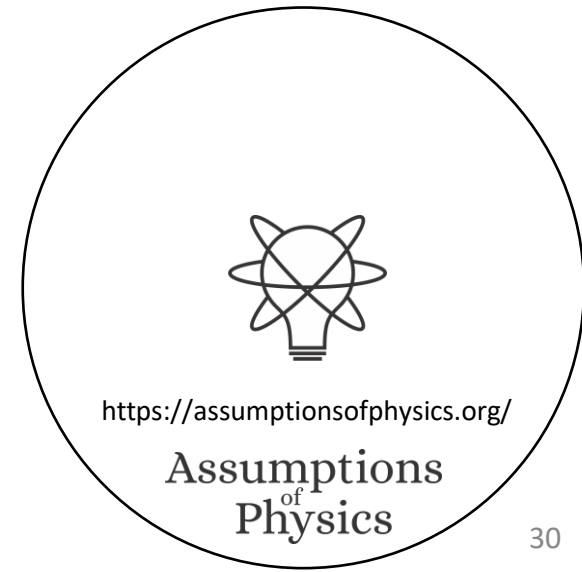
From now on, unless otherwise stated, by statement we mean an equivalence class of statements

**Corollary 1.16.** All certainties are equivalent. All impossibilities are equivalent.

**Corollary 1.18.** Statement equivalence satisfies the following properties:

- reflexivity:  $s \equiv s$
- symmetry: if  $s_1 \equiv s_2$  then  $s_2 \equiv s_1$
- transitivity: if  $s_1 \equiv s_2$  and  $s_2 \equiv s_3$  then  $s_1 \equiv s_3$

and is therefore an **equivalence relationship**.



**Definition 1.20.** Given two statements  $s_1$  and  $s_2$ , we say that:

- $s_1$  is narrower than  $s_2$  (noted  $s_1 \preceq s_2$ ) if  $s_2$  is true whenever  $s_1$  is true simply because of their content. That is, for all  $a \in \mathcal{A}_S$  if  $a(s_1) = \text{TRUE}$  then  $a(s_2) = \text{TRUE}$ .
- $s_1$  is broader than  $s_2$  (noted  $s_1 \succeq s_2$ ) if  $s_2 \preceq s_1$ .
- $s_1$  is compatible to  $s_2$  (noted  $s_1 \approx s_2$ ) if their content allows them to be true at the same time. That is, there exists  $a \in \mathcal{A}_S$  such that  $a(s_1) = a(s_2) = \text{TRUE}$ .

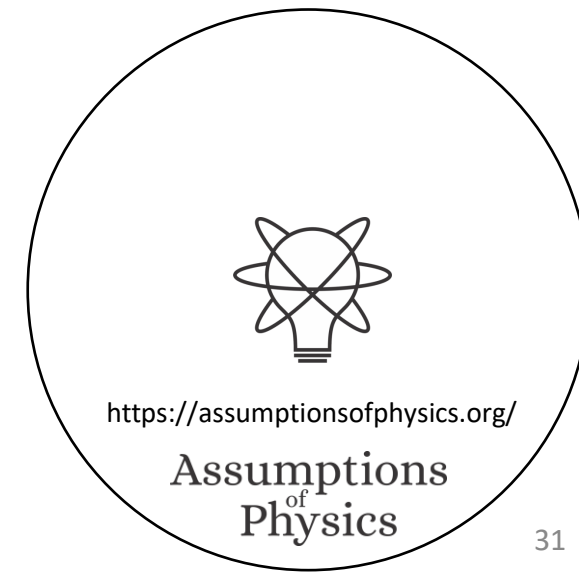
The negation of these properties will be noted by  $\not\preceq$ ,  $\not\succeq$ ,  $\not\approx$  respectively.

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| T     | T     | T     | T     | F     | T     | T     | ... |
| F     | F     | F     | T     | T     | T     | T     | ... |
| F     | F     | F     | F     | T     | T     | T     | ... |
| T     | F     | F     | T     | F     | T     | F     | ... |
| T     | F     | T     | F     | F     | T     | T     | ... |
| ...   | ...   | ...   | ...   | ...   | ...   | ...   | ... |



That animal is a mammal  $\approx$  That animal lays eggs

That animal is a cat  $\preceq$  That animal is a mammal



**Proposition 1.23.** *Statement narrowness satisfies the following properties:*

- *reflexivity:  $s \preceq s$*
- *antisymmetry: if  $s_1 \preceq s_2$  and  $s_2 \preceq s_1$  then  $s_1 \equiv s_2$*
- *transitivity: if  $s_1 \preceq s_2$  and  $s_2 \preceq s_3$  then  $s_1 \preceq s_3$*

*and is therefore a **partial order**.*

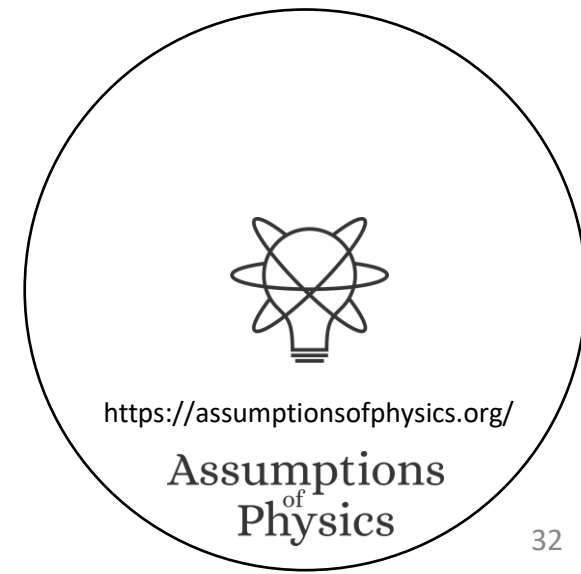
Narrowness  $\preceq$  is related to material implication  $\rightarrow$  but:

Material implication is a logical operation that returns a new statement:

$$a \rightarrow b = \neg a \vee b \text{ (i.e. NOT(a) OR b )}$$

Narrowness  $\preceq$  is a binary relationship between statements

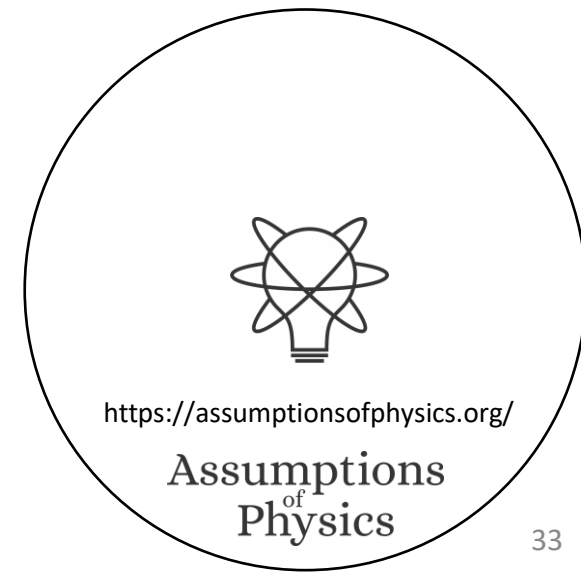
The order imposed by narrowness allows us to understand the context as an order theoretic structure





NOTE: AND, OR and NOT ( $\wedge, \vee, \neg$ )  
are operations within the context

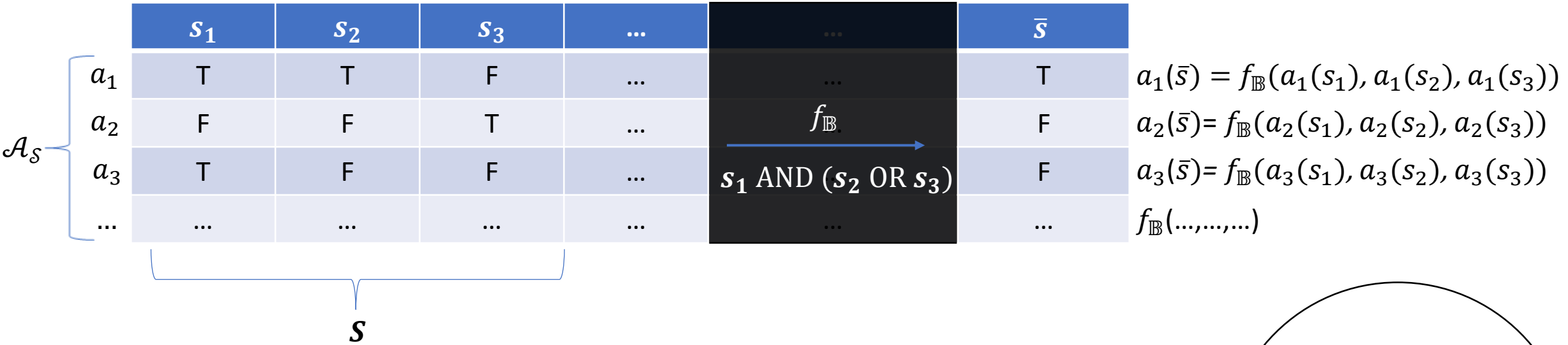
Equivalence, narrowness, compatibility, ... ( $\equiv, \preceq, \hat{\sim}, \dots$ )  
are not: they describe the context (i.e. metalanguage)



**Definition 1.8.** Let  $\bar{s} \in \mathcal{S}$  be a statement and  $S \subseteq \mathcal{S}$  be a set of statements. Then  $\bar{s}$  **depends on**  $S$  (or it is a function of  $S$ ) if we can find an  $f_{\mathbb{B}} : \mathbb{B}^S \rightarrow \mathbb{B}$  such that

$$a(\bar{s}) = f_{\mathbb{B}}(\{a(s)\}_{s \in S})$$

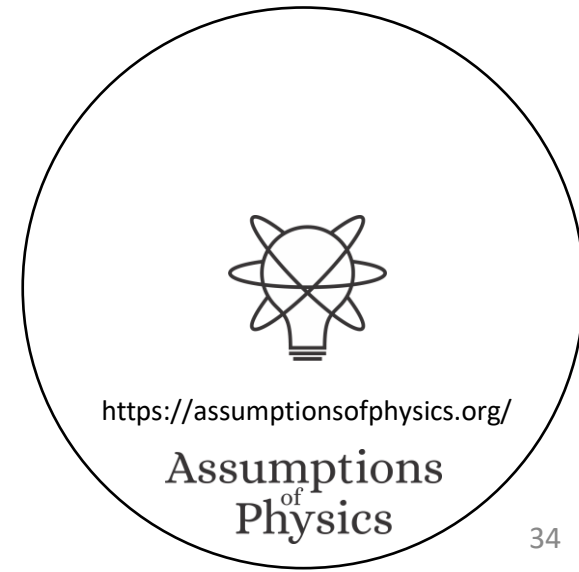
for every possible assignment  $a \in \mathcal{A}_S$ . We say  $\bar{s}$  depends on  $S$  **through**  $f_{\mathbb{B}}$ . The relationship is illustrated by the following diagram:



$s_1$  = "that animal is a cat"

$s_2$  = "that animal is a mammal"

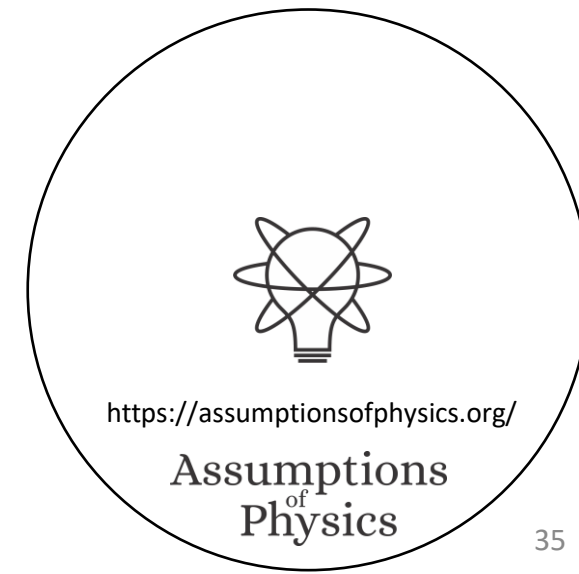
$s_3$  = "that animal is a bird"



**Axiom 1.9** (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let  $S \subseteq \mathcal{S}$  be a set of statements and  $f_{\mathbb{B}} : \mathbb{B}^S \rightarrow \mathbb{B}$  an arbitrary function from an assignment of  $S$  to a truth value. Then we can always find a statement  $\bar{s} \in \mathcal{S}$  that depends on  $S$  through  $f_{\mathbb{B}}$ .*

| $s_1$ | $s_2$ | $s_3$ |   | $\bar{s}$ | ... | $s_{\dots}$ | ... |
|-------|-------|-------|---|-----------|-----|-------------|-----|
| T     | T     | F     | $f_{\mathbb{B}}$<br>$\longrightarrow$<br>$s_1 \text{ AND } (s_2 \text{ OR } s_3)$ | T         | ... | T           | ... |
| F     | F     | T     |   | F         | ... | T           | ... |
| F     | F     | T     |   | F         | ... | F           | ... |
| T     | F     | F     |   | F         | ... | T           | ... |
| T     | F     | T     |   | F         | ... | F           | ... |
| ...   | ...   | ...   |   | ...       | ... | ...         | ... |

Not sure whether it is needed as an axiom: the closure may be proven to exist and be unique.

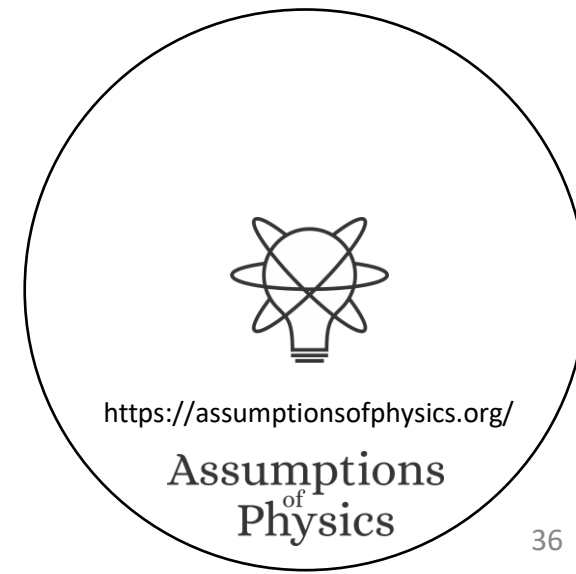


**Corollary 1.10.** *Functions on truth values induce functions on statements. Formally, let  $I$  be an index set and  $f_{\mathbb{B}} : \mathbb{B}^I \rightarrow \mathbb{B}$  be a function. There exists a function  $f : \mathcal{S}^I \rightarrow \mathcal{S}$  such that*

$$a(f(\{s_i\}_{i \in I})) = f_{\mathbb{B}}(\{a(s_i)\}_{i \in I})$$

for every indexed set  $\{s_i\}_{i \in I} \subseteq \mathcal{S}$  and possible assignment  $a \in \mathcal{A}_{\mathcal{S}}$ .

| $s_1$ | $s_2$ | $s_3$ | $f(s_1, s_2, s_3)$   | $\bar{s}$ | ... | $s_{\dots}$ | ... |
|-------|-------|-------|--|-----------|-----|-------------|-----|
| T     | T     | F     | $f_{\mathbb{B}}$<br>$s_1 \text{ AND } (s_2 \text{ OR } s_3)$ | T         | ... | T           | ... |
| F     | F     | T     |  | F         | ... | T           | ... |
| F     | F     | T     |  | F         | ... | F           | ... |
| T     | F     | F     |  | F         | ... | T           | ... |
| T     | F     | T     |  | F         | ... | F           | ... |
| ...   | ...   | ...   |  | ...       | ... | ...         | ... |



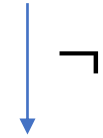
**Definition 1.11.** *The negation or logical NOT is the function  $\neg : \mathbb{B} \rightarrow \mathbb{B}$  that takes a truth value and returns its opposite. That is:  $\neg \text{TRUE} = \text{FALSE}$  and  $\neg \text{FALSE} = \text{TRUE}$ . We also call negation  $\neg : \mathcal{S} \rightarrow \mathcal{S}$  the related function on statements.*

| $t$ | $\neg t$ |
|-----|----------|
| T   | F        |
| F   | T        |

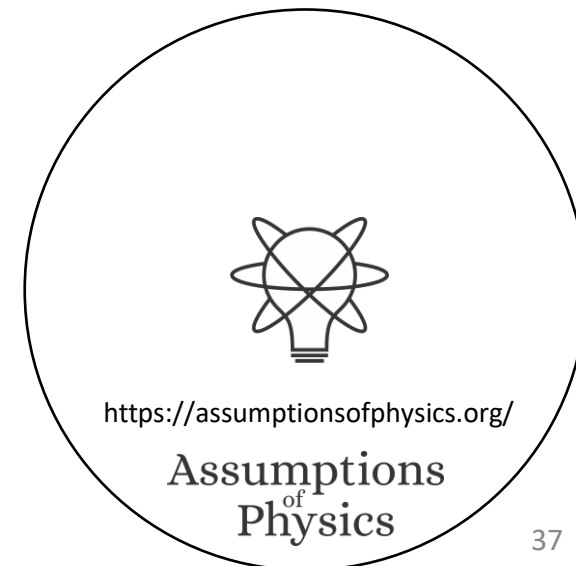
| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| T     | T     | F     | T     | F     | T     | T     | ... |
| F     | F     | T     | T     | T     | F     | F     | ... |
| F     | F     | T     | F     | T     | F     | F     | ... |
| T     | F     | F     | T     | F     | F     | T     | ... |
| T     | F     | T     | F     | F     | F     | T     | ... |
| ...   | ...   | ...   | ...   | ...   | ...   | ...   | ... |



That animal is a cat



That animal is not a cat



**Definition 1.12.** The *conjunction or logical AND* is the function  $\wedge : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$  that returns TRUE only if all the arguments are TRUE. That is:  $\text{TRUE} \wedge \text{TRUE} = \text{TRUE}$  and  $\text{TRUE} \wedge \text{FALSE} = \text{FALSE} \wedge \text{TRUE} = \text{FALSE} \wedge \text{FALSE} = \text{FALSE}$ . We also call conjunction  $\wedge : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  the related function on statements.

| $t_1$ | $t_2$ | $t_1 \wedge t_2$ |
|-------|-------|------------------|
| T     | T     | T                |
| T     | F     | F                |
| F     | T     | F                |
| F     | F     | F                |

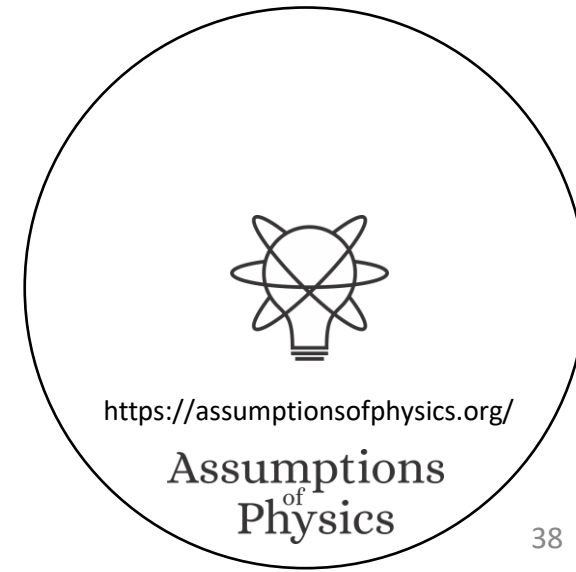
| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| T     | T     | F     | T     | F     | T     | T     | ... |
| F     | F     | T     | T     | T     | F     | F     | ... |
| F     | F     | T     | F     | T     | F     | F     | ... |
| T     | F     | F     | T     | F     | F     | T     | ... |
| T     | F     | T     | F     | F     | F     | T     | ... |
| ...   | ...   | ...   | ...   | ...   | ...   | ...   | ... |



That animal is a cat      That animal is black

$\wedge$

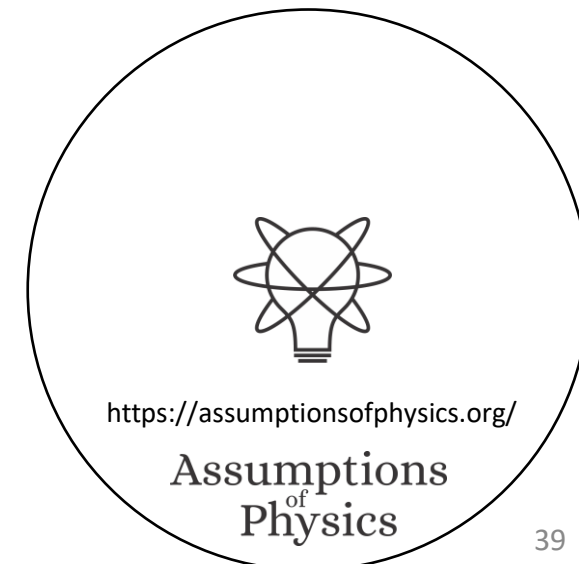
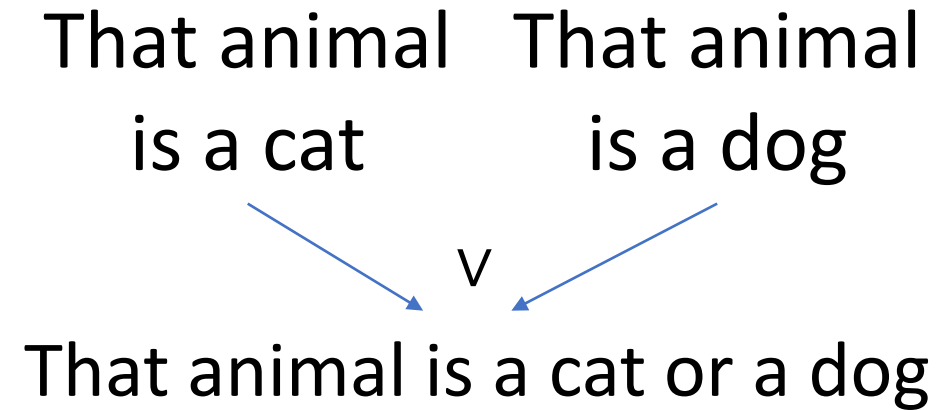
That animal is a black cat



**Definition 1.13.** The *disjunction or logical OR* is the function  $\vee : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$  that returns FALSE only if all the arguments are FALSE. That is:  $\text{FALSE} \vee \text{FALSE} = \text{FALSE}$  and  $\text{TRUE} \vee \text{FALSE} = \text{FALSE} \vee \text{TRUE} = \text{TRUE} \vee \text{TRUE} = \text{TRUE}$ . We also call disjunction  $\vee : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  the related function on statements.

