Assumptions of Physics Summer School 2024

Foundational Structures

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https://assumptionsofphysics.org



Main goal of the project

Identify a handful of physical starting points from which the basic laws can be rigorously derived

For example:





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This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based \Rightarrow scientific theory must be characterized by experimentally verifiable statements \Rightarrow topology and σ -algebras





If physics is about creating models of empirical reality, the foundations of physics should be a theory of models of empirical reality

Requirements of experimental verification, assumptions of each theory, realm of validity of assumptions, ...





Reverse physics: Start with the equations, reverse engineer physical assumptions/principles

Found Phys **52**, 40 (2022)



Goal: find the right overall physical concepts, "elevate" the discussion from mathematical constructs to physical principles

Physical mathematics: Start from scratch and rederive all mathematical structures from physical requirements



Goal: get the details right, perfect one-to-one map between mathematical and physical objects



This session

Physical Mathematics: Foundational Structures

Assumptions of Physics, Michigan Publishing (v2 2023)



Physics

Formal system for physics



Formal system:

e.g. Euclidean geometry

Basic objects that are taken as-is, without definition in terms of other objects

formal language

primitive notions

Symbols and rules to write sentences in the formal system

axioms

Statements about primitive objects that are to be taken as true

E.g. Points and lines

E.g. A, B, C for points \overline{AB} for segment

E.g. Given two points, there is a line that joins them



Formal system for all of mathematics:

Sets + first-order logic + Zermelo–Fraenkel axioms (+ axiom of choice)

Formal system for all of physics:

???



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Problems in formalizing physical concepts



Guiding principle What should our primitive "informal" notion be? Principle of scientific objectivity: science is universal, non-contradictory and evidence based.

Universal \rightarrow same for everybody

Suggest logic as fundamental ...

like mathematics!

Non-contradictory \rightarrow something is either true or false

Evidence based \rightarrow truth is determined experimentally

... with some extensions

⇒ Logic of experimentally verifiable statements!



Not "verifiable statements"

Chocolate tastes good (not universal)

It is immoral to kill one person to save ten (not universal and/or evidence based)

The number 4 is prime (not evidence based)

This statement is false (not non-contradictory)

The mass of the photon is exactly 0 eV (not verifiable due to infinite precision)

"Verifiable statements"

The mass of the photon is less than 10^{-13} eV

If the height of the mercury column is between 24 and 25 millimeters then its temperature is between 24 and 25 Celsius

If I take 2 \pm 0.01 Kg of Sodium-24 and wait 15 \pm 0.01 hours there will be only 1 \pm 0.01 Kg left

A scientific theory needs "at least" the concept of a verifiable statement: good primitive notion



Takeaways

- A good part of physics must remain informal
- Formal part is "precise" because it represents only an idealized part
- Pragmatic considerations as to what is formalized
- We take verifiable statements as the basic building blocks of our formal system



Space of the well-posed scientific theories



Physical theories

Specializations of the general theory under the different assumptions

Assumptions

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Axioms of logic



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Axiom 1.2 (Axiom of context). A statement s is an assertion that is either true or false. A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is



defined a function truth : $S \to \mathbb{B}$.

Axiom 1.4 (Axiom of possibility). A possible assignment for a logical context S is a map $a: S \to \mathbb{B}$ that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$ such that truth $\in \mathcal{A}_{\mathcal{S}}$. A map $a: \mathcal{S} \to \mathbb{B}$ is a possible assignment for \mathcal{S} if $a \in \mathcal{A}_{\mathcal{S}}$.

Axiom 1.9 (Axiom of closure). We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let $S \subseteq S$ be a set of statements and $f_{\mathbb{B}}: \mathbb{B}^S \to \mathbb{B}$ an arbitrary function from an assignment of S to a truth value. Then we can always find a statement $\bar{s} \in S$ that depends on S through $f_{\mathbb{R}}$.





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Axiom 1.2 (Axiom of context). A statement s is an assertion that is either true or false. A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is defined a function truth : $S \rightarrow \mathbb{B}$.



Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed



Each axiom/definition has two parts:

- Informal part: tells us what elements in the physical world we are characterizing
- Formal part: how the elements are characterized mathematically

Each axiom/definition has a justification: argues why the mathematical characterization follows from the physical one



Axiom 1.2 (Axiom of context). A *statement* s *is an assertion that is either true or false.* A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is defined a function truth : $S \to \mathbb{B}$.



Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value A priori we only assume these objects exist simply because we cannot proceed



Axiom: brings objects from the informal to the formal Definition: further specializes formal objects

Axioms/definitions should be formulated so that they are easy to justify... ... not so that they follow trends in mathematics



Axiom 1.2 (Axiom of context). A statement s is an assertion that is either true or false. A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is defined a function truth : $S \rightarrow \mathbb{B}$.



Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed



Physical objects are made "mathematically precise" by throwing out everything that can't be made precise

Syntax, grammar, meaning, ... can't be made precise, so are not part of the formal system

⇒ Statements are primitive objects



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Axiom 1.2 (Axiom of context). A statement s is an assertion that is either true or false. A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is defined a function truth : $S \rightarrow \mathbb{B}$.

Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed

In mathematics, primitive objects (i.e. those that are left unspecified) must be elements of a set. The logical context, then, has two functions:

1) in the formal system, it is the "container" for the primitive objects (i.e. the statements)

2) in the informal system, consistency/semantics/... are properties of groups of statements (i.e. of the context)



Informal part

Formal part

Axiom 1.2 (Axiom of context). A statement s is an assertion that is either true or false. A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is defined a function truth : $S \rightarrow \mathbb{B}$.

Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed



A statement here represents the assertion and not the sentence that declares the assertion. Therefore the translation of a sentence into another language represents the same statement.

Technically, we only assume the existence of valid statements for doing science. Therefore statements are also primitives in the informal system. But if they exist, they must follow the axioms we are going to specify.



Axiom 1.2 (Axiom of context). A statement s is an assertion that is either true or false. A logical context S is a collection of statements with well defined logical relationships. Formally, a logical context S is a collection of elements called statements upon which is defined a function truth : $S \rightarrow \mathbb{B}$.

Justification. As science is universal and non-contradictory, it must deal with assertions that have clear meaning, well-defined logical relationships and are associated with a unique truth value. A priori, we only assume these objects exist, simply because we cannot proceed

The existence of a truth function stems from the assumption of non-contradiction and universality. Every statement must be either true or false for everybody.

SR

Т

Sg

F

...

S₇

Т

hig table	whore	statamants	0 KO	

S

S₅

S₄

S₃

F

Context \Rightarrow big table where statements are columns

S₆

F

S1

truth

S₂





Note: the semantic content constrains the possible combinations of truth values

"that animal is a cat"	"that animal is a mammal"	"that animal is a bird"		
	Т	Т	···· ·	
Т	Т	F		
—_ 	F	т		impossible
— T	F	F		
F	Т	Т	•••	
F	Т	F	•••	
F	F	т		
F	F	F		

The only semantics captured by the formal system is the set of possible combinations of truth values



Axiom 1.4 (Axiom of possibility). A possible assignment for a logical context S is a map $a : S \to \mathbb{B}$ that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set $\mathcal{A}_S \subseteq \mathbb{B}^S$ such that truth $\in \mathcal{A}_S$. A map $a : S \to \mathbb{B}$ is a possible assignment for S if $a \in \mathcal{A}_S$.



Possible assignments are those assignments consistent with the meaning (semantics) of the statements

Context \Rightarrow big table where statements are columns and possible assignments are rows



Definition 1.6. Statements are categorized based on their possible assignments.

- A certain statement, or **certainty**, is a statement \top that must be true simply because of its content. Formally, $a(\top) = \text{TRUE}$ for all possible assignments $a \in \mathcal{A}_S$.
- An impossible statement, or **impossibility**, is a statement \perp that must be false simply because of its content. Formally, $a(\perp) = \text{FALSE}$ for all possible assignments $a \in \mathcal{A}_S$.
- A statement is **contingent** if it is neither certain nor impossible.

Corollary 1.7. A statement $s \in S$ can only be exactly one of the following: impossible, contingent, certain.

"that cat is a mammal"	"that mammal is a cat"	"that mammal is a bird"
Т	Т	F
Т	F	F
certain	contingent	impossible

Certainties and impossibilities have the same truth value in all rows



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- A statement is **contingent** if it is neither certain nor impossible.

Corollary 1.7. A statement $s \in S$ can only be exactly one of the following: impossible, contingent, certain.

Whether a statement is certain or contingent depends on context!

the mass of the electron is 510 \pm 0.5 KeV

Contingent when measuring the mass of the electron

Certain when performing particle identification



Some statements depend on other statements



⇒ possible assignments determine the logical relationship



Equivalent

Т

 \checkmark

×

F

×

 \checkmark

F

 \checkmark

 \checkmark

"that animal has feathers"	"that animal is a bird"	Т
Т	Т	F
T	F	
F	T	
F	F	

"that animal is a cat"	"that animal is black"	Т
т	Т	F
т	F	
F	Т	
F	F	



			_
"that animal is a	"that animal is a		Т
mammal"	bird″	Т	×
T	T	F	✓
т	F		
· · ·	I		
F	Т		
F	F		

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Т

 \checkmark

 \checkmark

F

✓

 \checkmark

Definition 1.15. Two statements s_1 and s_2 are *equivalent* $s_1 \equiv s_2$ if they must be equally true or false simply because of their content. Formally, $s_1 \equiv s_2$ if and only if $a(s_1) = a(s_2)$ for all possible assignments $a \in A_S$.