| $t_1$ | $t_2$ | $t_1 \vee t_2$ |
|-------|-------|----------------|
| T     | T     | T              |
| T     | F     | T              |
| F     | T     | T              |
| F     | F     | F              |

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| T     | T     | F     | T     | F     | T     | T     | ... |
| F     | F     | T     | T     | T     | F     | F     | ... |
| F     | F     | T     | F     | T     | F     | F     | ... |
| T     | F     | F     | T     | F     | F     | T     | ... |
| T     | F     | T     | F     | F     | F     | T     | ... |
| ...   | ...   | ...   | ...   | ...   | ...   | ...   | ... |



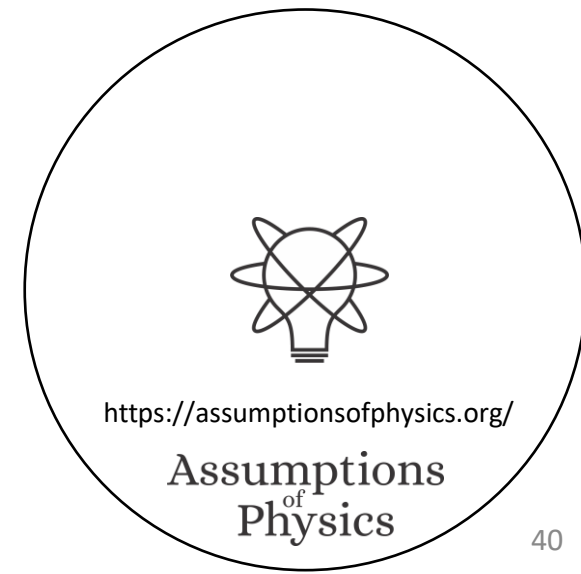
**Corollary 1.19.** *A logical context  $\mathcal{S}$  satisfies the following properties:*

- *associativity:*  $a \vee (b \vee c) \equiv (a \vee b) \vee c$ ,  $a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$
- *commutativity:*  $a \vee b \equiv b \vee a$ ,  $a \wedge b \equiv b \wedge a$
- *absorption:*  $a \vee (a \wedge b) \equiv a$ ,  $a \wedge (a \vee b) \equiv a$
- *identity:*  $a \vee \perp \equiv a$ ,  $a \wedge \top \equiv a$
- *distributivity:*  $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$ ,  $a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$
- *complements:*  $a \vee \neg a \equiv \top$ ,  $a \wedge \neg a \equiv \perp$
- *De Morgan:*  $\neg a \vee \neg b \equiv \neg(a \wedge b)$ ,  $\neg a \wedge \neg b \equiv \neg(a \vee b)$

*Therefore  $\mathcal{S}$  is a **Boolean algebra** by definition.*

## Recovers the standard structure for classical logic

Note how many properties are part of the definition of a Boolean algebra: if that had been our starting point, we would have had to justify every single one, which is cumbersome and not particularly enlightening





# Functions in a Boolean algebra have a standard representation important for us

| $s_1$ | $s_2$ | $s_3$ | $\bar{s}$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|-------|-------|-------|-----------|-------|-------|-------|-------|
| T     | T     | F     | T         | T     | F     | F     | F     |
| F     | F     | T     | T         | F     | T     | F     | F     |
| F     | T     | T     | F         | F     | F     | T     | F     |
| T     | F     | F     | T         | F     | F     | F     | T     |

$m_1, m_2, m_3, m_4$  each picks a line of the table, and can be expressed as the conjunction that takes  $s_1, s_2, s_3$  once, negated or not

$$m_1 = s_1 \wedge s_2 \wedge \neg s_3$$

$$m_2 = \neg s_1 \wedge \neg s_2 \wedge s_3$$

$$m_3 = \neg s_1 \wedge s_2 \wedge s_3$$

$$m_4 = s_1 \wedge \neg s_2 \wedge \neg s_3$$

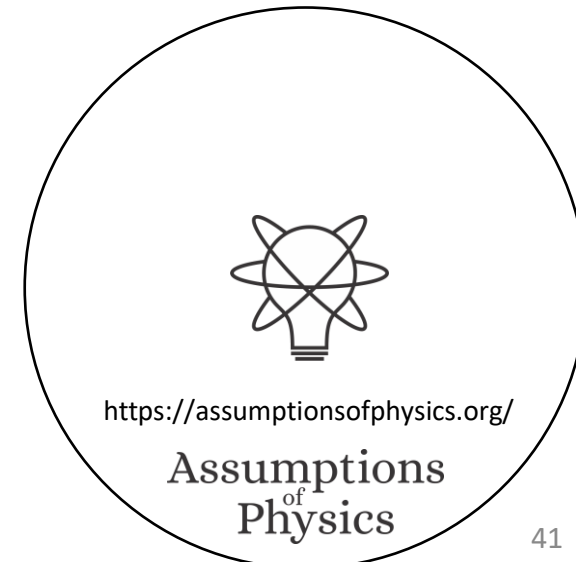
**minterms**

$\bar{s}$  is a function of  $s_1, s_2, s_3$

$\bar{s}$  is the disjunction of  $m_1, m_2, m_4$

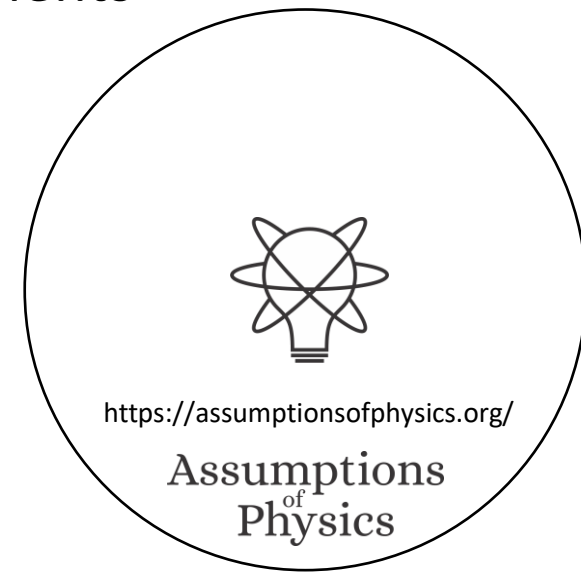
$$\bar{s} = (s_1 \wedge s_2 \wedge \neg s_3) \vee (\neg s_1 \wedge \neg s_2 \wedge s_3) \vee (s_1 \wedge \neg s_2 \wedge \neg s_3)$$

**disjunctive normal form**

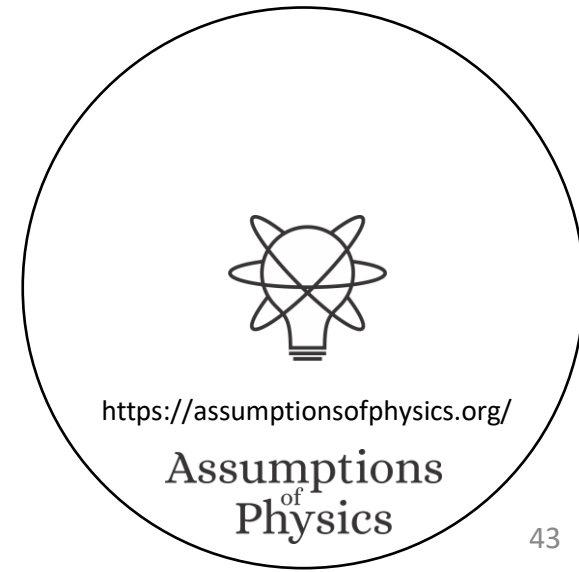


# Takeaways

- Semantics define which assignments are possible on a given context
- The possible assignments define the logical relationships and operations
- The possible assignments describe “what could happen”, which is inherently tied to the model
  - Certainty, equivalence, narrowness, etc... are all metaconcepts about the theory
- TODOs
  - Statement equivalence should be defined before functions on statements (technically, they should be operations on equivalence classes)



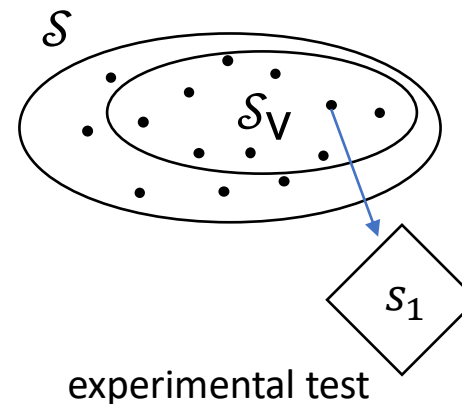
# Axioms of verifiability



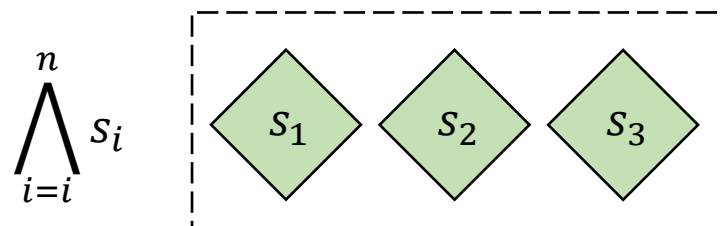
**Axiom 1.27** (Axiom of verifiability). A *verifiable statement* is a statement that, if true, can be shown to be so experimentally. Formally, each logical context  $\mathcal{S}$  contains a set of statements  $\mathcal{S}_V \subseteq \mathcal{S}$  whose elements are said to be verifiable. Moreover, we have the following properties:

- every certainty  $\top \in \mathcal{S}$  is verifiable
- every impossibility  $\perp \in \mathcal{S}$  is verifiable
- a statement equivalent to a verifiable statement is verifiable

*Remark.* The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



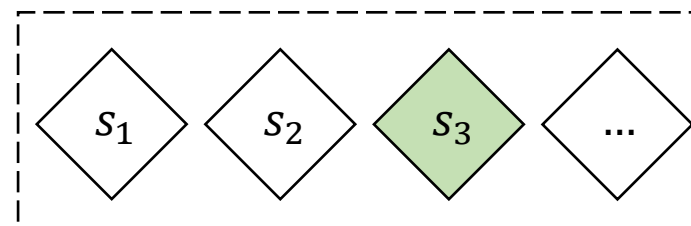
| $S_1$ | Test Result              |
|-------|--------------------------|
| T     | SUCCESS (in finite time) |
| F     | FAILURE (in finite time) |
|       | UNDEFINED                |



All tests must succeed

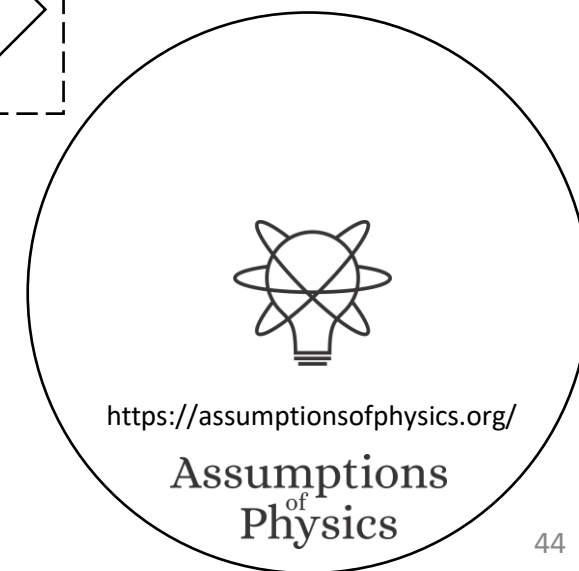
**Axiom 1.31** (Axiom of finite conjunction verifiability). The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let  $\{s_i\}_{i=1}^n \subseteq \mathcal{S}_V$  be a finite collection of verifiable statements. Then the conjunction  $\bigwedge_{i=1}^n s_i \in \mathcal{S}_V$  is a verifiable statement.

$$\bigvee_{i=1}^{\infty} S_i$$



One successful test is sufficient

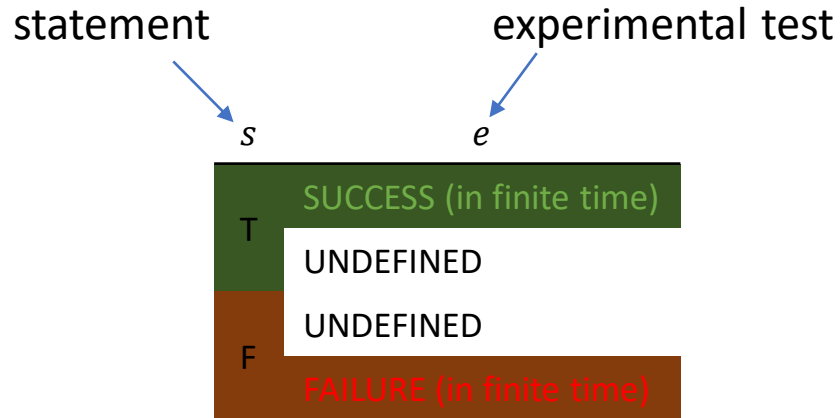
**Axiom 1.32** (Axiom of countable disjunction verifiability). The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let  $\{s_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_V$  be a countable collection of verifiable statements. Then the disjunction  $\bigvee_{i=1}^{\infty} s_i \in \mathcal{S}_V$  is a verifiable statement.



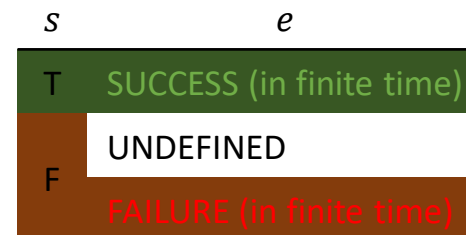
**Axiom 1.27** (Axiom of verifiability). A *verifiable statement* is a statement that, if true, can be shown to be so experimentally. Formally, each logical context  $\mathcal{S}$  contains a set of statements  $\mathcal{S}_v \subseteq \mathcal{S}$  whose elements are said to be verifiable. Moreover, we have the following properties:

- every certainty  $\top \in \mathcal{S}$  is verifiable
- every impossibility  $\perp \in \mathcal{S}$  is verifiable
- a statement equivalent to a verifiable statement is verifiable

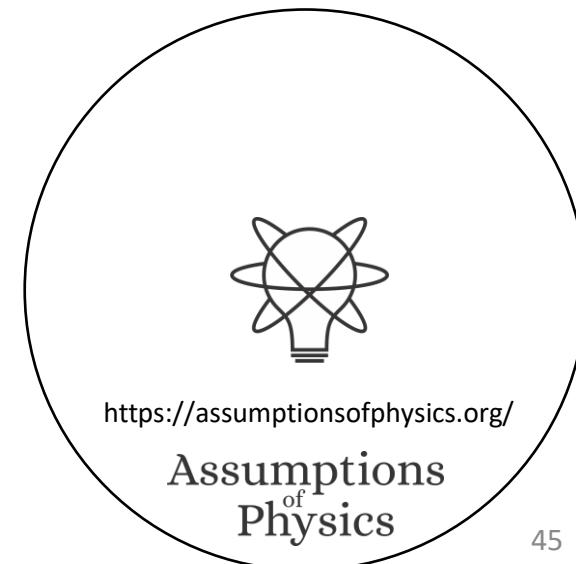
New axiom to bring in the idea that some statements are experimentally verifiable



Statements are verifiable if there is a test that always terminates successfully if the statement is true



Tests may or may not terminate



**Axiom 1.27** (Axiom of verifiability). A *verifiable statement* is a statement that, if true, can be shown to be so experimentally. Formally, each logical context  $\mathcal{S}$  contains a set of statements  $\mathcal{S}_v \subseteq \mathcal{S}$  whose elements are said to be verifiable. Moreover, we have the following properties:

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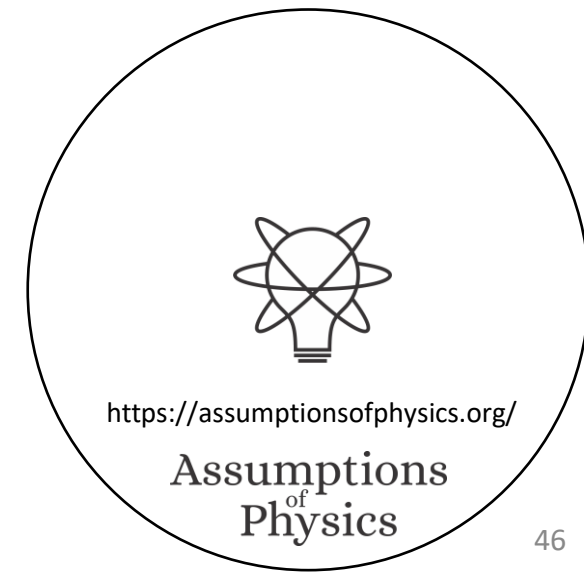
## The tests are not objects in the mathematical framework

Defining tests formally is cumbersome

Capturing which statements are verifiable is enough

Formally we are only “tagging” which statements are verifiable

Only need to tag the verifiable statements:  
all other tests can be constructed from those

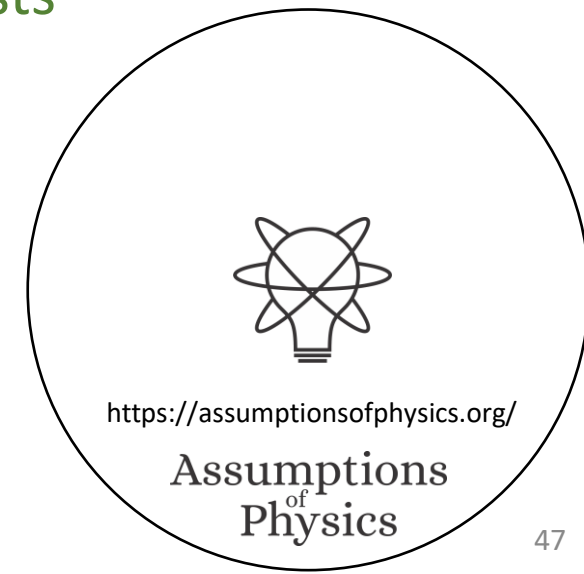


**Axiom 1.27** (Axiom of verifiability). A *verifiable statement* is a statement that, if true, can be shown to be so experimentally. Formally, each logical context  $\mathcal{S}$  contains a set of statements  $\mathcal{S}_v \subseteq \mathcal{S}$  whose elements are said to be verifiable. Moreover, we have the following properties:

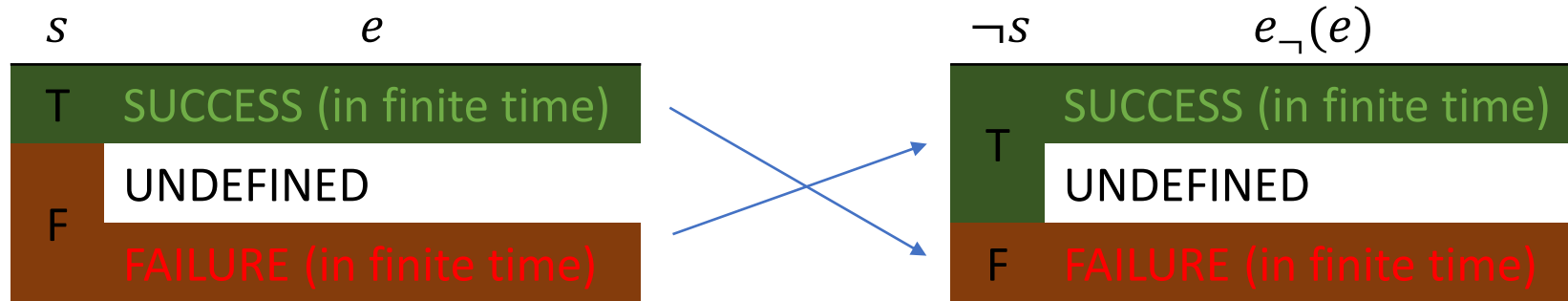
- every certainty  $\top \in \mathcal{S}$  is verifiable
- every impossibility  $\perp \in \mathcal{S}$  is verifiable
- a statement equivalent to a verifiable statement is verifiable

Certainties and impossibilities are defined to be true and false, therefore a trivial test that always succeeds or fails is adequate

If two statements are equivalent, the termination conditions on the tests are the same  $\Rightarrow$  we can use the same test



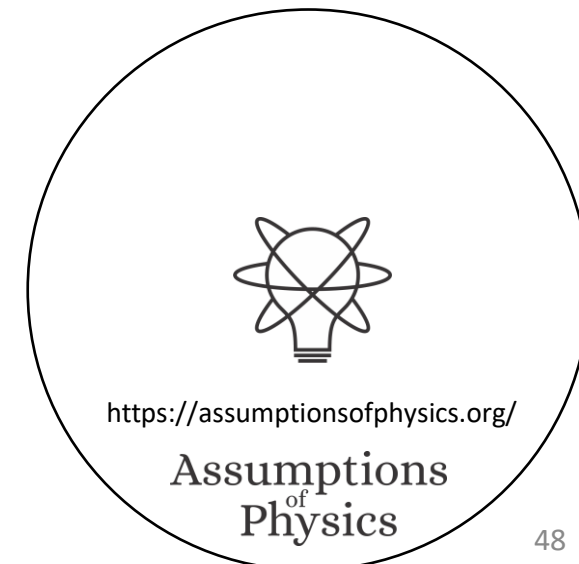
*Remark.* The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



- $e_{\neg}(e)$ :
1. Run test  $e$
  2. If  $e$  fails, return SUCCESS
  3. If  $e$  succeeds, return FAILURE

From  $e$ , we can construct the test  $e_{\neg}(e)$  that switches SUCCESS with FAILURE, but the non-termination remains

**$\Rightarrow$  the logic of verifiable statements does not include negation!**





**Definition 1.28.** A *falsifiable statement* is a statement that, if false, can be shown to be so experimentally. Formally, a statement  $s$  is falsifiable if its negation  $\neg s \in \mathcal{S}_V$  is a verifiable statement.

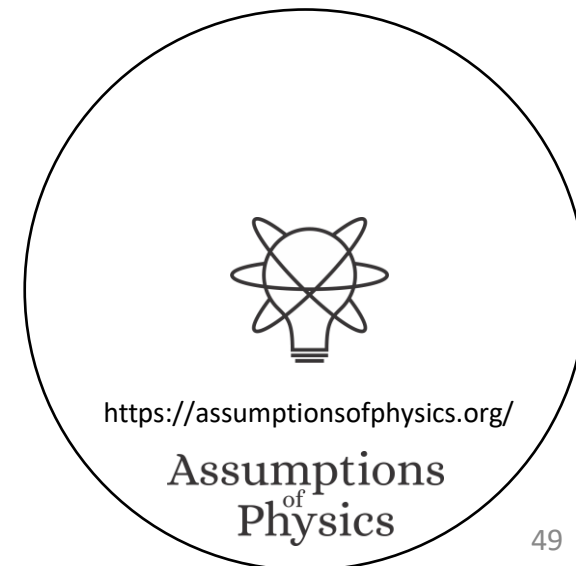
| $s$ | $e$                      |
|-----|--------------------------|
| T   | SUCCESS (in finite time) |
|     | UNDEFINED                |
| F   | FAILURE (in finite time) |

Statements are falsifiable if there is a test that always terminates with failure if the statement is true

Note that formally falsifiable is defined to be the negation of verifiable statements

Reduces the number of primitive concepts

⇒ The justification must show these definitions to be equivalent



**Definition 1.28.** A *falsifiable statement* is a statement that, if false, can be shown to be so experimentally. Formally, a statement  $s$  is falsifiable if its negation  $\neg s \in \mathcal{S}_V$  is a verifiable statement.

Suppose  $\neg s$  is verifiable. Then we can find a test such that

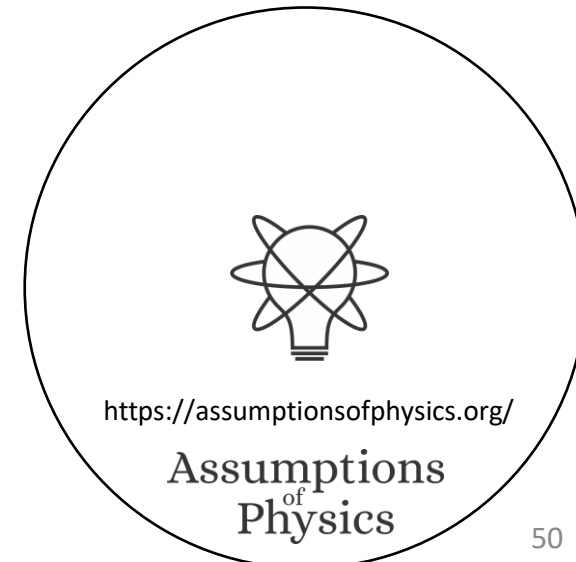
From  $e$ , we can construct the test  $e_{\neg}(e)$  that switches SUCCESS with FAILURE

| $s$ | $\neg s$ | $e$                      | $e_{\neg}(e)$            |
|-----|----------|--------------------------|--------------------------|
| T   | F        | FAILURE (in finite time) | SUCCESS (in finite time) |
|     |          | UNDEFINED                | UNDEFINED                |
| F   | T        | SUCCESS (in finite time) | FAILURE (in finite time) |

$e_{\neg}(e)$ :

1. Run test  $e$
2. If  $e$  fails, return SUCCESS
3. If  $e$  succeeds, return FAILURE

$\Rightarrow$  If the negation of a statement is verifiable, then the statement is falsifiable



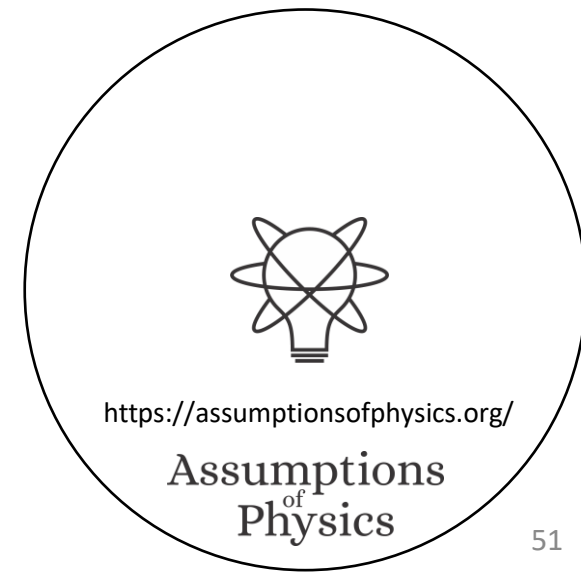
**Definition 1.29.** A *decidable statement* is a statement that can be shown to be either true or false experimentally. Formally, a statement  $s$  is decidable if  $s \in \mathcal{S}_V$  and  $\neg s \in \mathcal{S}_V$ . We denote  $\mathcal{S}_D \subseteq \mathcal{S}_V$  the set of all decidable statements.

| $s$ | $e$                      |
|-----|--------------------------|
| T   | SUCCESS (in finite time) |
| F   | FAILURE (in finite time) |

Statements are decidable if there is a test that always terminates

Note that formally decidable statements are verifiable statements whose negation is verifiable

⇒ The justification must show these definitions to be equivalent



**Definition 1.29.** A *decidable statement* is a statement that can be shown to be either true or false experimentally. Formally, a statement  $s$  is decidable if  $s \in \mathcal{S}_V$  and  $\neg s \in \mathcal{S}_V$ . We denote  $\mathcal{S}_D \subseteq \mathcal{S}_V$  the set of all decidable statements.

Suppose  $s$  is verifiable. Then we can find a test such that

| $s$ | $\neg s$ | $e$                      | $e_{\neg}$               | $\hat{e}(e, e_{\neg})$   |
|-----|----------|--------------------------|--------------------------|--------------------------|
| T   | F        | SUCCESS (in finite time) | FAILURE (in finite time) | SUCCESS (in finite time) |
| F   | T        | UNDEFINED                | UNDEFINED                | SUCCESS (in finite time) |
| F   | T        | FAILURE (in finite time) | SUCCESS (in finite time) | FAILURE (in finite time) |

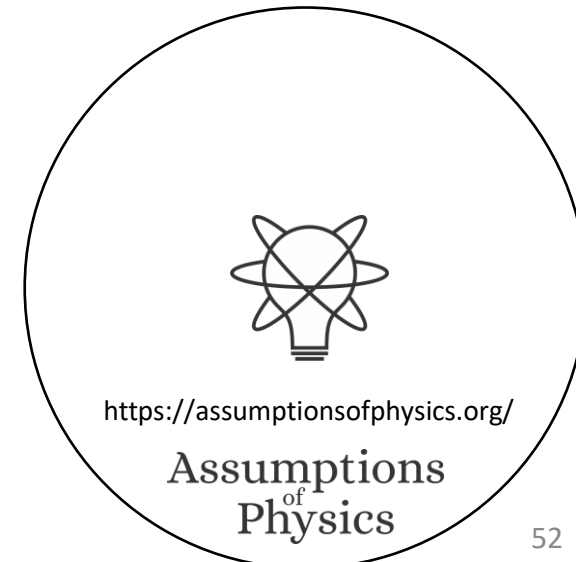
Construct the test  $\hat{e}(e, e_{\neg})$

$\hat{e}(e, e_{\neg})$ :

1. Initialize  $n$  to 1
2. Run  $e$  for  $n$  seconds
3. If  $e$  succeeds, return SUCCESS
4. Run  $e_{\neg}$  for  $n$  seconds
5. If  $e_{\neg}$  succeeds, return FAILURE
6. Increment  $n$  and go to 2

Suppose  $\neg s$  is verifiable. Then we can find a test such that

$\Rightarrow$  If  $s$  and  $\neg s$  verifiable, then the statement is decidable



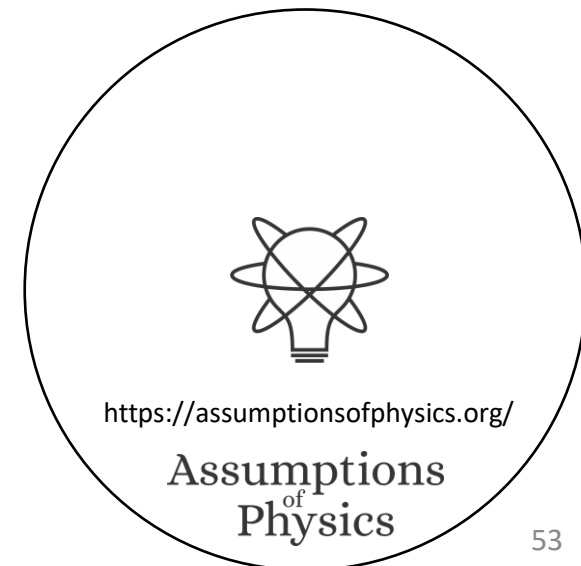
**Definition 1.29.** *A **decidable statement** is a statement that can be shown to be either true or false experimentally. Formally, a statement  $s$  is decidable if  $s \in \mathcal{S}_V$  and  $\neg s \in \mathcal{S}_V$ . We denote  $\mathcal{S}_d \subseteq \mathcal{S}_V$  the set of all decidable statements.*

**Corollary 1.30.** *Certainties and impossibilities are decidable statements.*

Certainties and impossibilities are true and false by definition.  
Yet, we can make trivial tests for them.

$e_{\top}$ :  
1. return SUCCESS

$e_{\perp}$ :  
1. return FAILURE



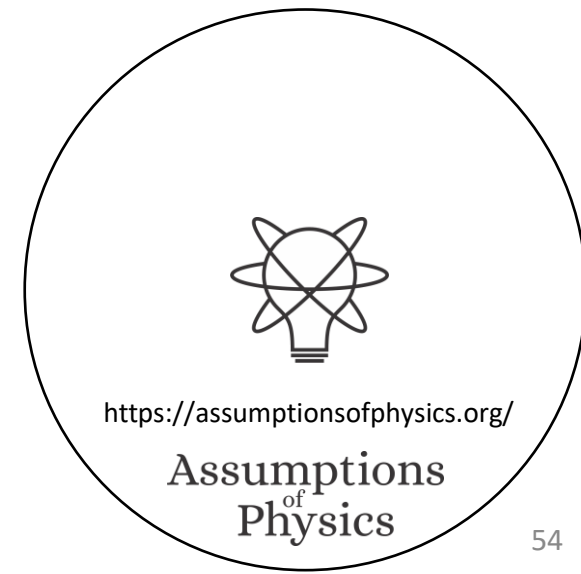
**Axiom 1.31** (Axiom of finite conjunction verifiability). *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let  $\{s_i\}_{i=1}^n \subseteq \mathcal{S}_v$  be a finite collection of verifiable statements. Then the conjunction  $\bigwedge_{i=1}^n s_i \in \mathcal{S}_v$  is a verifiable statement.*

Conjunction (AND) of verifiable statements:  
check that all tests terminate successfully

$\wedge (e_i)$ :

1. Run all  $e_i$
2. If all succeed, return SUCCESS
3. Return FAILURE

$\Rightarrow$  Only finite conjunction is guaranteed to terminate



**Axiom 1.32** (Axiom of countable disjunction verifiability). *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let  $\{s_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_v$  be a countable collection of verifiable statements. Then the disjunction  $\bigvee_{i=1}^{\infty} s_i \in \mathcal{S}_v$  is a verifiable statement.*

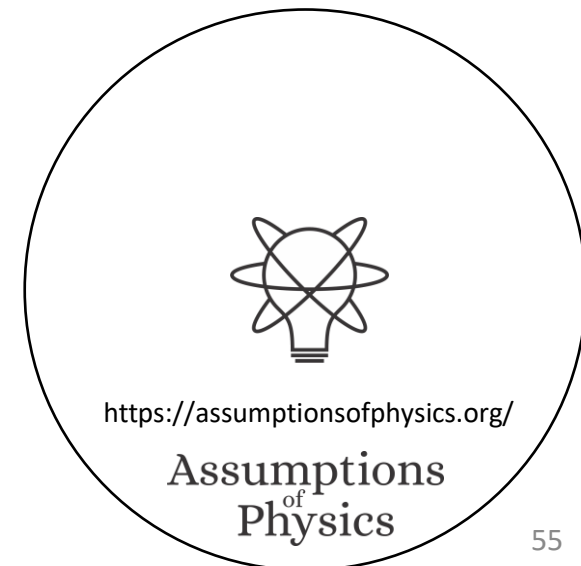
Disjunction (OR) of verifiable statements:  
check that ONE test terminates successfully

watch out for non-termination!

⇒ Only countable disjunction can reach all tests

$\vee (e_i)$ :

1. Initialize  $n$  to 1
2. For each  $i = 1 \dots n$ 
  - a) Run  $e_i$  for  $n$  seconds
  - b) If  $e_i$  succeeds, return SUCCESS
3. Increment  $n$  and go to 2

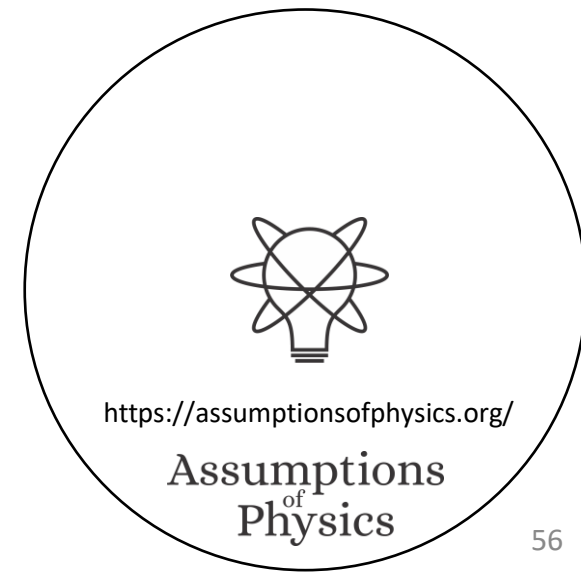


**Proposition 1.33.** *The conjunction and disjunction of a finite collection of decidable statements are decidable. Formally, let  $\{s_i\}_{i=1}^n \subseteq \mathcal{S}_d$  be a finite collection of decidable statements. Then the conjunction  $\bigwedge_{i=1}^n s_i \in \mathcal{S}_d$  and the disjunction  $\bigvee_{i=1}^n s_i \in \mathcal{S}_d$  are decidable statements.*

For decidable statements, we need both the statement and its negation to be verifiable

$$\neg \bigwedge e_i = \bigvee \neg e_i \quad \neg \bigvee e_i = \bigwedge \neg e_i$$

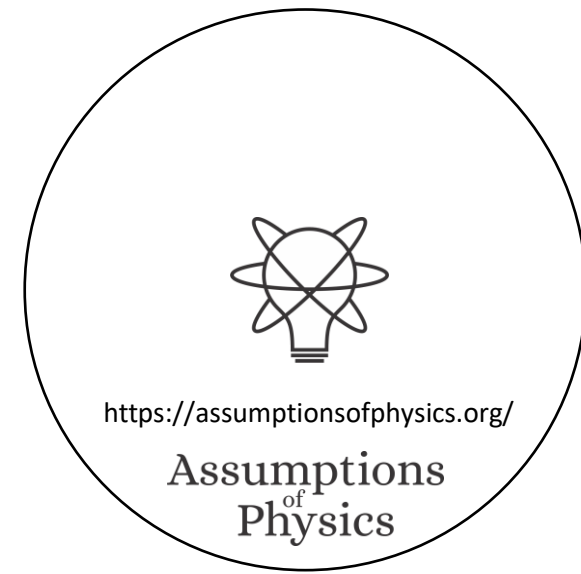
Using De Morgan properties, we can construct tests using test for negation





# Takeaways

- Adding the notion of verifiability only requires tagging which statements are verifiable
- We are essentially modeling procedures that output success/failure (i.e. one bit) and may not terminate
- These are the only axioms at this point
  - Everything else is a construction on top of this



# Topology and the logic of experimental verifiability



<https://assumptionsofphysics.org/>

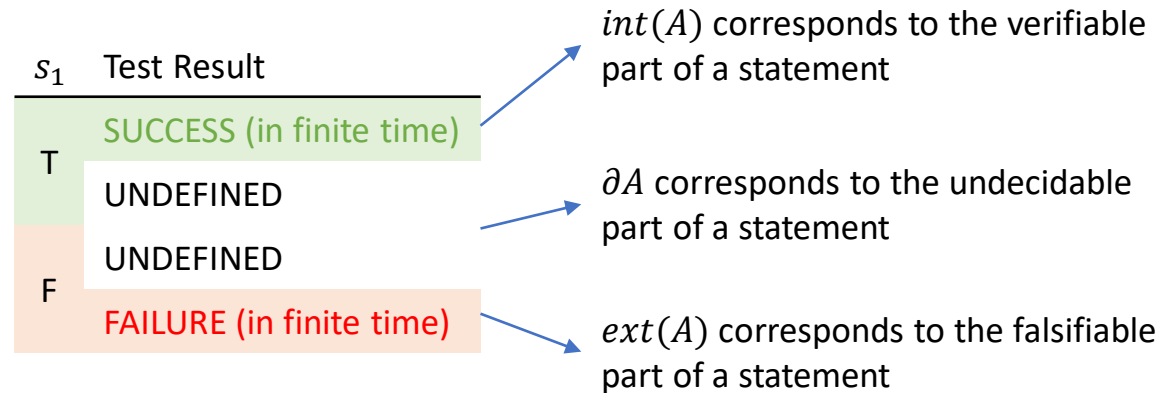
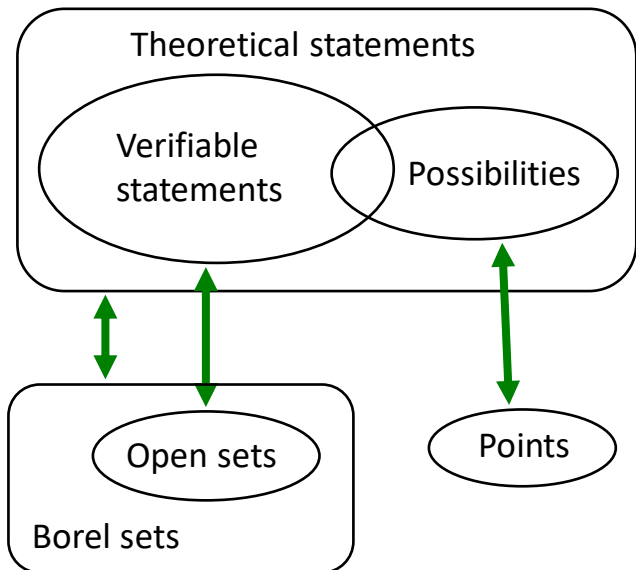
Assumptions  
of  
Physics

# Topology and $\sigma$ -algebra

Experimental verifiability  $\Rightarrow$   
 topology and  $\sigma$ -algebras  
 (foundation of geometry,  
 probability, ...)

Perfect map  
 between math and  
 physics

NB: in physics, topology and  
 $\sigma$ -algebra are parts of the  
**same** logic structure

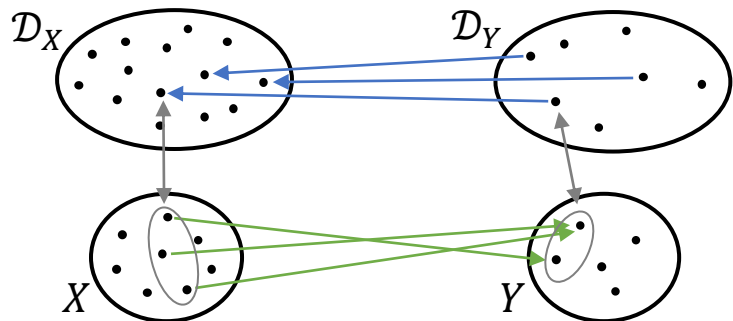


Open set  $(509.5, 510.5) \Leftrightarrow$  Verifiable "the mass of the electron is  $510 \pm 0.5$  KeV"

Closed set  $[510] \Leftrightarrow$  Falsifiable "the mass of the electron is exactly 510 KeV"

Borel set  $\mathbb{Q} (int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset) \Leftrightarrow$  Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Inference relationship  $r: \mathcal{D}_Y \rightarrow \mathcal{D}_X$  such that  $r(s) \equiv s$



Inference relationship

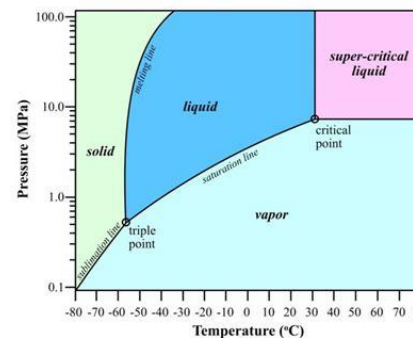


Causal relationship

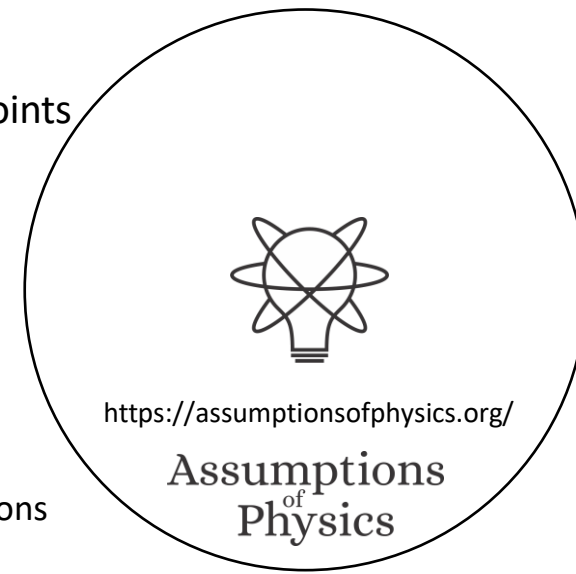
Relationships must be  
 topologically continuous

Causal relationship  $f: X \rightarrow Y$  such that  $x \leq f(x)$

Topologically continuous consistent  
 with analytic discontinuity on isolated points



Phase transition  $\Leftrightarrow$  Topologically isolated regions



<https://assumptionsofphysics.org/>

Assumptions  
 of  
 Physics

What is the largest set of verifiable statements  
it makes sense to consider?