Corollary 1.16. All certainties are equivalent. All impossibilities are equivalent.

Corollary 1.18. Statement equivalence satisfies the following properties:

- reflexivity: $s \equiv s$
- symmetry: if $s_1 \equiv s_2$ then $s_2 \equiv s_1$
- transitivity: if $s_1 \equiv s_2$ and $s_2 \equiv s_3$ then $s_1 \equiv s_3$

and is therefore an equivalence relationship.

From now on, unless otherwise stated, by statement we mean an equivalence class of statements



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Definition 1.20. Given two statements s_1 and s_2 , we say that:

- s_1 is narrower than s_2 (noted $s_1 \leq s_2$) if s_2 is true whenever s_1 is true simply because of their content. That is, for all $a \in \mathcal{A}_S$ if $a(s_1) = \text{TRUE}$ then $a(s_2) = \text{TRUE}$.
- s_1 is broader than s_2 (noted $s_1 \ge s_2$) if $s_2 \le s_1$.
- s₁ is compatible to s₂ (noted s₁ ≈ s₂) if their content allows them to be true at the same time. That is, there exists a ∈ A_S such that a(s₁) = a(s₂) = TRUE.

The negation of these properties will be noted by \leq , \geq , \neq respectively.



That animal is a mammal \curvearrowright That animal lays eggs

That animal is a cat \leq That animal is a mammal



Proposition 1.23. Statement narrowness satisfies the following properties:

- reflexivity: $s \leq s$
- antisymmetry: if $s_1 \leq s_2$ and $s_2 \leq s_1$ then $s_1 \equiv s_2$
- transitivity: if $s_1 \leq s_2$ and $s_2 \leq s_3$ then $s_1 \leq s_3$

and is therefore a partial order.

Narrowness \leq is related to material implication \rightarrow but:

Material implication is a logical operation that returns a new statement: $a \rightarrow b = \neg a \lor b$ (i.e. NOT(a) OR b)

Narrowness \leq is a binary relationship between statements

The order imposed by narrowness allows us to understand the context as an order theoretic structure



NOTE: AND, OR and NOT (Λ, V, \neg) are operations within the context

Equivalence, narrowness, compatibility, ... (\equiv , \preccurlyeq , \approx , ...) are not: they describe the context (i.e. metalanguage)



Definition 1.8. Let $\bar{s} \in S$ be a statement and $S \subseteq S$ be a set of statements. Then \bar{s} depends on S (or it is a function of S) if we can find an $f_{\mathbb{B}} : \mathbb{B}^S \to \mathbb{B}$ such that

 $a(\bar{\mathsf{s}}) = f_{\mathbb{B}}(\{a(\mathsf{s})\}_{\mathsf{s}\in S})$

for every possible assignment $a \in \mathcal{A}_{\mathcal{S}}$. We say \bar{s} depends on S through $f_{\mathbb{B}}$. The relationship is illustrated by the following diagram:



Axiom 1.9 (Axiom of closure). We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let $S \subseteq S$ be a set of statements and $f_{\mathbb{B}}: \mathbb{B}^S \to \mathbb{B}$ an arbitrary function from an assignment of S to a truth value. Then we can always find a statement $\bar{s} \in S$ that depends on S through $f_{\mathbb{B}}$.

<i>s</i> ₁	<i>s</i> ₂	<i>s</i> ₃		Ī		<i>s</i>	•••
Т	Т	F		Т	•••	Т	
F	F	Т	$f_{\mathbb{B}}$	F	•••	Т	
F	F	Т	s_1 AND (s_2 OR s_3)	F		F	
Т	F	F		F		Т	
т	F	Т		F		F	
				•••			

Not sure whether it is needed as an axiom: the closure may be proven to exist and be unique.



Corollary 1.10. Functions on truth values induce functions on statements. Formally, let I be an index set and $f_{\mathbb{B}} : \mathbb{B}^I \to \mathbb{B}$ be a function. There exists a function $f : S^I \to S$ such that

$$a(f({\mathsf{s}_i}_{i\in I})) = f_{\mathbb{B}}({a(\mathsf{s}_i)}_{i\in I})$$

for every indexed set $\{s_i\}_{i \in I} \subseteq S$ and possible assignment $a \in A_S$.