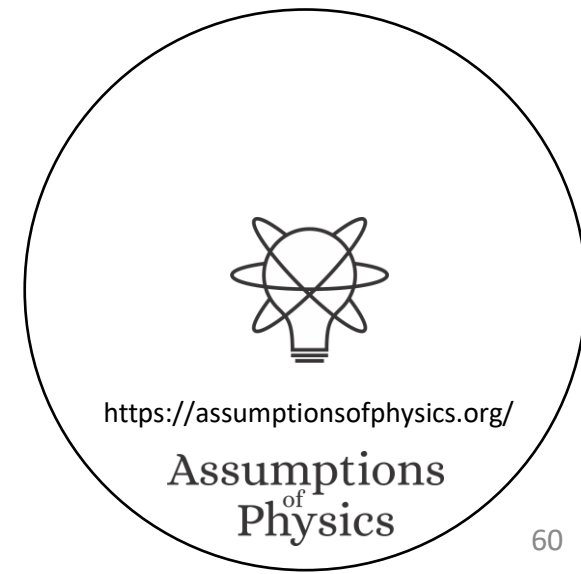
$S_1, S_2, S_3, \dots, S_n, \dots$

Note: even assuming an indeterminate amount of  
time, we can only run up to countably many tests

$S_1 \vee S_2, S_1 \wedge S_2, \dots$

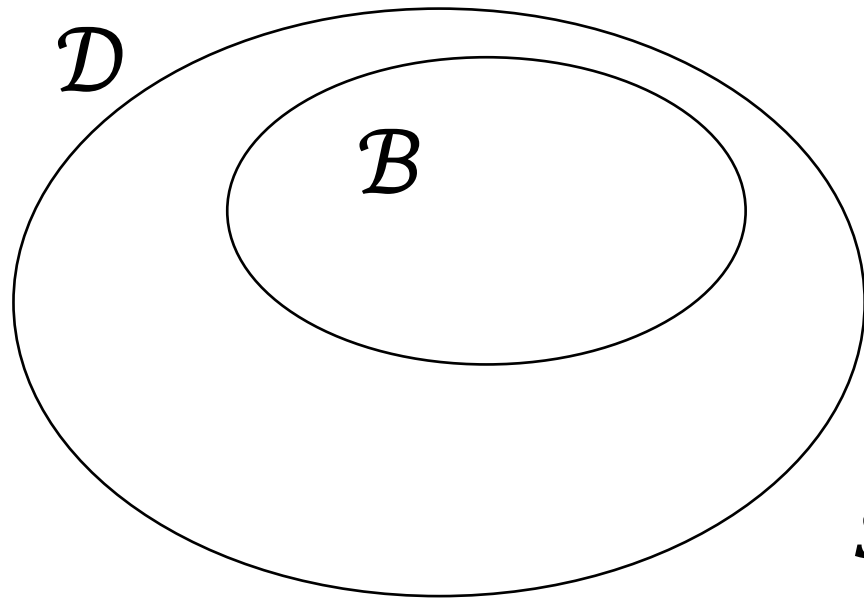
However, testing those statements implicitly tests  
all other statements that depend on those

⇒ Set of verifiable statements whose truth can  
be verified by running countably many tests



**Definition 1.34.** Given a set  $\mathcal{D}$  of verifiable statements,  $\mathcal{B} \subseteq \mathcal{D}$  is a **basis** if the truth values of  $\mathcal{B}$  are enough to deduce the truth values of the set. Formally, all elements of  $\mathcal{D}$  can be generated from  $\mathcal{B}$  using finite conjunction and countable disjunction.

**Definition 1.35.** An **experimental domain**  $\mathcal{D}$  represents a set of verifiable statements that can be tested and possibly verified in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the certainty, the impossibility, and a set of verifiable statements that can be generated from a countable basis.

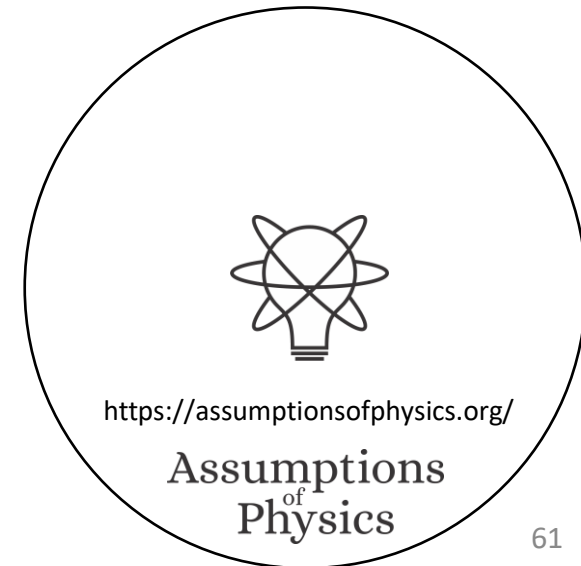


$$\mathcal{B} = \{e_1, e_2, e_3, \dots\}$$

Countable basis

Only finite conjunction and countable disjunction

$$s_1 = (e_1 \vee e_3) \wedge e_2 \dots$$



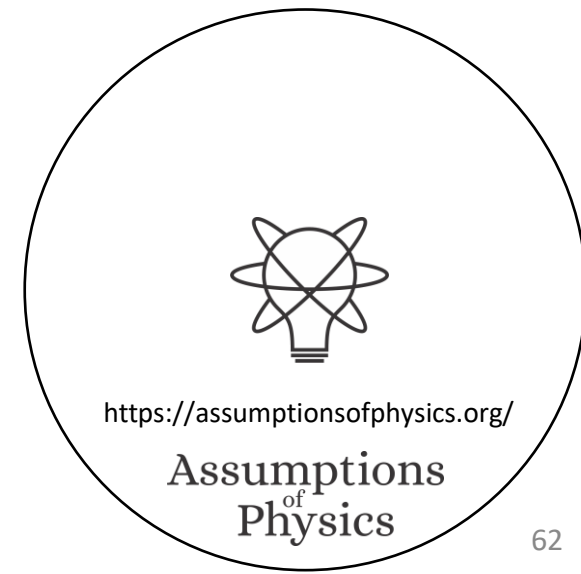
**Definition 1.34.** Given a set  $\mathcal{D}$  of verifiable statements,  $\mathcal{B} \subseteq \mathcal{D}$  is a **basis** if the truth values of  $\mathcal{B}$  are enough to deduce the truth values of the set. Formally, all elements of  $\mathcal{D}$  can be generated from  $\mathcal{B}$  using finite conjunction and countable disjunction.

**Definition 1.35.** An **experimental domain**  $\mathcal{D}$  represents a set of verifiable statements that can be tested and possibly verified in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the certainty, the impossibility, and a set of verifiable statements that can be generated from a countable basis.

## Every physical theory must be fully characterized by an experimental domain

All its content must be expressible in terms of verifiable statements

The theory must be fully explorable with a countable set of tests

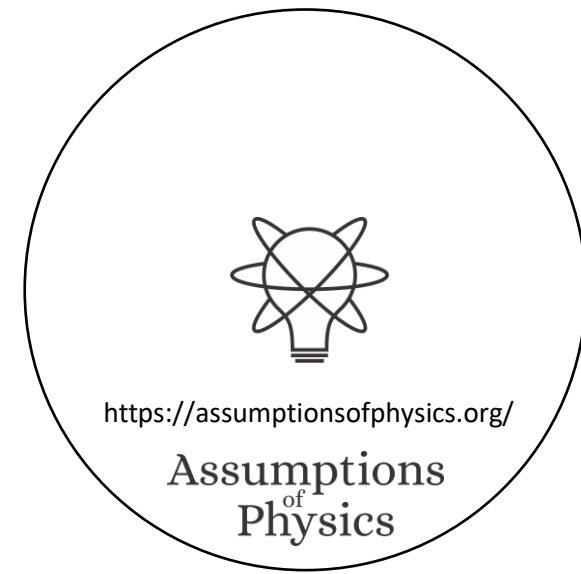


**Definition 1.36.** *The **theoretical domain**  $\bar{\mathcal{D}}$  of an experimental domain  $\mathcal{D}$  is the set of statements constructed from  $\mathcal{D}$  to which we can associate a test regardless of termination. We call **theoretical statement** a statement that is part of a theoretical domain. More formally,  $\bar{\mathcal{D}}$  is the set of all statements generated from  $\mathcal{D}$  using negation, finite conjunction and countable disjunction.*

Extend the domain to include all statements that are associated with a test, regardless of termination.

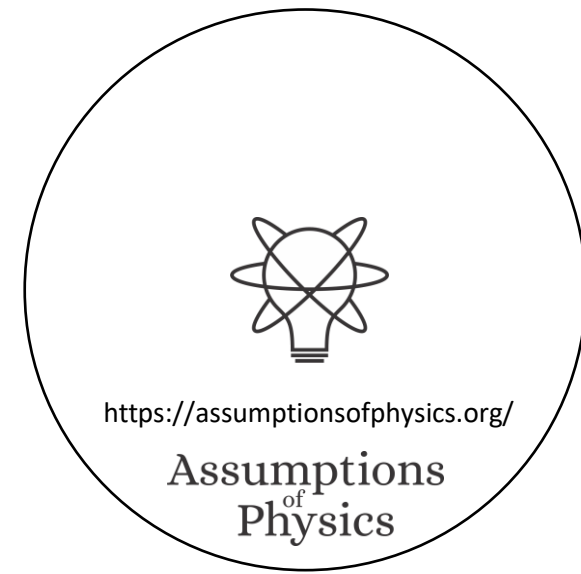
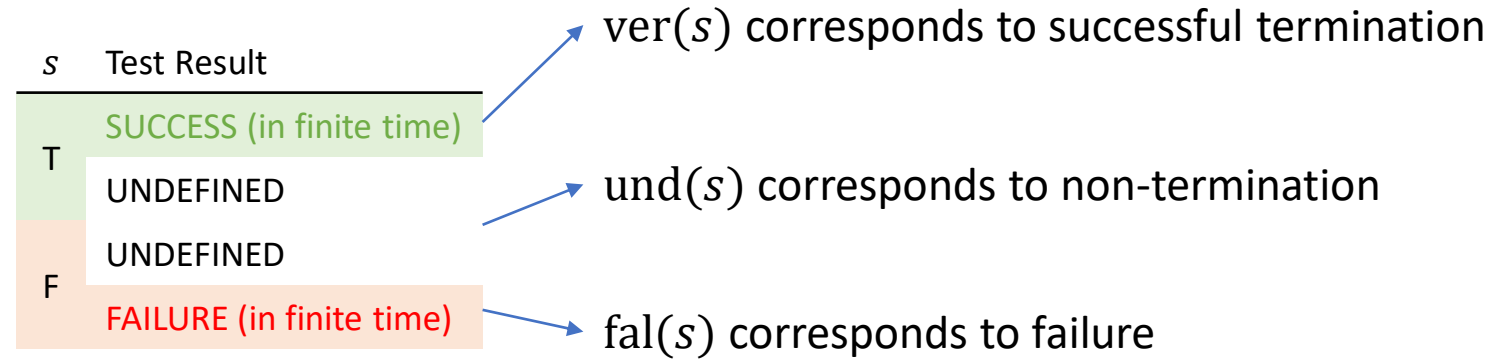
All statements depend on the verifiable statements  
(which depend on the basis)

No new information is captured



**Definition 1.39.** Let  $\bar{s} \in \bar{\mathcal{D}}$  be a theoretical statement. We call the **verifiable part**  $\text{ver}(\bar{s}) = \bigvee_{s \in \mathcal{D} \mid s \leq \bar{s}} s$  the broadest verifiable statement that is narrower than  $\bar{s}$ . We call the **falsifiable part**  $\text{fal}(\bar{s}) = \bigvee_{s \in \mathcal{D} \mid s \not\leq \bar{s}} s$  the broadest verifiable statement that is incompatible with  $\bar{s}$ . We call the **undecidable part**  $\text{und}(\bar{s}) = \neg \text{ver}(\bar{s}) \wedge \neg \text{fal}(\bar{s})$  the broadest statement incompatible with both the verifiable and the falsifiable part.

Formalizing successful termination is indeed enough to characterize termination





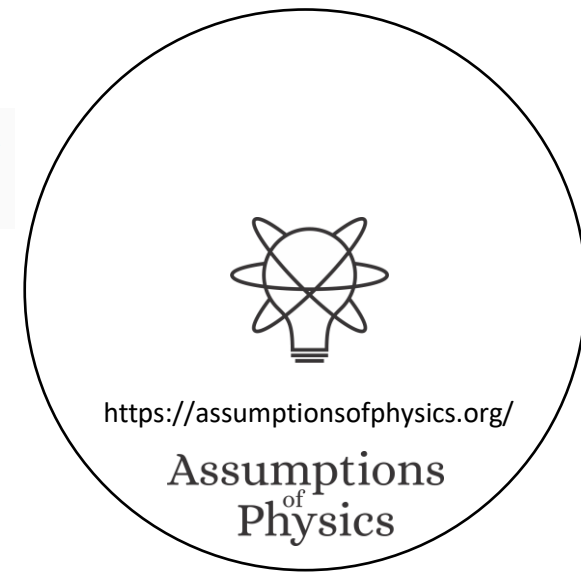
**Definition 1.47.** A *possibility* for an experimental domain  $\mathcal{D}$  is a statement  $x \in \bar{\mathcal{D}}$  that, when true, determines the truth value for all statements in the theoretical domain. Formally,  $x \neq \perp$  and for each  $s \in \bar{\mathcal{D}}$ , either  $x \leq s$  or  $x \not\leq s$ . The **full possibilities**, or simply the **possibilities**,  $X$  for  $\mathcal{D}$  are the collection of all possibilities.

A possibility of a domain is a statement that picks one assignment

Possibilities: experimentally defined alternative cases defined by the verifiable cases

| $s_1$ | $s_2$ | $s_3$ | ... | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-----|-------|-------|-------|-------|
| T     | T     | F     | T   | T     | F     | F     | F     |
| F     | F     | T     | T   | F     | T     | F     | F     |
| F     | T     | T     | F   | F     | F     | T     | F     |
| T     | F     | F     | T   | F     | F     | F     | T     |

**Proposition 1.48.** Let  $\mathcal{D}$  be an experimental domain. A possibility for  $\mathcal{D}$  is any minterm of a basis that is not impossible.



Start with a countable set of verifiable statements



Add all dependent verifiable statements (close under finite AND countable OR)



Add all statements with tests (close under negation as well)



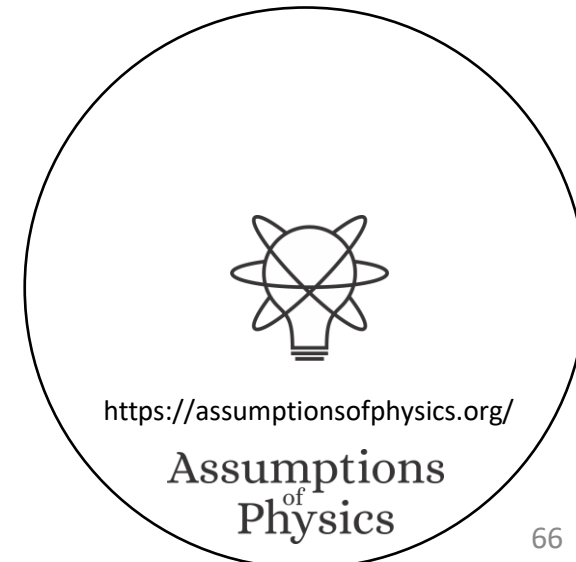
The points of the space (the possibilities, the distinguishable cases) are not given a priori but are constructed from the chosen verifiable statements

$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \dots$

For each possible assignment we have a theoretical statement that is true only in that case (minterm). We call these statements possibilities of the domain.



Fill in all possible assignments



Each column (statement) is also a set of possibilities  
 $S = \bigvee_{x \in U} x$

Finite AND and countable OR become finite intersection and countable union

Negation and countable AND become complement and countable union

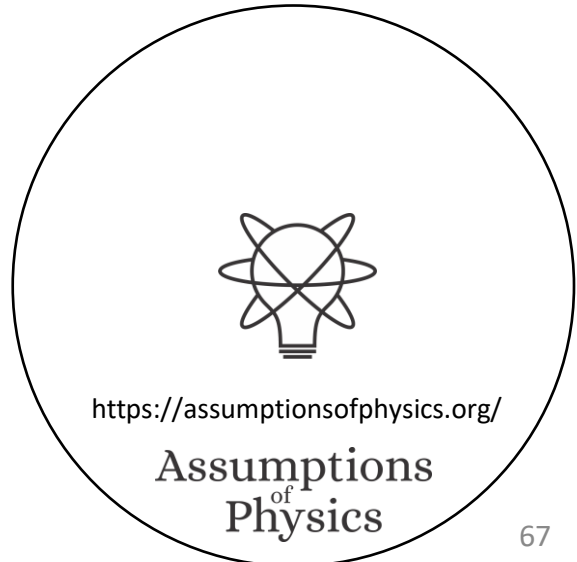
| Basis $\mathcal{B}$ |       |       |     | Experimental domain $\mathcal{D}_X$ |                        |     | Theoretical domain $\bar{\mathcal{D}}_X$ |                        |     |
|---------------------|-------|-------|-----|-------------------------------------|------------------------|-----|--|------------------------|-----|
| $e_1$               | $e_2$ | $e_3$ | ... | $s_1 = e_1 \vee e_2$                | $s_2 = e_1 \wedge e_3$ | ... | $\bar{s}_1 = e_1 \vee \neg e_2$          | $\bar{s}_2 = \neg e_1$ | ... |
| F                   | F     | F     | ... | F                                   | F                      | ... | T  | T                      | ... |
| ...                 | ...   | ...   | ... | ...                                 | ...                    | ... | ...                                      | ...                    | ... |
| F                   | T     | F     | ... | T                                   | F                      | ... | F  | T                      | ... |
| T                   | T     | F     | ... | T                                   | F                      | ... | T  | F                      | ... |
| ...                 | ...   | ...   | ... | ...                                 | ...                    | ... | ...                                      | ...                    | ... |

Possibilities  $X \subset \bar{\mathcal{D}}_X$

Topologies (needed for manifold/geometric spaces) and  $\sigma$ -algebras (needed for integration and probability spaces) naturally arise from requiring experimental verifiability

The experimental domain  $\mathcal{D}_X$  induces a topology on the possibilities  $X$ .

The theoretical domain  $\bar{\mathcal{D}}_X$  induces a (Borel)  $\sigma$ -algebra



# Topologies and $\sigma$ -algebras

All definitions and all proofs about these structures have precise physical meaning in this context

| $s_1$ | Test Result              |
|-------|--------------------------|
| T     | SUCCESS (in finite time) |
|       | UNDEFINED                |
| F     | FAILURE (in finite time) |

$int(A)$  corresponds to the verifiable part of a statement

$\partial A$  corresponds to the undecidable part of a statement

$ext(A)$  corresponds to the falsifiable part of a statement

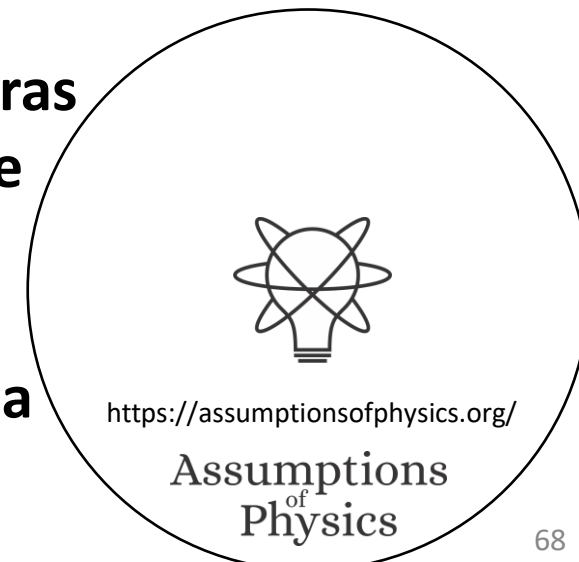
If  $U \subseteq X$  is an open set then “ $x$  is in  $U$ ” is a verifiable statement (e.g. “the mass of the electron is  $511 \pm 0.5$  KeV”)

If  $V \subseteq X$  is a closed set then “ $x$  is in  $V$ ” is a falsifiable statement (e.g. “the mass of the electron is exactly 511 KeV”)

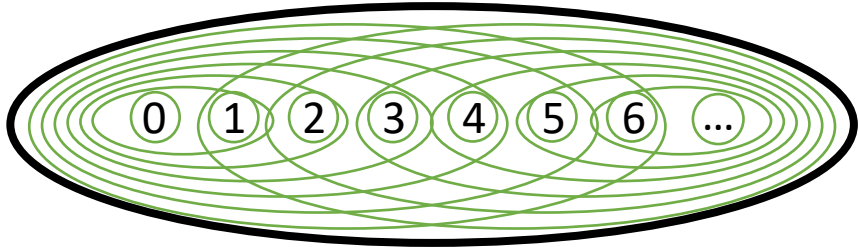
If  $A \subseteq X$  is a Borel set then “ $x$  is in  $A$ ” is a theoretical statement: a test can be created, though we have no guarantee of termination (e.g. “the mass of the electron in KeV is a rational number” is undecidable, the test will never terminate)

**Topologies and  $\sigma$ -algebras each capture part of the formal structure**

**For us, they are part of a single unified structure**



# Examples



## *Standard topology on integers*

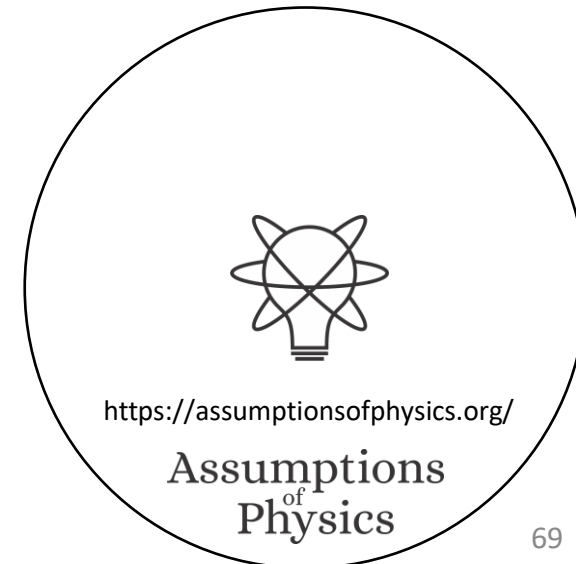
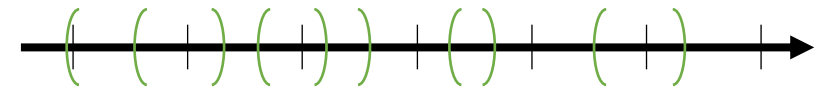
*Decidable domain (all statements are decidable)*

Discrete topology (every set is clopen); topology and  $\sigma$ -algebra both coincide with the power set

## *Standard topology on the reals*

*Finite precision measurements (open intervals are verifiable)*

Topology generated by open intervals (coincides with order and metric topology); separable, complete, connected (no clopen sets except full and empty set);  $\sigma$ -algebra is the Borel algebra (strict subset of power set)

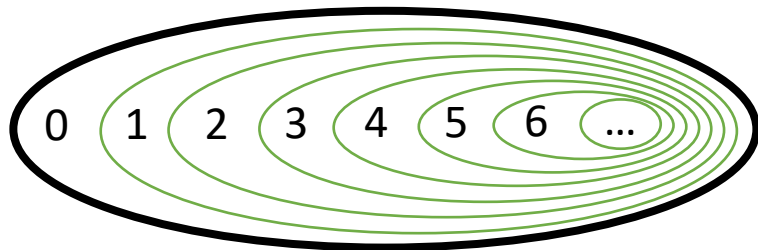
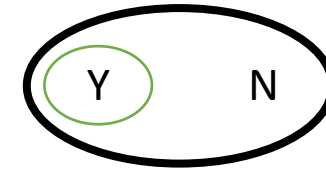


# Examples

Does extra-terrestrial life exist?

*Semi-decidable question*

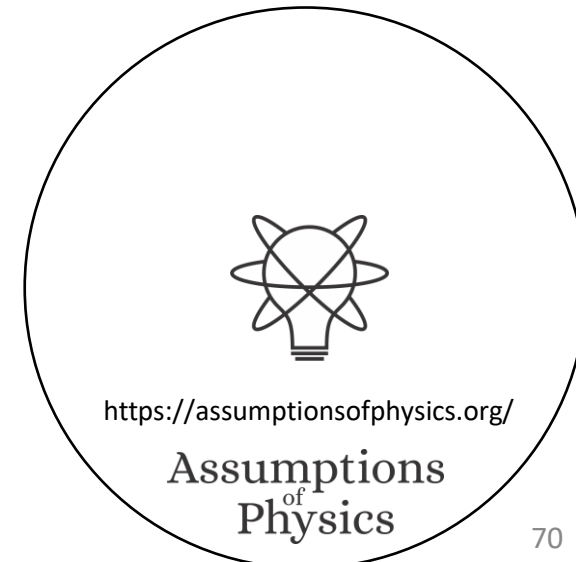
Topology  $\{\emptyset, \{Y\}, \{Y, N\}\}$  is strictly  $T_0$ ;  $\sigma$ -algebra is the power set



*How many leptons (electron-like particles) are there?  
(through direct observation)*

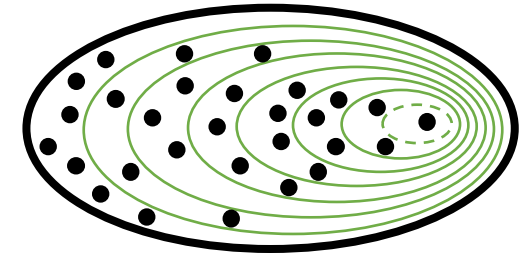
*Can only measure lower bound (e.g. "there are at least  $i$ ")*

Topology contains empty set and  $\{i, i + 1, i + 2, \dots\}$  for all  $i$ ; strictly  $T_0$ ;  $\sigma$ -algebra is the power set

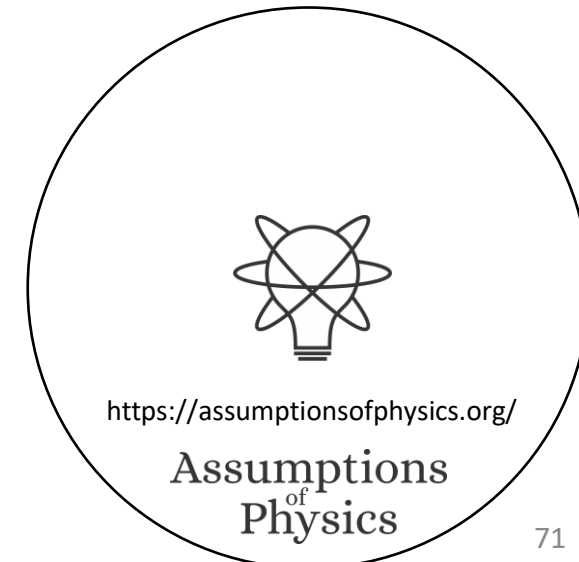


# Physical meaning of separation axioms

- All topologies are Kolmogorov (i.e.  $T_0$ )
  - Possibilities are experimentally well-defined  
i.e. possibilities constructible from a base by countable AND/OR and NOT (singletons in the  $\sigma$ -algebra)
- The topology is  $T_1$  if all possibilities are approximately verifiable
  - Possibilities are the limit of a sequence of verifiable statements  
i.e. possibilities are the countable conjunction of verifiable statements
- The topology is Hausdorff (i.e.  $T_2$ ) if all possibilities are pairwise experimentally distinguishable
  - Given two possibilities, we can find a test that confirms one and excludes the other
  - i.e. for any  $x_1, x_2 \in X$  there is a statement  $s \in \bar{\mathcal{D}}_X$  such that  $x_1 \preceq \text{ver}(s)$  and  $x_2 \preceq \text{fal}(s)$

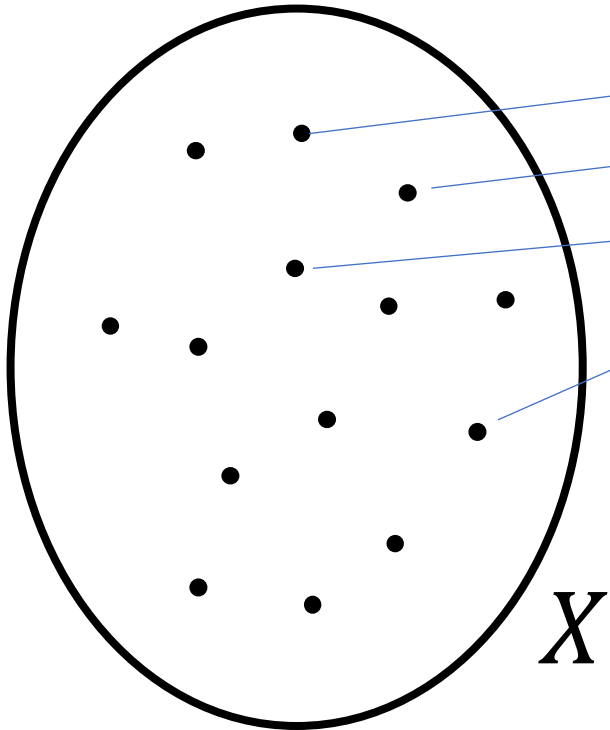


| s | Test Result              | $x_1$ | $x_2$ |
|---|--------------------------|-------|-------|
| T | SUCCESS (in finite time) | T     | F     |
|   | UNDEFINED                | F     | F     |
| F | FAILURE (in finite time) | F     | T     |



# Maximum cardinality of distinguishable cases $\mathbb{R}$

Set of distinguishable cases



Test results for countable basis

FTFFFTTTFTFTT...  
 TFFTTFTTFFFTF...  
 FTFFFTTFTFFTF...  
 FTTFTFTTFTFFT...

Correspond to binary expansion

0.0100011101011...  
 0.1001101100010...  
 0.0100011010010...  
 0.0110101101001...

$\mathbb{R}$

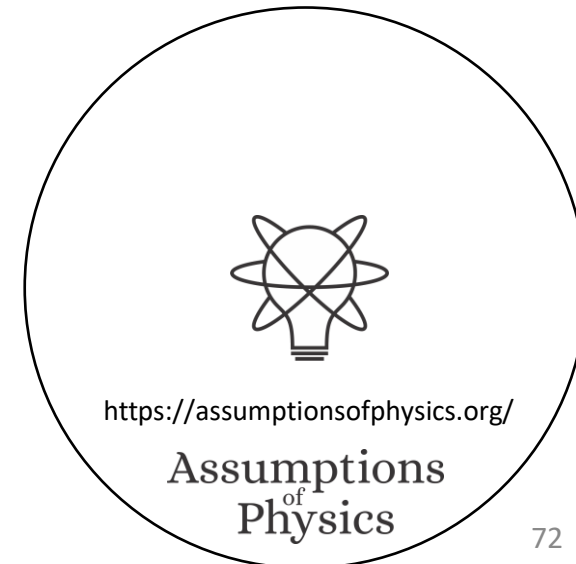


Correspondence to binary sequence

0100011101011...  
 1001101100010...  
 0100011010010...  
 0110101101001...

$$|X| \leq |\mathbb{R}|$$

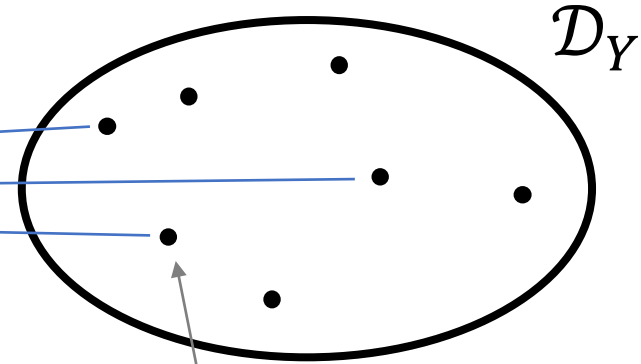
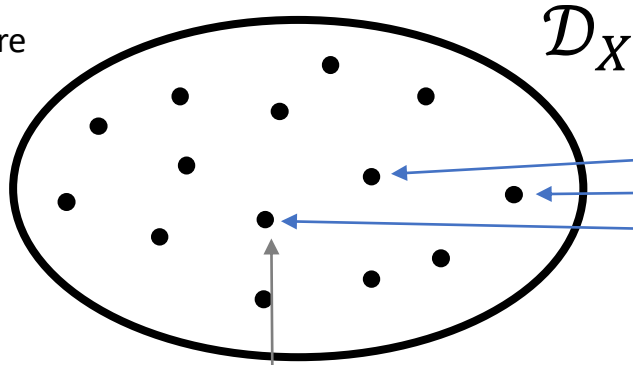
- Sets with greater cardinality (e.g. the set of all discontinuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ ) cannot represent physical objects
- Issues about higher infinities (e.g. large cardinals) are not relevant, but those surrounding the continuum hypothesis may be





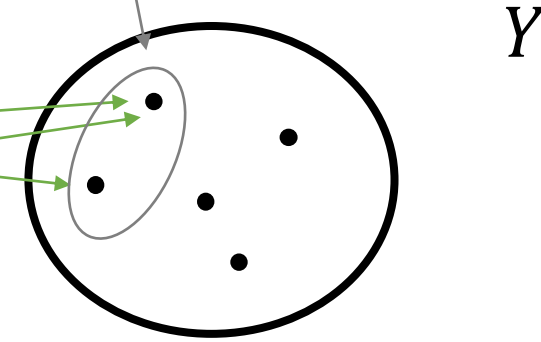
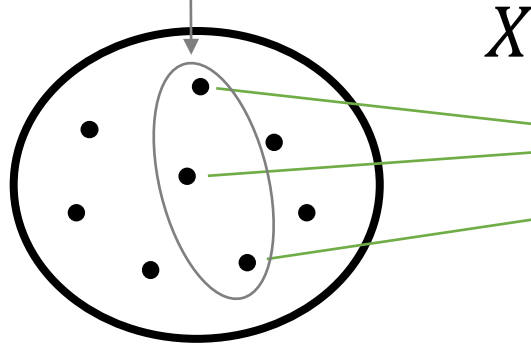
An **inference relationship** is a map  $r: \mathcal{D}_Y \rightarrow \mathcal{D}_X$  such that  $r(s) \equiv s$

e.g. the water temperature is between 0 and 0.52 Celsius or between 7.6 and 9.12 Celsius



e.g. the water density is between 999.8 and 999.9 kg/m<sup>3</sup>

e.g. the water temperature is exactly 4 Celsius

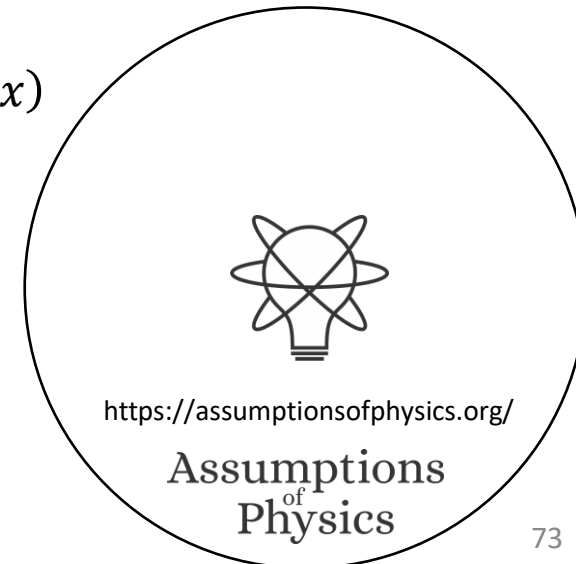


e.g. the water density is exactly 1 kg/m<sup>3</sup>

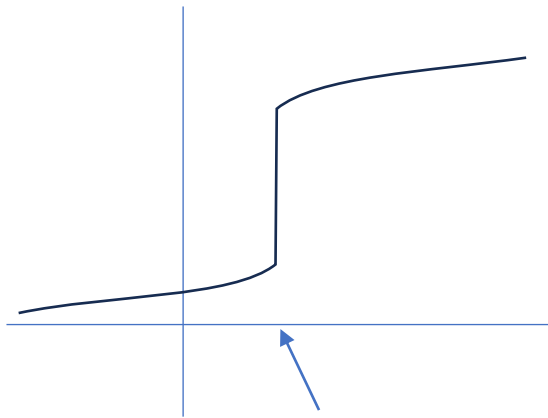
A **causal relationship** is a map  $f: X \rightarrow Y$  such that  $x \preceq f(x)$

Two general and important results:

- 1) Two domains admit an inference relationship if and only if they admit a causal relationship
- 2) The causal relationship must be a continuous map in the natural topology



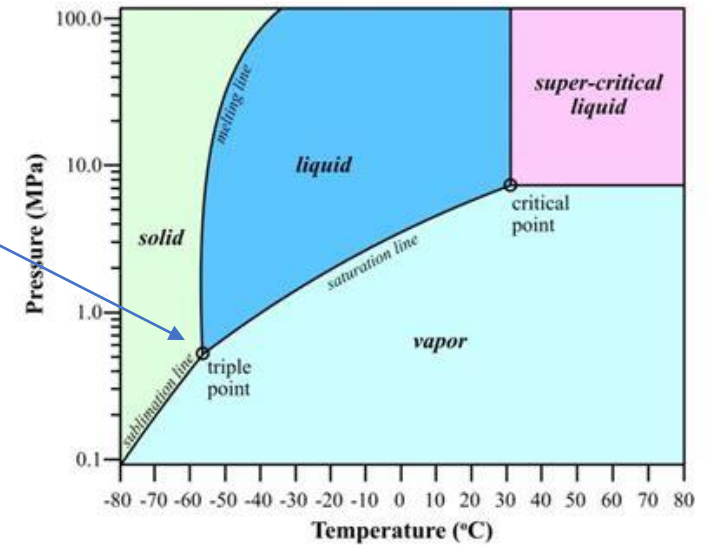
# Functions in physics must be “well-behaved”



Topologically continuous function can be analytically discontinuous at a topologically isolated point

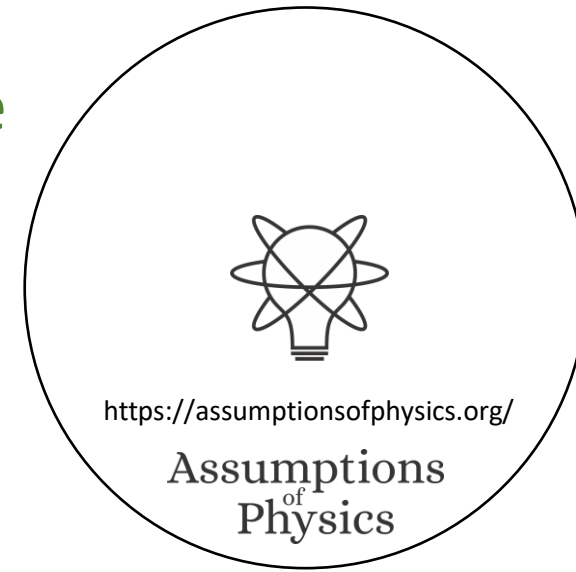
- Second countable space
- ⇒ up to countably many isolated points
- ⇒ up to countably many discontinuity
- ⇒ “well-behaved”

We can verify we are in the triple point ⇒ topologically isolated point



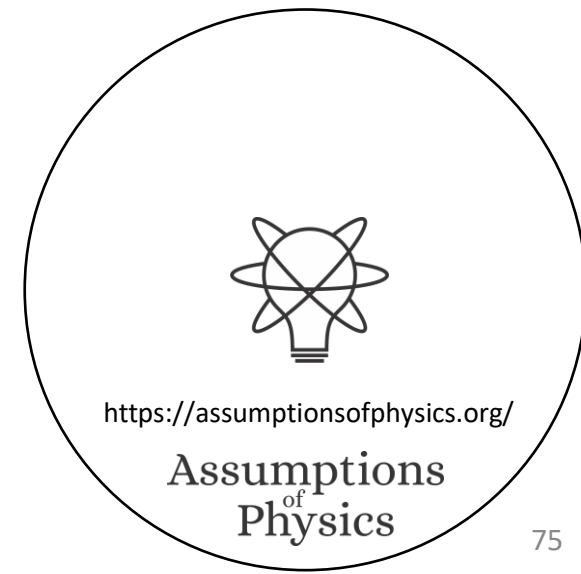
Phase transition ⇔ Topologically isolated regions

Internal energy can change discontinuously through phase transitions

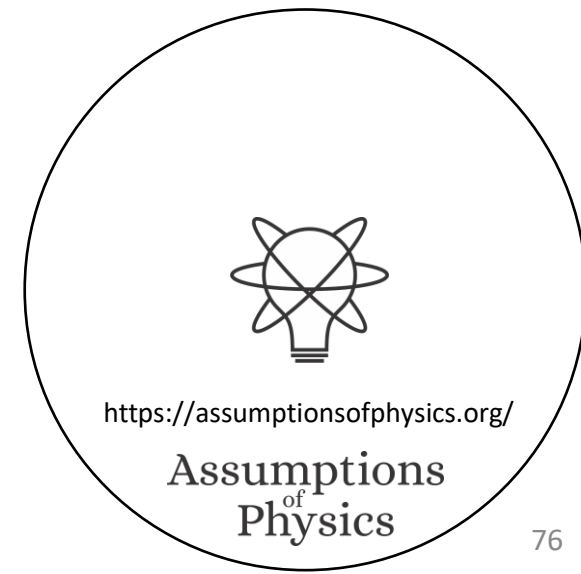


# Takeaway

- The most fundamental mathematical structures (topology and  $\sigma$ -algebra) are there to capture the logic of experimental verifiability
  - Precise science/math dictionary
  - “Well-behaved” mathematical objects are really “well-defined” physical objects
- Experimental verifiability is the basis for scientifically well-defined objects
- TODOs:
  - Space of possible composite experimental domains
  - Approximations of domains
  - Projections to domains



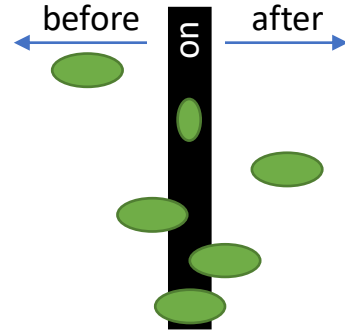
# Quantities and ordering



# Quantities and ordering

Goal: deriving the notion of quantities and numbers (i.e. integers, reals, ...) from an operational (metrological) model

A **reference** (i.e. a tick of a clock, notch on a ruler, sample weight with a scale) is something that allows us to distinguish between a before and an after

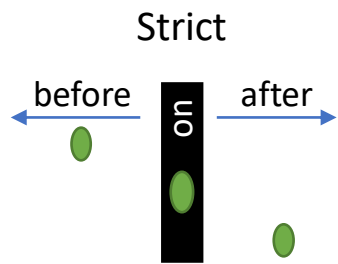


Mathematically, it is a triple  $(b, o, a)$  such that:

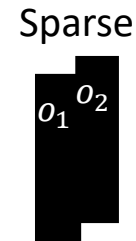
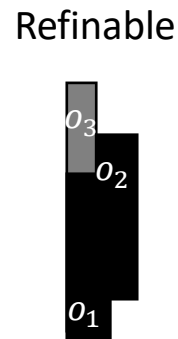
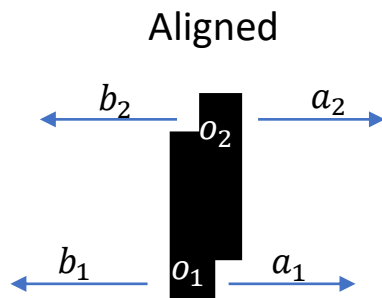
- $b$  and  $a$  are verifiable
- The reference has an extent ( $o \neq \perp$ )
- If it's not before or after, it is on ( $\neg b \wedge \neg a \leq o$ )
- If it's before and after, it is on ( $b \wedge a \leq o$ )

Numbers defined by metrological assumptions, NOT by ontological assumptions

To define an **ordered** sequence of possibilities, the references must be (nec/suff conditions):



$\Rightarrow (X, \leq)$



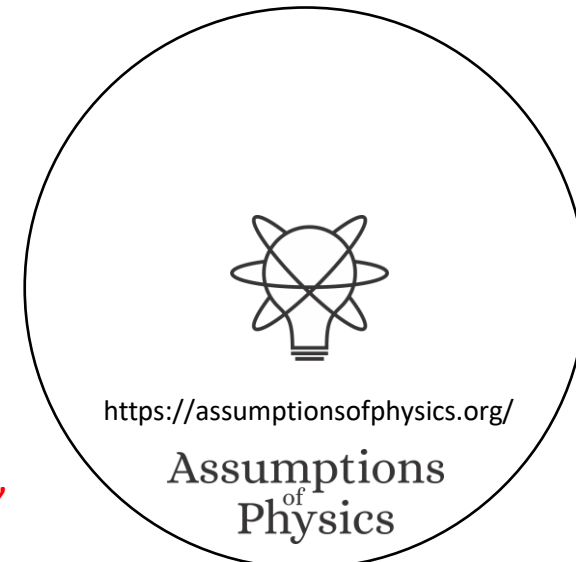
Dense

$\Rightarrow (X, \leq) \cong (\mathbb{R}, \leq)$

$\Rightarrow (X, \leq) \cong (\mathbb{Z}, \leq)$

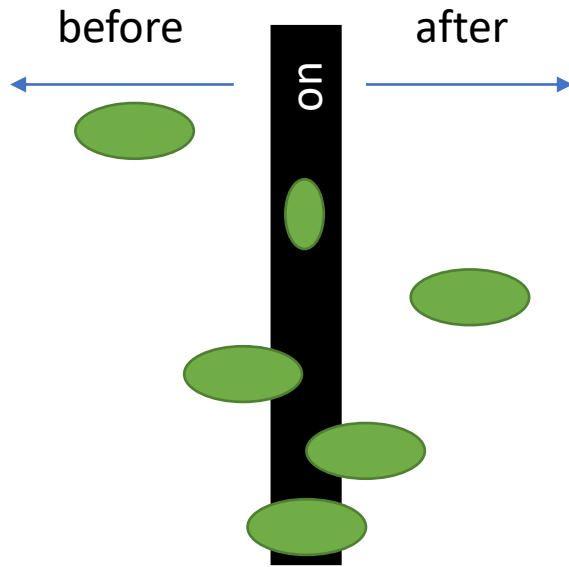
The hard part is to recover ordering. After that, recovering reals and integers is simple.

Assumptions untenable at Planck scale:  
no consistent **ordering**: no "objective" "before" and "after"



# How do we formally model a quantity?

A **reference** (e.g. a tick of a clock) is something that allows us to distinguish between a before and an after

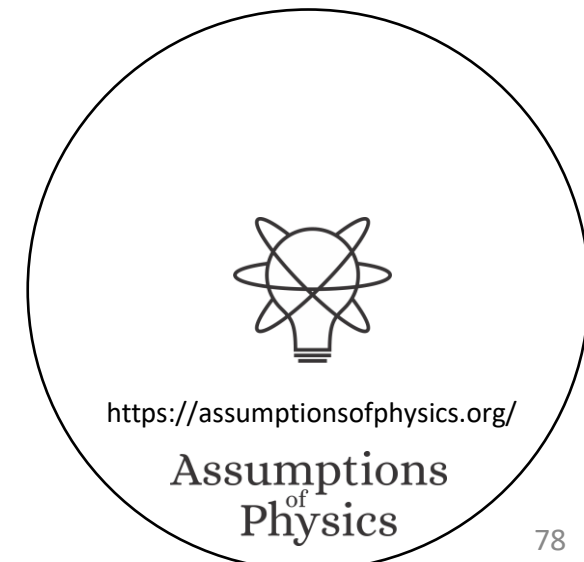


Mathematically, it is a triple  $(b, o, a)$  such that:

- $b$  and  $a$  are verifiable
- The reference has an extent ( $o \neq \perp$ )
- If it's not before or after, it is on ( $\neg b \wedge \neg a \leq o$ )
- If it's before and after, it is on ( $b \wedge a \leq o$ )

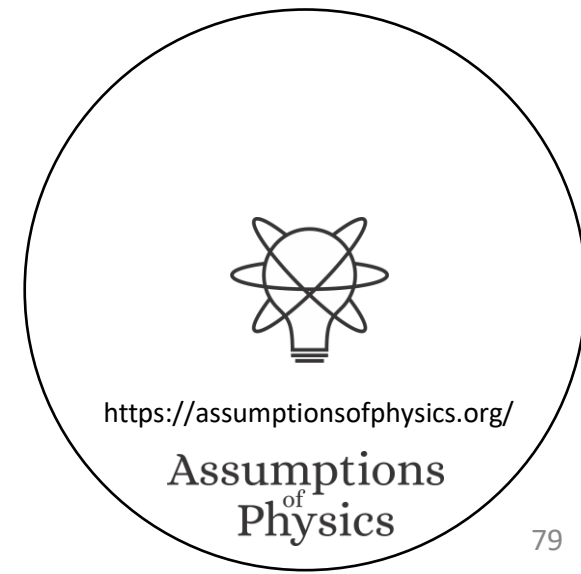
| Before | On | After |
|--------|----|-------|
| T      | F  | F     |
| F      | T  | F     |
| F      | F  | T     |
| T      | T  | F     |
| F      | T  | T     |
| T      | T  | T     |

The experimental domain for a quantity is a collection of references



Imagine collecting the references of all possible clocks into a single logical structure. What are the necessary and sufficient conditions such that they identify a point on the real line?

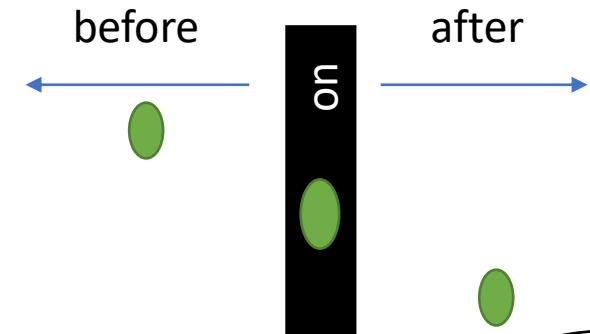
Intuitively, we would need clocks at higher and higher resolutions, all perfectly synchronized, ...



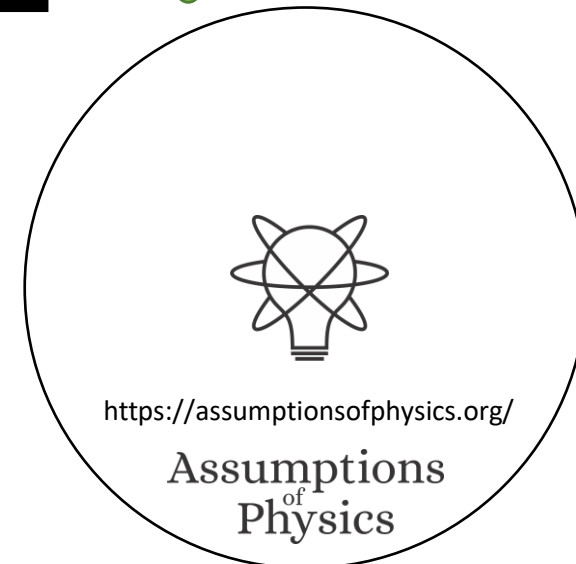
# 1. Strict references

A reference is strict if before/on/after are mutually exclusive

| Before | On | After |
|--------|----|-------|
| T      | F  | F     |
| F      | T  | F     |
| F      | F  | T     |



Physically, the extent of what we measure is assumed to be smaller than the extent of our reference

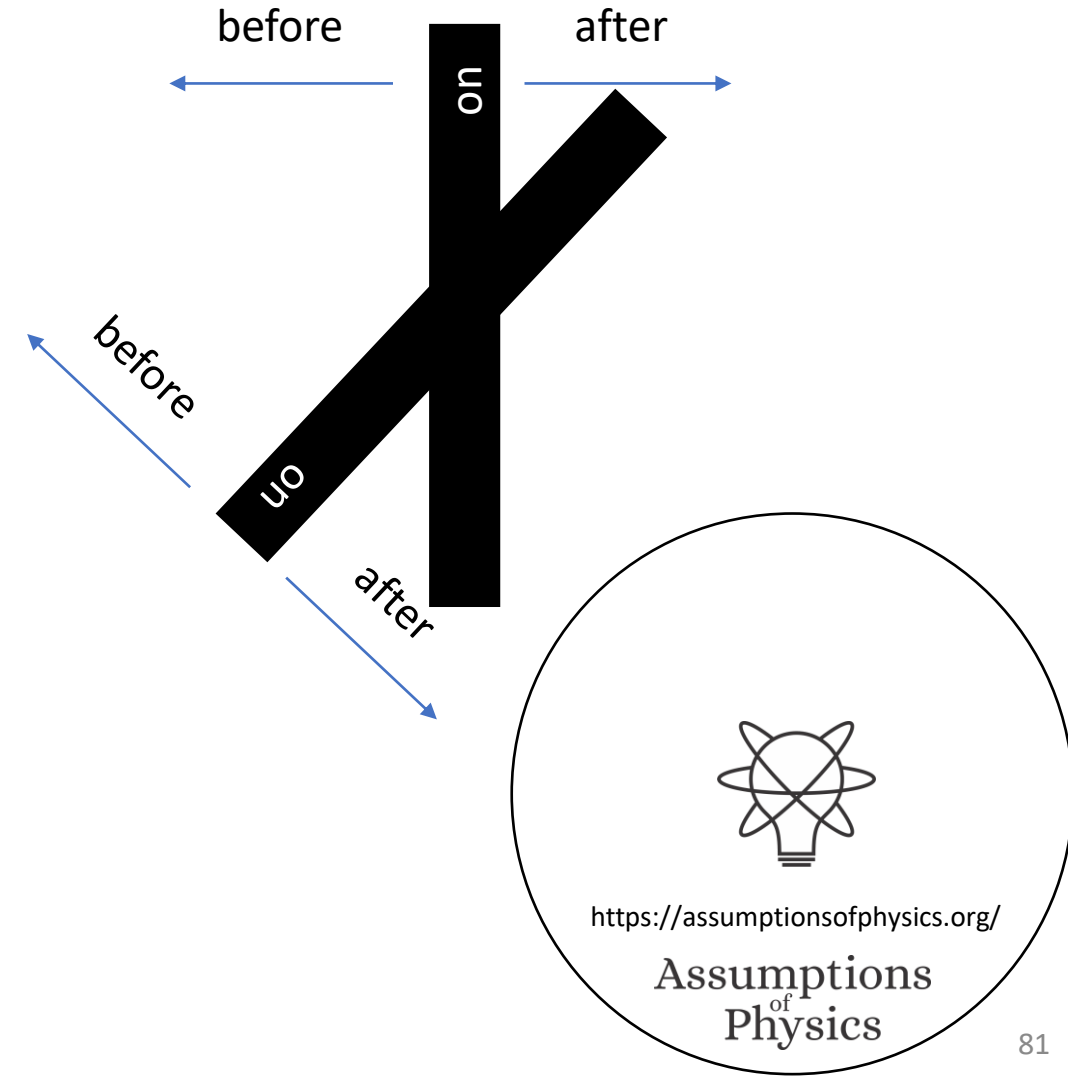




# Multiple references

Without further constraints, references would not lead to a linear order

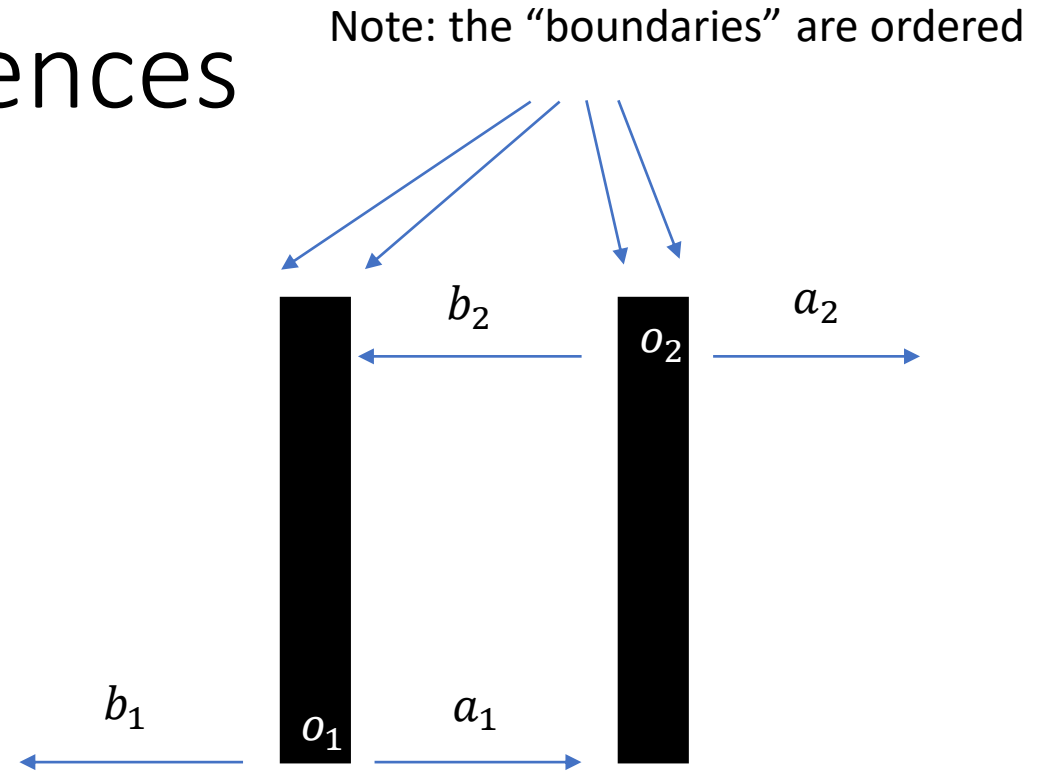
|       | $b_2$ | $o_2$ | $a_2$ |
|-------|-------|-------|-------|
| $b_1$ | ✓     | ✓     | ✓     |
| $o_1$ | ✓     | ✓     | ✓     |
| $a_1$ | ✓     | ✓     | ✓     |



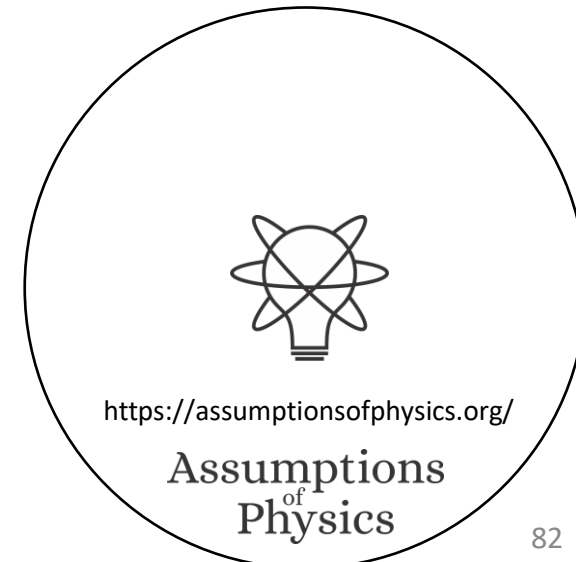
# Multiple references

The fact that a reference is “before” or “after” another is captured by the statements’ logical relationship

|       | $b_2$ | $o_2$ | $a_2$ |
|-------|-------|-------|-------|
| $b_1$ | ✓     | ✗     | ✗     |
| $o_1$ | ✓     | ✗     | ✗     |
| $a_1$ | ✓     | ✓     | ✓     |



Order relationship between references is too restrictive



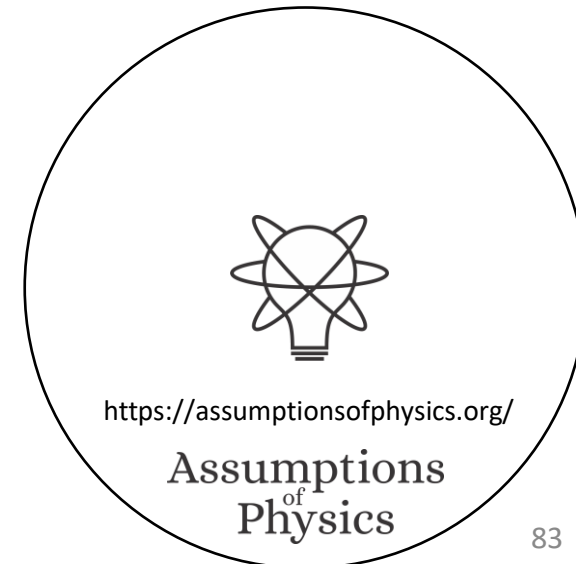
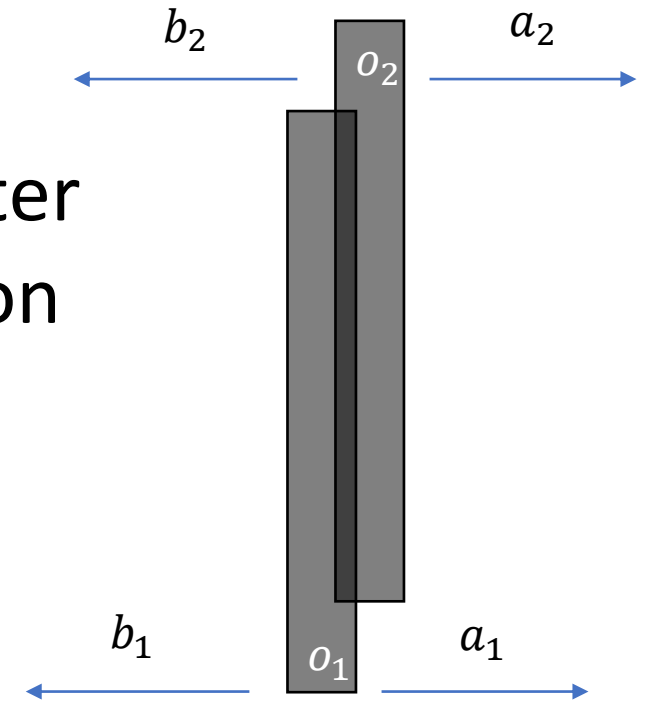
## 2. Aligned references

Two references are aligned if the before and not-after statement can be ordered by narrowness/implication

For example,  $b_1 \preceq b_2 \preceq \neg a_1 \preceq \neg a_2$

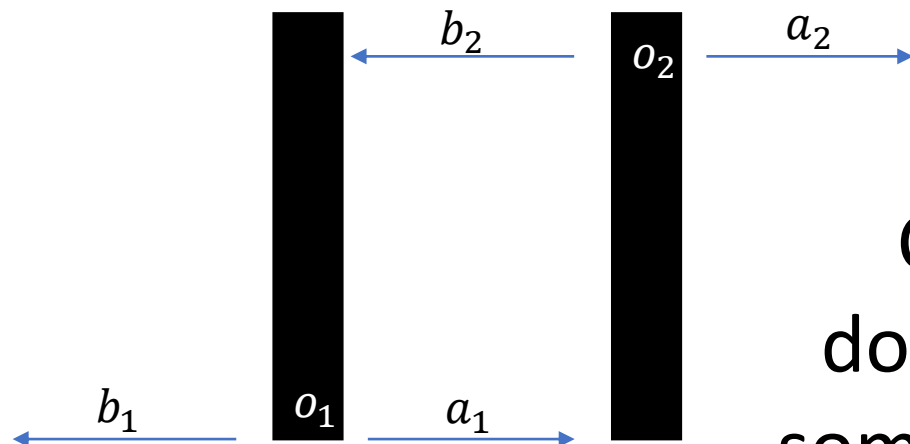
$\preceq$  Means that if the first statement is true  
then the second statement will be true as well

That is, the first statement is narrower, more specific

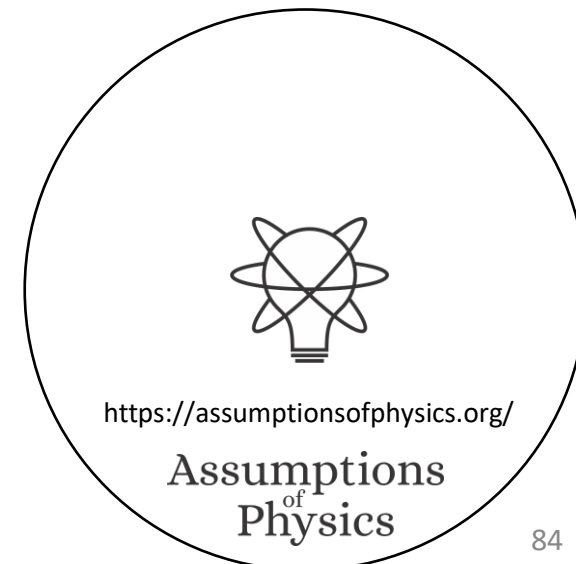
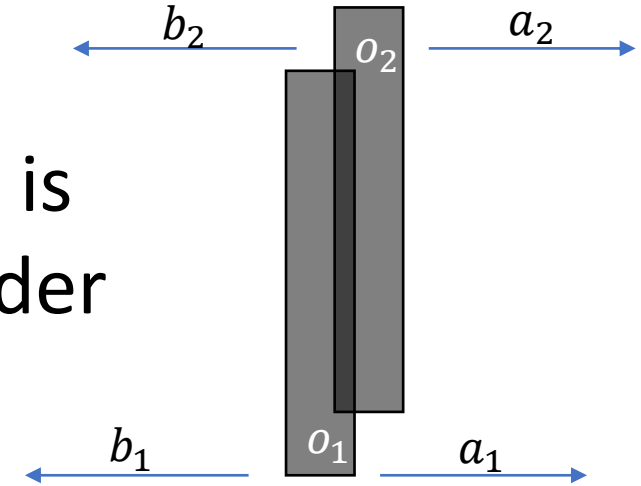


# Filling the whole region

If two different references overlap, we can't say one is before the other: we can't fully resolve the linear order



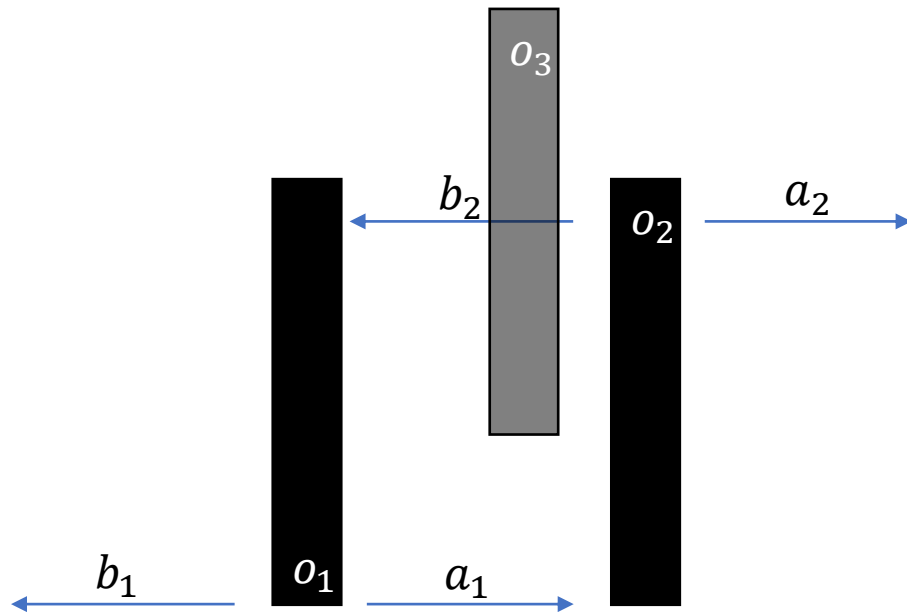
Conversely, if two references don't overlap and there can be something in between, we must be able to put a reference there



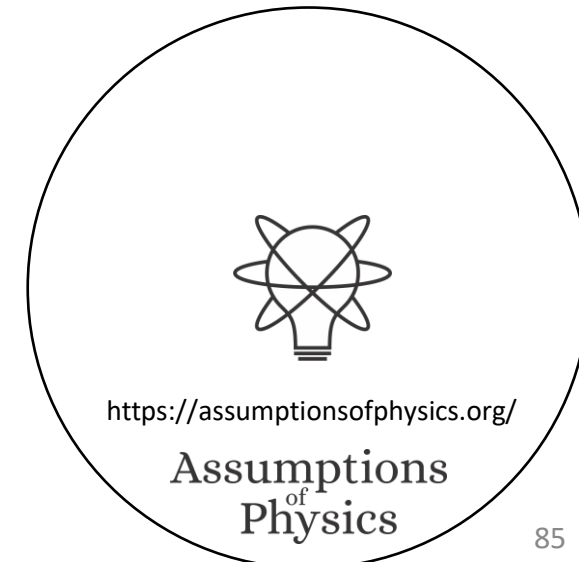
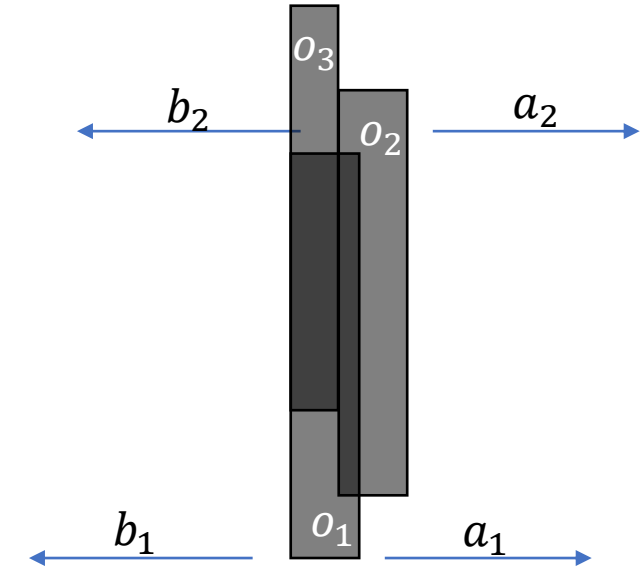
### 3. Refinable references

A set of references is refinable if we can address the previous two problems and resolve the full space

If two references overlap, we can find a reference that refines the overlap



If something can be found between two references, then there must be another reference in between



the possibilities themselves can be ordered, and how this ordering, in the end, is ultimately characterized by statement narrowness: 10 is less than 42 because "the quantity is less than 10" is narrower than "the quantity is less than 42".

As the defining characteristic for a quantity is the ability to compare to values, then the values must be ordered in some fashion from smaller to greater. Therefore, given two different values, one must be before the other. Mathematically, we call this order an order with such a characteristic as we can imagine the elements positioned along a line. Note that vectors are not linearly ordered; no direction is greater than the other. Therefore, in this context, a vector will not strictly be a quantity but a collection of quantities. We also have to define how this order can be experimentally verified. The idea is that we should, at least, be able to verify that the value of a given quantity is before or after a set value. This allows us to construct domains such as "the mass of the electron is  $1.1 \pm 0.5 \times 10^{-31}$ " which we take to be equivalent to "the mass of the electron is more than  $10.5 \times 10^{-32}$  kg but less than  $3.1 \times 10^{-31}$  kg". For integers, this also allows us to verify particular numbers as "the earth is one natural number" is equivalent to the "the earth has more than one natural satellite and fewer than two". Therefore we will define the order topology as the one generated by set of the type  $(a, \infty)$  and  $(-\infty, a)$ .

The quantity, then, is an ordered property with the order topology.

- Definition 3.4.** A linear order on a set  $Q \in \mathcal{Q}$  is a relationship  $\leq : Q \times Q \rightarrow \mathbb{B}$  such that:
- (antisymmetry) if  $q_1 \leq q_2$  and  $q_2 \leq q_1$  then  $q_1 = q_2$
  - (transitivity) if  $q_1 \leq q_2$  and  $q_2 \leq q_3$  then  $q_1 \leq q_3$
  - (total) at least  $q_1 \leq q_2$  or  $q_2 \leq q_1$

A set together with a linear order is called a linearly ordered set.

**Definition 3.5.** Let  $(Q, \leq)$  be a linearly ordered set. The order topology is the topology generated by the collection of sets of the form:

- $(a, \infty) := \{q \in Q \mid q < a\}$
- $(-\infty, a) := \{q \in Q \mid q > a\}$

**Definition 3.6.** A property for an experimental domain  $D_X$  is a linearly ordered proper formally, it is a triple  $(Q, \leq, \alpha)$  where  $(Q, \leq)$  is a property  $\leq : Q \times Q \rightarrow \mathbb{B}$  as in the first case and  $Q$  is a topological space with the order topology. We require that:

- $\alpha$  is proper, the quantity values are first attributes used to label the different cases.
- $\alpha$  correspond to the degree of a set of words ordered alphabetically.

The units are not captured by the numbers themselves; they are captured by the function  $\alpha$ .

As for properties, the quantity values are first attributes used to label the different cases (more granular) and as "measure" (e.g. mass, quantity, etc.). In the most meaning of "unit" that is explained here, the units are not captured by the numbers themselves; they are captured by the function  $\alpha$ . For example, the mass of the electron is  $1.1 \pm 0.5 \times 10^{-31}$  kg could mean that the mass is in kg, instead of some arbitrary base that would construct a different meaning of the order meaning in units by themselves. The units are not captured by the numbers themselves; they are captured by the function  $\alpha$ .

When remaining the dictionary, we note the fact that we can experimentally tell whether the word is looking for a before or after in the way mentioned above.

which returns elements of the original set and therefore reduces to countable conjunctions. Therefore, when forming  $D_X$  the only new elements will be the countable disjunctions.

Consider two countable sets  $B_1, B_2 \subseteq \mathbb{B}$ . Their disjunctions  $b_1 \vee b_2$  and  $b_1 \wedge b_2$  represent the narrowest statement that is broader than all elements of the respective set. Suppose that for each element of  $B_1$  we can find a broader element in  $B_2$ . Then  $b_2$  being broader than all elements of  $B_1$  will be broader than all elements of  $B_1$ . But since  $b_1$  is the narrowest element that is broader than all elements in  $B_1$ , we have  $b_1 \leq b_2$ . Conversely, suppose there is some element in  $B_2$  for which there is no broader element in  $B_1$ . Since the initial set is fully ordered, it means that that element of  $B_2$  is broader than all the elements in  $B_2$ . This means that element is broader than  $b_2$  and since  $b_1$  is broader than all elements in  $B_1$ , we have  $b_1 \leq b_2$ . Therefore the domain  $D_X$  generated by  $B_1$  is linearly ordered by narrowness.

Now we show that  $(D_X, \leq)$  is linearly ordered. The basic  $B_i$  is linearly ordered by broadness because the negation of its elements are part of  $B$  and are ordered by narrowness. Note that broadness is the opposite order of narrowness and therefore a set linearly ordered by one is linearly ordered by the other. Therefore  $B_i$  is also linearly ordered by narrowness and so is  $D_X$  by the previous argument. Therefore  $D_X$  is ordered by broadness.

To show that  $D_X \cup D_X$  is linearly ordered by narrowness, we only need to show that the countable disjunction of elements of  $B_i$  are either narrower or broader than countable conjunctions of the negations of elements of  $B_i$ . Let  $B_1 \in \mathcal{B}_1$  and  $A_2$  disjunction  $b_1 \vee b_2$  represents the narrowest statement that is broader than all of  $B_1$  while the conjunction  $\neg a_1 \wedge \neg a_2 = \neg(a_1 \vee a_2)$  represents the broadest statement is narrower than all elements of  $\neg(A_2)$ . Suppose that for one element of  $\neg(A_2)$  find a broader statement in  $B_1$ . Then  $b_1$  being broader than all elements in  $B_1$  is broader than that one element in  $\neg(A_2)$ . But since  $\neg a_1 \wedge \neg a_2$  is narrower than all of  $\neg(A_2)$ , we have  $\neg a_1 \wedge \neg a_2 \leq b_1$ . Conversely, suppose that for no element of  $\neg(A_2)$  we can find a broader statement in  $B_1$ . As  $B_1$  is linearly ordered, it means that all elements in  $B_1$  are broader than all elements in  $\neg(A_2)$ . This means that all elements in  $\neg(A_2)$  are broader than  $b_1$  and therefore  $b_1 \leq \neg a_1 \wedge \neg a_2$ . Therefore  $D_X$  is linearly ordered by narrowness.

**Theorem 3.6.** (Domain ordered theorem). An experimental domain  $D_X$  is ordered if and only if it is the combination of two experimental domains  $D_1 = D_X$  and  $D_2 = D_X$ .

- (i)  $D_1 = D_X \cup D_2$  is linearly ordered by narrowness
- (ii) all elements of  $D_1$  are part of a pair  $(s_i, \neg s_i)$  such that  $s_i \in D_1, \neg s_i \in D_2$  and  $s_i$  is the immediate successor of  $\neg s_i$  in  $D_1$  or  $s_i = \neg s_i$
- (iii) if  $s_i$  has an immediate successor, then  $\neg s_i \in D_2$

**Proof.** Let  $D_X$  be a naturally ordered experimental domain. Let  $B_1$  and  $B_2$  be in  $\mathcal{B}_1$  and  $\mathcal{B}_2$  respectively. Let  $B_1 \in \mathcal{B}_1$  and  $D_1$  be the domain generated by  $B_1$ . Let  $D_2$  be the domain generated by  $B_2$ . If generated by  $D_1$  and  $D_2$  by finite conjunction and countable disjunction and  $D_X = D_1 \cup D_2$ .

that allows us to map statements to numbers and vice-versa.

As we want to understand quantities better, we concentrate on those experimental domains that are fully characterized by a quantity. For example, the domain for the mass of a system will be fully characterized by a real number greater than or equal to zero. Each possibility will be identified by a number which will correspond to the mass expressed in a particular unit, say in kg. As the values of the mass are ordered, we can also say that the possibilities themselves are ordered. Thus, the "the mass of the system is  $t$ " kg precedes "the mass of the system is  $t'$ " kg. This ordering of the possibilities will be linked to the natural topology as "the mass of the system is less than  $t$ " kg is "the disjunction of all possibilities that are below a particular possibility, in a veridical statement".

We call a natural order for the possibility a linear order on them such that the order is linked to the natural topology as "the mass of the system is less than  $t$ " kg if and only if it is naturally ordered and that quantity is ordered in the same way; it is order isomorphic. In other words, we can only assign a quantity to an experimental domain if it is fully linked with a natural ordering of the same type.

mass of the system is more than  $t$  kg  $K_2^t$  is also ordered by narrowness but with the reverse ordering of the possibilities values. These are the very statements whose veridical sets define the order topology and therefore jointly constitute the basis for the experimental domain.

Now consider the statement  $s_1$  "the mass of the system is less than or equal to  $t$ " kg with  $t_1$  "the mass of the system is less than  $t'$ " kg. We have  $t_1 \leq t$ . In fact, if we replace the value in  $s_1$  with anything less than  $t$  kg we'll still have  $s_1 \leq t_1$ . Instead we use a value greater than  $t$  kg we'll still have  $s_1 \leq t_1$ . In other words, if we call  $B$  the sets includes both the less-than-or-equal and less-than statements that are ordered by narrowness. Then "the mass of the system is less than or equal to  $t$ " kg is equivalent to "the mass of the system is greater than  $t'$ ". In other words,  $B_1 = B_2$ . Note that all the statements like "the mass of the system is less than or equal to  $t$ " kg and "the mass of the system is greater than  $t'$ " kg are all linearly ordered by narrowness.

This ordering of  $B$  can be further characterized. Note that  $s_1$  "the mass of the system is less than or equal to  $t$ " kg is the immediate successor of  $s_2$  "the mass of the system is less than  $t'$ " kg. That is, they are different and there can't be any other statement in  $B$  that has an immediate successor of  $s_2$  since they differ by  $t_1$ . In fact, if we replace the value in  $s_2$  with anything less than  $t_1$  since they differ by  $t_1$  we'll get  $s_2$ . Similarly, let  $s_3$  "the mass of the system is less than  $t''$ " kg while its immediate successor is the form "the mass of the system is less than or equal to  $t_1$ " kg. The immediate successor of  $s_3$  is the property associated with  $q_1$ . Therefore statements in  $B$  that have an immediate successor must be in  $B_1$  and well.

The main result is that the above characterization of the basis of the domain is necessary and sufficient to order the possibilities. If an experimental domain has a basis composed of

To prove (i), let us have  $B_1$  and  $B_2$  are linearly ordered by 3.14. We need to show that the linear ordering holds across the set  $D_X$ . Let  $x \in D_X$  consider the two statements " $x \leq x'$ " and " $x \leq x''$ " and " $x \leq x''$ ". As  $x$  is linearly ordered, either  $(x \leq x') \wedge (x \leq x'')$  or  $(x \leq x') \wedge \neg(x \leq x'')$  or  $(x \leq x'') \wedge \neg(x \leq x')$  or  $(x \leq x') \wedge \neg(x \leq x'')$  or  $(x \leq x'') \wedge \neg(x \leq x')$ . Which is ordered by 3.14. The set  $D_X \cup D_X$  is also linearly ordered.

To prove (ii), let  $x \in D_1$ . Take  $s_1 \in D_1$  such that  $\neg s_1$  is the narrowest statement in  $\neg(D_2)$  that is broader than  $s_1$ . We can find  $\neg s_1$  by infinite disjunction  $\neg a_1 \vee \neg a_2 \vee \dots$ . Let  $s_2$  be the set of possibilities compatible with  $\neg s_1$  but not compatible with  $s_1$ . The set  $s_2$  cannot have more than one element, or we could find an element  $s_3 \in s_2$  such that  $s_3 \leq s_2 \leq \neg s_1$ . As  $D_1$  contains one possibility, then  $\neg s_1$  is the immediate successor. If that part is empty then  $s_1 = \neg s_1$ . Similarly, we can start with  $x \in D_2$  and find  $s_1 \in D_2$  such that  $s_1$  is the broadest statement in  $D_1$  that is narrower than  $s_1$ . Let  $X_1$  be the set of possibilities compatible with  $s_1$  but not compatible with  $\neg s_1$ . If  $X_1$  contains one possibility, then  $\neg s_1 = X_1$  is the immediate successor and if  $X_1$  is empty then  $s_1 = X_1$ .

To prove (iii), let  $s_1, s_2 \in D$  such that  $s_2$  is the immediate successor of  $s_1$ . This means we can write  $s_2 = \neg a_1 \vee \dots \vee \neg a_n$  for some  $n \in \mathbb{N}$ . This means  $s_1 \leq \neg a_1$  while  $s_2 \leq \neg a_2$  and therefore  $s_1 \leq \neg a_1$ .

that allows us to map statements to numbers and vice-versa.

As we want to understand quantities better, we concentrate on those experimental domains that are fully characterized by a quantity. For example, the domain for the mass of a system will be fully characterized by a real number greater than or equal to zero. Each possibility will be identified by a number which will correspond to the mass expressed in a particular unit, say in kg. As the values of the mass are ordered, we can also say that the possibilities themselves are ordered. Thus, the "the mass of the system is  $t$ " kg precedes "the mass of the system is  $t'$ " kg. This ordering of the possibilities will be linked to the natural topology as "the mass of the system is less than  $t$ " kg is "the disjunction of all possibilities that are below a particular possibility, in a veridical statement".

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means we can find  $r_2 = (b_1 \wedge b_2 \wedge \neg a_2)$  for some  $b_i \in D_2$  such that  $r_2 < r_1$  and therefore  $\neg a_1 \leq b_1 \wedge b_2$ .

For the third, suppose  $a_1 \in D_1$  and  $b_1 \in D_2$  such that  $\neg a_1 < b_1$ . Then  $r_1 = (\neg a_1 \wedge a_2)$  and  $r_2 = (b_1 \wedge \neg a_2)$  are strict references aligned with the domain such that  $r_2 < r_1$  but  $r_2$  is not an immediate successor of  $r_1$ . This means we can find  $r_3 = (b_1 \wedge \neg a_1 \wedge a_2)$  such that  $r_3 < r_2$  and therefore  $\neg a_1 \leq b_1 \wedge \neg a_2$ .

**Proposition 3.37.** Let  $D$  be an experimental domain generated by a set of finitely aligned strict references. Then all elements of  $D$  are part of a pair  $(s_i, \neg s_i)$  such that  $s_i \in D_1$ ,  $\neg s_i \in D_2$  and  $s_i$  is the immediate successor of  $\neg s_i$  in  $D$  or  $s_i = \neg s_i$ . Moreover if  $s_i$  has an immediate successor, then  $\neg s_i \in D_2$ .

**Proof.** Let  $D$  be an experimental domain generated by a set of finitely aligned strict references. Let  $s_1 \in D_1$ . Let  $A = \{a \in D_2 \mid a \leq s_1\}$ . Let  $s_2 = \neg a_1$ . First we show that  $s_2 \leq \neg a_1$ . We have  $s_1 \wedge \neg a_1 \leq s_1 \wedge \neg a_2 \leq s_1 \wedge \neg a_3 \wedge \dots \leq \neg a_1 \wedge a_2$ . For all  $a \in A$  we have  $s_1 \wedge \neg a \leq \neg a_1 \wedge a_2$  which means  $s_2 \leq \neg a_1$  because of the total order of  $D$ . This means that  $s_1 \wedge \neg a_1 \leq a$  for all  $a \in A$ , therefore  $s_1 \wedge \neg a_1 \leq s_1$  and  $s_2 \leq s_1$ .

Next we show that no statement  $s \in D$  is such that  $s_1 < s < s_2$ . Let  $a \in D_2$  such that  $s_1 < a < \neg a_1$ . Then  $s_1 \wedge a \in D_1$  and therefore  $s_1 \wedge a \leq s_2$ . Therefore we can't have  $s_1 < a < s_2$ . We also can't have  $b \in D_2$  such that  $s_1 < b < \neg a_1$ . By 3.36 we find  $a \in D_2$  such that  $s_1 < a < \neg a_1$  which was ruled out. So there are two cases. Either  $s_1 \wedge \neg a_1 \leq s_2$  or  $s_1 < \neg a_1$ . We also have  $s_1 \wedge \neg a_1 \leq s_1$  and  $s_2 \leq s_1$ .

The same reasoning can be applied starting from  $s_2 \in D_2$  to find  $a \in D_1$  such that  $s_1$  is the immediate predecessor of  $\neg s_1$  or an equivalent statement. This shows that all elements of  $D$  are paired.

To show that if a statement in  $D$  has a successor then it must be a before statement, let  $s_1, s_2 \in D$  such that  $s_2$  is the immediate successor of  $s_1$ . By 3.36, in all cases where  $s_1 \in D_1$  and  $s_2 \in D_2$  we can always find another statement between the two. Then we must have that  $s_1 \in D_2$  and  $s_2 \in D_1$ .

**Theorem 3.38.** (Reference ordering theorem). An experimental domain is naturally ordered if and only if it can be generated by a set of finitely aligned strict references.

**Proof.** Suppose  $D_X$  is a naturally ordered domain generated by a set of finitely aligned strict references. Then by 3.34 and 3.37 the domain satisfies the requirement of theorem 3.16 and therefore is naturally ordered.

Now suppose  $D_X$  is naturally ordered. Define the set of  $B_1$  elements  $B_1$  and  $D$  as in 3.12. Let  $R = \{(b_1 \wedge \neg a_1 \wedge a_2) \in B_1, a_2 \in B_2, b_1 < a_2\}$  be the set of all references constructed from the basis. First let us verify they are references. The before and after statements are veridical since they are part of the basis. The on statement  $\neg b_1 \wedge a_2$  is also a contradiction since  $b_1 < \neg a_1$  means  $b_1 \wedge a_2$  and  $b_1 \wedge a_2$  on the statement is broader than  $\neg b_1 \wedge a_2$  as they are equivalent and it is broader than  $b_1 \wedge a_2$  as it is a contradiction since  $b_1 < \neg a_1$ . Therefore  $R$  is a set of references. Since the before and after statements of  $R$  coincide with the basis of the domain,  $D_X$  is generated by  $R$ .

- we can verify whether the object is before or after the reference; and  $a$  and  $a'$  are veridical statements
- the object can be on the reference:  $a \leq a'$
- it's not before or after, it's also on the reference:  $\neg b_1 \wedge a_2$
- it's before and after, it's also on the reference:  $b_1 \wedge a_2$

A beginning reference has nothing before it. That is,  $b_1 = \perp$ . An ending reference has nothing after it. That is,  $a_2 = \top$ . A terminal reference is both beginning or ending.

Proof. By definition, we have  $\neg b_1 \wedge a_2 \leq 0$  and by 1.23  $\neg(b_1 \wedge a_2) \vee 0 \leq \top \wedge b_1 \vee a_2$ .

**Definition 3.19.** A reference  $r_1 = (b_1, a_1)$  is finer than another reference  $r_2 = (b_2, a_2)$  if  $b_1 \leq b_2$ ,  $a_1 \leq a_2$  and  $a_1 \geq a_2$ .

**Corollary 3.20.** The finer relationship between references is a partial order.

**Proof.** As the finer relationship is directly based on narrowness, it inherits its reflexivity, antisymmetry and transitivity properties and is therefore a partial order.

**Definition 3.21.** A reference is strict if its before, and/or after statements are incompatible. Formally,  $r = (b, a)$  is strict if  $b \wedge a$  and  $\neg b \wedge \neg a$ . A reference is loose if it is not strict.

**Remark.** In general, we can't turn a loose reference into a strict one. The on statement can be made strict by replacing it with  $\neg b \wedge a$ . This is possible because it is not required to be veridical. The before (and after) statements would need to be replaced with statements like  $b \wedge a$ , which are not in general verifiable because of the negation.

To measure a quantity we will have many references one after the other; a ruler will have many marks, a scale will have many reference weights, a clock will keep ticking. What does it mean that a reference comes after another in terms of the before/after statements? If reference  $r_1$  is before reference  $r_2$  we expect that if the value measured is before the first it will also be before the second, and if it is after the second it will also be after the first. Note that this is not enough, though, because as references have an extent they may overlap. And if they overlap one can't be after the other. To have an ordering properly defined we must have that the first reference is strictly before the second. That is, if the value measured

is on the first it will be before the second.

Mathematically, this type of order before and strictly after. It does not.

One may be tempted to define the order by the before/after statements for the references refining the references and, in the original or refined references, not the original or refined references.

**Definition 3.22.** A reference is the first can be made an after the first can be made an after the first.

- Proposition 3.23.** Reference order
- irreflexivity: not  $r < r$
  - transitivity: if  $r_1 < r_2$  and  $r_2 < r_3$  then  $r_1 < r_3$

and therefore a strict partial order.

**Proof.** For irreflexivity, since  $r$  and therefore  $b \vee \neg b \vee a \vee \neg a$ . Therefore irreflexive.

**Proof.** Let  $r_1 < r_2$ . By 3.27, we have  $\neg a_1 \leq a_2$ . Conversely, let  $a_1 \leq a_2$ . Then  $a_1 \wedge \neg a_2$ . Because the references are strict,  $\neg a_1 \geq b_1 \vee 0$  and  $\neg b_2 \vee a_2 \geq a_2$ . Therefore  $b_1 \vee 0 \leq a_2 \vee a_2$  and  $r_1 < r_2$  by definition.

**Definition 3.24.** A reference is the immediate predecessor of another if nothing can be found before the second and after the first. Formally,  $r_1, r_2$  and  $a_1 \leq a_2$ . Two references are consecutive if one is the immediate successor of the other.

**Proposition 3.30.** Let  $r_1 = (b_1, a_1)$  and  $r_2 = (b_2, a_2)$  be two references. If  $r_1$  is immediately before  $r_2$  then  $b_1 \leq b_2$ .

**Proof.** Let  $r_1$  be immediately before  $r_2$ . Then  $a_1 \leq a_2$  which means  $b_2 \leq \neg a_1$ . By 3.27 we also have  $\neg a_1 \leq b_1$ . Therefore  $b_2 \leq b_1$ .

**Proposition 3.31.** Let  $r_1 = (b_1, a_1)$  and  $r_2 = (b_2, a_2)$  be two strict references. Then  $r_1$  is immediately before  $r_2$  if and only if  $b_1 \leq a_2$ .

**Proof.** Let  $r_1$  be immediately before  $r_2$ . Then  $b_2 \leq \neg a_1$  by 3.30. Conversely, let  $b_1 \leq a_2$ . Then  $r_1 < r_2$  by 3.28. We also have  $a_1 \leq \neg a_2$ , therefore  $a_1 \leq b_2$  and  $r_1$  immediately before  $r_2$  by definition.

means we can find  $r_3 = (b_1 \wedge b_2 \wedge \neg a_2)$  for some  $b_i \in D_2$  such that  $r_3 < r_2$  and therefore  $\neg a_1 \leq b_1 \wedge b_2$ .

For the third, suppose  $a_1 \in D_1$  and  $b_1 \in D_2$  such that  $\neg a_1 < b_1$ . Then  $r_1 = (\neg a_1 \wedge a_2)$  and  $r_2 = (b_1 \wedge \neg a_2)$  are strict references aligned with the domain such that  $r_2 < r_1$  but  $r_2$  is not an immediate successor of

# Reference ordering theorem

To define an **ordered** sequence (e.g. of “instants”), the references must be (nec/suff conditions):

- Strict – an event is strictly before/on/after the reference (doesn't extend over the tick)
- Aligned – shared notion of before and after (logical relationship between statements)
- Refinable – overlaps can always be resolved

Additionally:

Between any two references we can always have another reference  $\Rightarrow$  **real numbers**

Only finitely many references between any two references  $\Rightarrow$  **integers**

**For time/space, these conditions are idealizations**



<https://assumptionsofphysics.org/>

Assumptions  
of  
Physics

# How does this model break down?

The ticks of a clock have an extent and so do the events (references not strict)

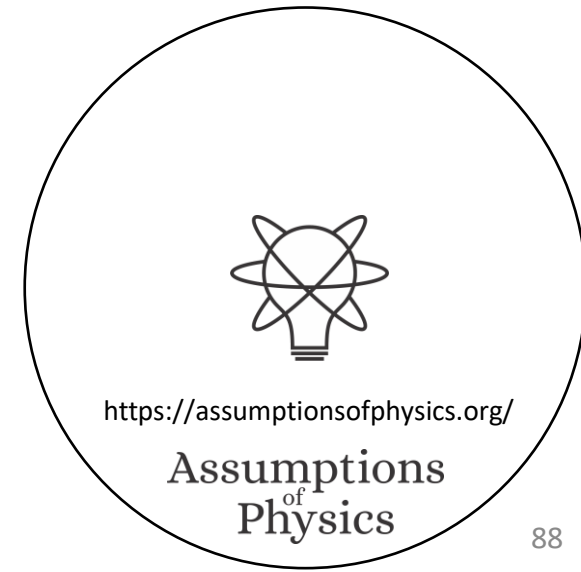
If clocks have jitter, they cannot achieve perfect synchronization (references not aligned)

We cannot make clock ticks as narrow as we want (references not refinable)

**No consistent ordering: no “objective” “before” and “after”**

In relativity, different observers measure time differently, but the order is the same. We should expect this to fail at “small” scales.

A better understanding of space-time means  
creating a more realistic formal model that  
accounts for those failures

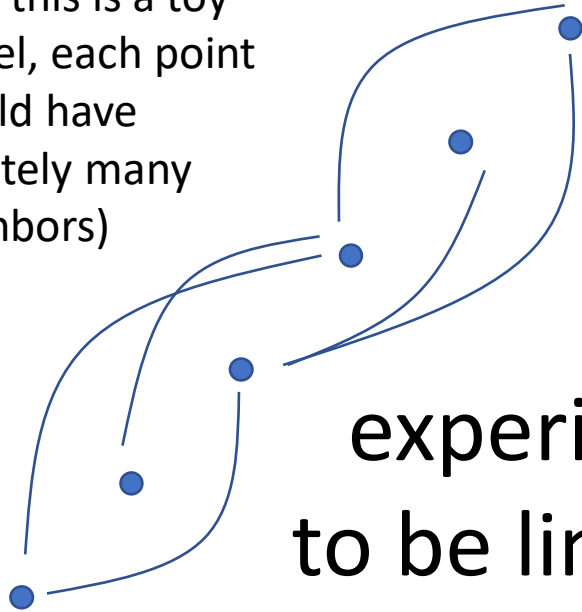




# What type of models should we use?

Hard to say, but we can argue from necessity

(N.B. this is a toy model, each point should have infinitely many neighbors)

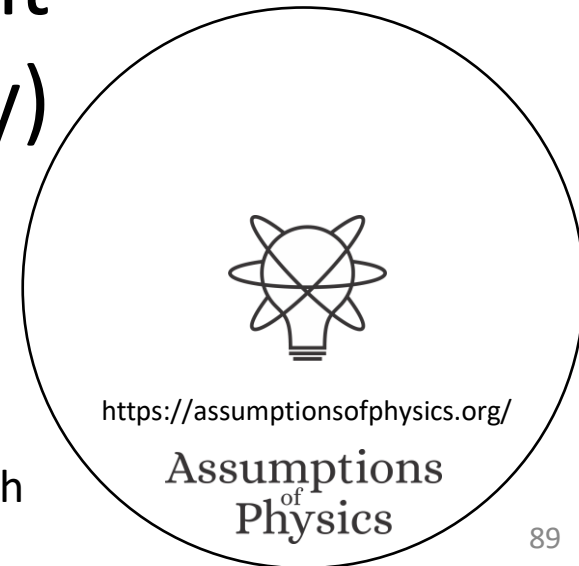


Lack of order at small scales,  
order at large enough scale

What we can distinguish experimentally (i.e. topology) seems to be linked to how precisely we want to distinguish (i.e. geometry)

Current mathematical tools have a hard division between topology and geometry

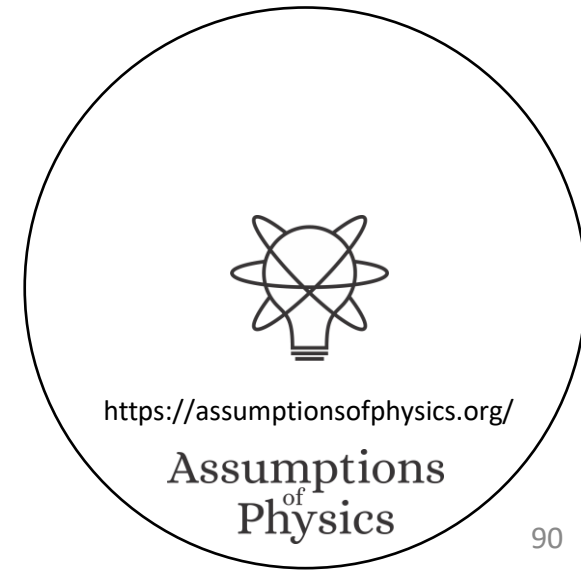
Likely need new math



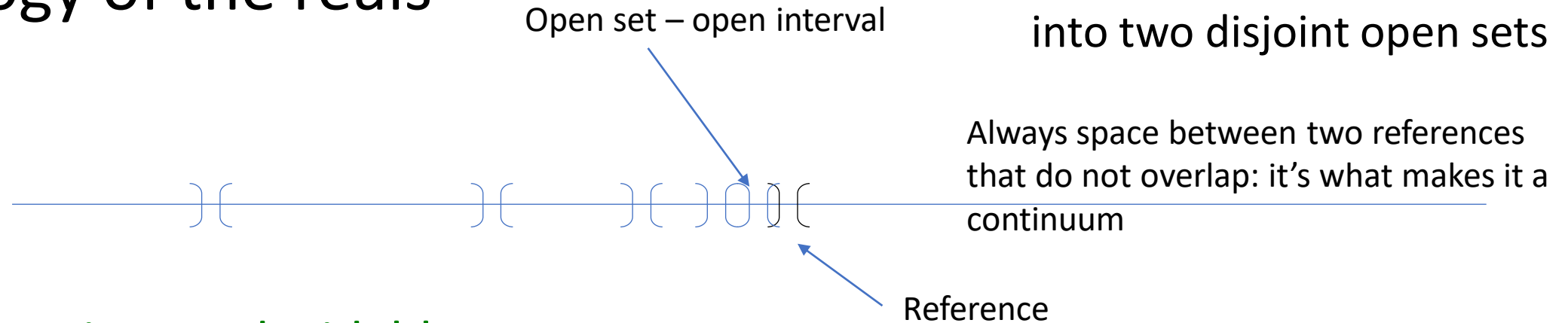
Our reasoning contradicts the expectations of many that time is simply “discrete” at the smallest scale

This intuition is based on the idea that the continuum is like the discrete but “with more points”

This idea (though extremely common in physics) is flawed



# Topology of the reals



No contingent decidable statements

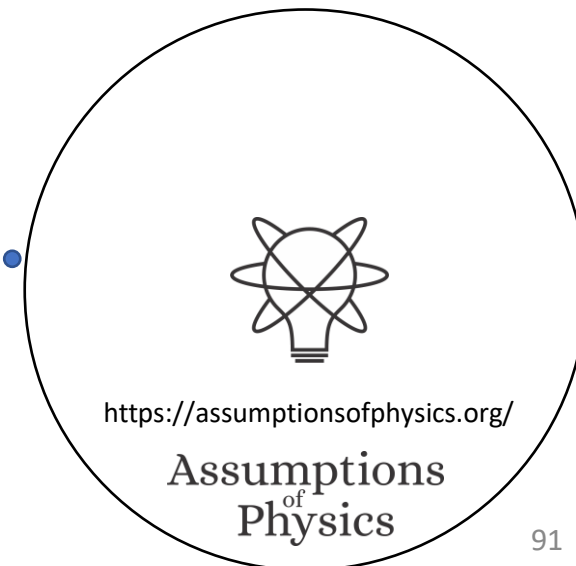
# Topology of the integers

No space between two consecutive references



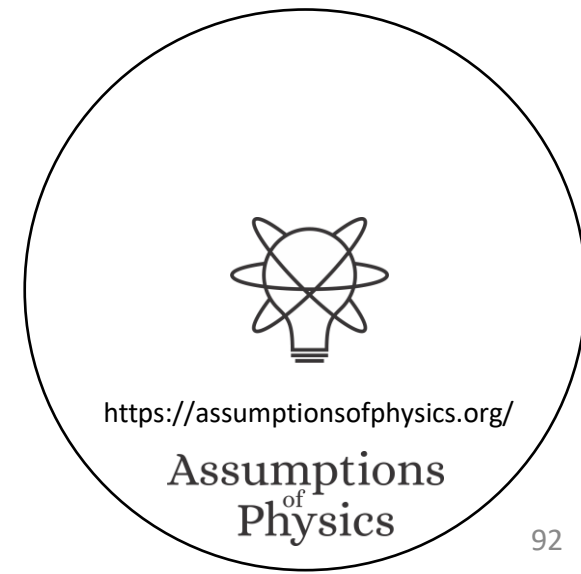
Disconnected: can be divided into two disjoint open sets

All contingent statements are decidable

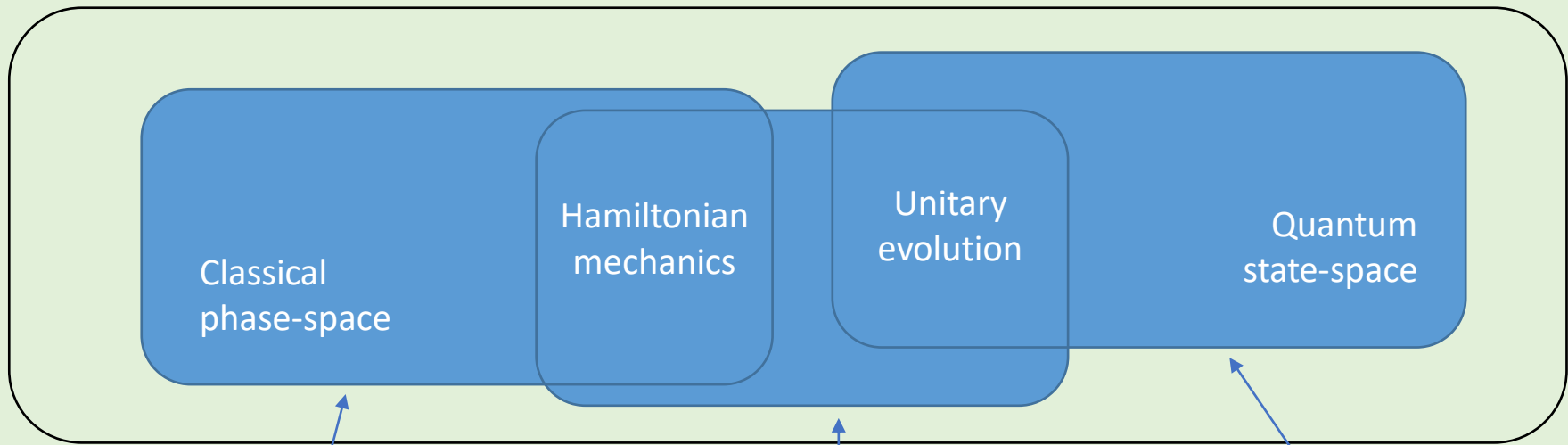


# Takeaway

- Ordering, the defining features of quantities, is a logical structure
  - $3 \leq 5$  precisely because “there are less than 3 items”  $\Leftarrow$  “there are less than 5 items”
- TODOs:
  - Find whether one can construct topological spaces that are not locally metrizable but are “sort of metrizable” on long “distances”



Space of the well-posed scientific theories



## Physical theories

Specializations of the general theory under the different assumptions

## Assumptions

Infinitesimal  
reducibility

Determinism/  
reversibility

Irreducibility

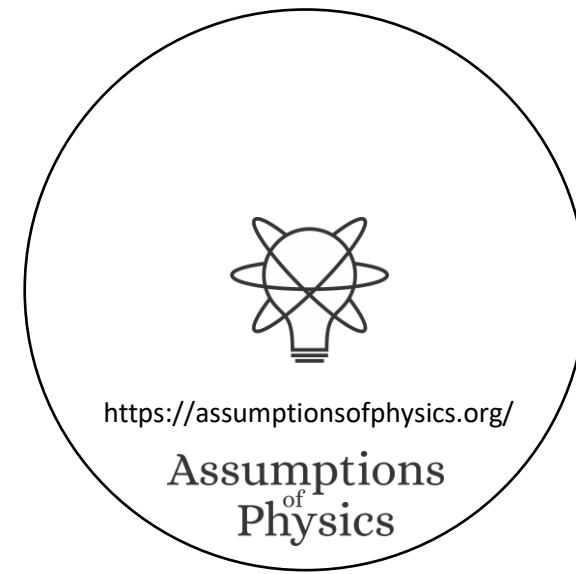
States and processes

Information granularity

Experimental verifiability

## General theory

Basic requirements and definitions valid in all theories



# Information granularity

Logical relationships  $\Leftrightarrow$  Topology/ $\sigma$ -algebra

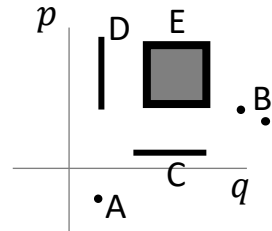
- “The position of the object is between 0 and 1 meters”  $\preceq$  “The position of the object is between 0 and 1 kilometers”
- “The fair die landed on 1”  $\preceq$  “The fair die landed on 1 or 2”
- “The first bit is 0 and the second bit is 1”  $\preceq$  “The first bit is 0”

Granularity relationships  $\Leftrightarrow$  Geometry/Probability/Information

- “The position of the object is between 0 and 1 meters”  $\preceq$  “The position of the object is between 2 and 3 kilometers”
- “The fair die landed on 1”  $\preceq$  “The fair die landed on 3 or 4”
- “The first bit is 0 and the second bit is 1”  $\preceq$  “The third bit is 0”

$\Rightarrow$  Measure theory, geometry, probability theory, information theory,  
... all quantify the level of granularity of different statements

A partially ordered set allows us to compare size at different level of infinity and to keep track of incommensurable quantities (i.e. physical dimensions)



$$A \preceq B \preceq C \preceq E$$

$$C \not\preceq D$$

$$D \not\preceq C$$

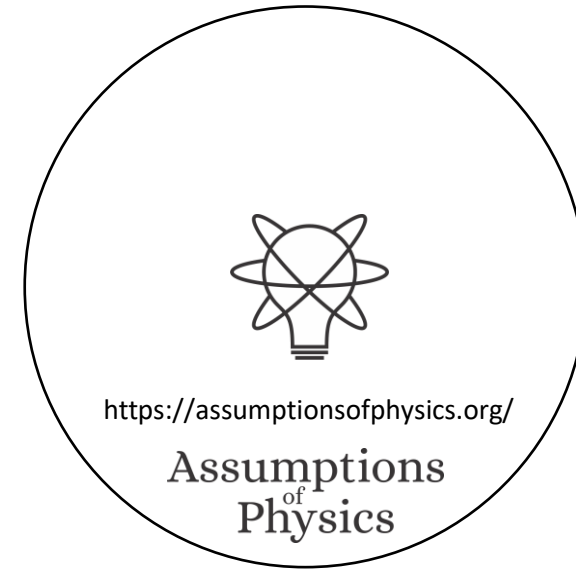
Once a “unit” is chosen, a measure quantifies the granularity of another statement with respect to the unit

$$\mu_u: \bar{\mathcal{D}} \rightarrow \mathbb{R}$$

$$\mu_u(u) = 1$$

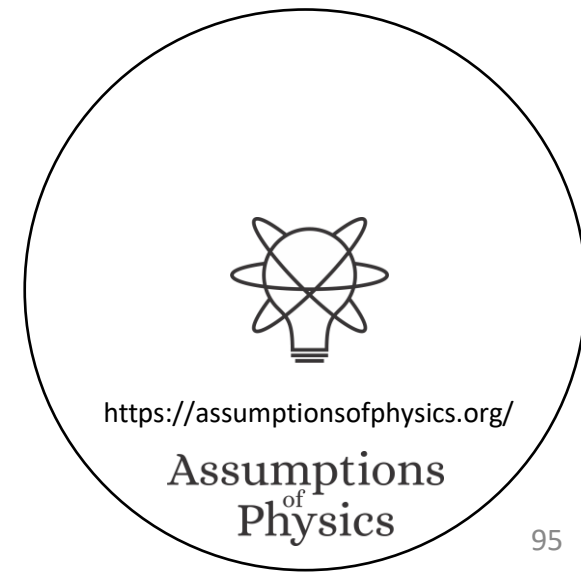
$$s_1 \preceq s_2 \Rightarrow \mu_u(s_1) \leq \mu_u(s_2)$$

$$\mu_u(s_1 \vee s_2) = \mu_u(s_1) + \mu_u(s_2) \text{ if } s_1 \text{ and } s_2 \text{ are incompatible}$$



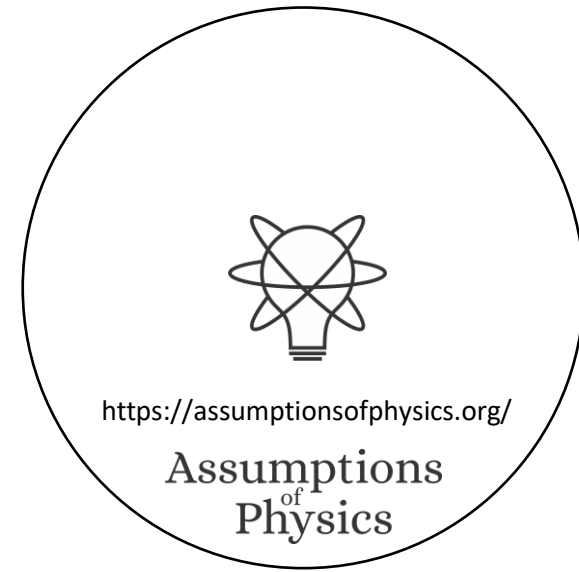
# Takeaway

- Only rough ideas at this point
- TODOs:
  - Find “right” basic axioms by reverse engineering measure theory

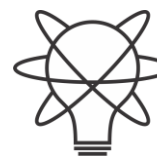


# Wrapping it up

- We have a good foundational layer done that recovers topological structures from requiring experimental verifiability
  - Though some elements can still be developed and better understood
- The layer to describe more quantitative elements (geometry, probability, ...) is still to be understood







<https://assumptionsofphysics.org/>

**Assumptions  
of  
Physics**