<i>s</i> ₁	<i>s</i> ₂	s 3	$\xrightarrow{f(s_1, s_2, s_3)}$	Ī	 s	
Т	Т	F		Т	 Т	
F	F	Т	$f_{\mathbb{B}}$	F	 Т	
F	F	Т	s_1 AND (s_2 OR s_3)	F	 F	
Т	F	F		F	 Т	
Т	F	Т		F	 F	


Definition 1.11. The negation or logical NOT is the function $\neg : \mathbb{B} \to \mathbb{B}$ that takes a truth value and returns its opposite. That is: $\neg \text{TRUE} = \text{FALSE}$ and $\neg \text{FALSE} = \text{TRUE}$. We also call negation $\neg : S \to S$ the related function on statements.



<i>s</i> ₁	<i>s</i> ₂	s 3	<i>s</i> ₄	<i>s</i> ₅	<i>s</i> ₆	<i>s</i> ₇	•••
Т	Т	F	Т	F	Т	Т	
F	F	Т	Т	T T F		F	
F	F	Т	F	Т	F	F	
Т	F	F	Т	T F		Т	
Т	F	Т	F	F	F	Т	
	•••						
		_		/			





Definition 1.12. The conjunction or logical AND is the function $\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ that returns TRUE only if all the arguments are TRUE. That is: TRUE \wedge TRUE = TRUE and TRUE \wedge FALSE = FALSE \wedge TRUE = FALSE \wedge FALSE = FALSE. We also call conjunction $\wedge : S \times S \to S$ the related function on statements.

t_1	<i>t</i> ₂	$t_1 \wedge t_2$
Т	Т	Т
Т	F	F
F	Т	F
F	F	F

<i>s</i> ₁	<i>s</i> ₂	<i>s</i> ₃	<i>s</i> ₄	<i>s</i> ₅	<i>s</i> ₆	<i>s</i> ₇		
Т	Т	F	Т	F	Т	Т		
F	F	Т	Т	Т	F	F		
F	F	Т	F	Т	F	F		
Т	F	F	Т	F	F	Т		
Т	F	Т	F	F	F	Т		
				•••	•••	•••		
Λ								



Λ

That animal is a black cat



Definition 1.13. The disjunction or logical OR is the function $\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ that returns FALSE only if all the arguments are FALSE. That is: FALSE \vee FALSE = FALSE and TRUE \vee FALSE = FALSE \vee TRUE = TRUE \vee TRUE = TRUE. We also call disjunction $\vee : S \times S \to S$ the related function on statements.

<i>t</i> ₁	<i>t</i> ₂	$t_1 \lor t_2$
Т	Т	Т
Т	F	Т
F	Т	Т
F	F	F

<i>s</i> ₁	<i>s</i> ₂	<i>s</i> ₃	<i>s</i> ₄	<i>s</i> ₅	<i>s</i> ₆	<i>s</i> ₇	
Т	Т	F	Т	F	Т	Т	
F	F	Т	Т	Т	F	F	
F	F	Т	F	Т	F	F	
Т	F	F	Т	F	F	Т	
Т	F	Т	F	F	F	Т	
			•••	•••			
V							

That animal That animal is a cat is a dog

That animal is a cat or a dog



Corollary 1.19. A logical context S satisfies the following properties:

- associativity: $a \lor (b \lor c) \equiv (a \lor b) \lor c, a \land (b \land c) \equiv (a \land b) \land c$
- commutativity: $a \lor b \equiv b \lor a$, $a \land b \equiv b \land a$
- absorption: $a \lor (a \land b) \equiv a, a \land (a \lor b) \equiv a$
- *identity:* $a \lor \bot \equiv a$, $a \land \top \equiv a$
- distributivity: $a \lor (b \land c) \equiv (a \lor b) \land (a \lor c), a \land (b \lor c) \equiv (a \land b) \lor (a \land c)$
- complements: $a \lor \neg a \equiv \top$, $a \land \neg a \equiv \bot$
- De Morgan: $\neg a \lor \neg b \equiv \neg (a \land b), \ \neg a \land \neg b \equiv \neg (a \lor b)$

Therefore S is a **Boolean algebra** by definition.

Recovers the standard structure for classical logic

Note how many properties are part of the definition of a Boolean algebra: if that had been our starting point, we would have had to justify every single one, which is cumbersome and not particularly enlightening



Functions in a Boolean algebra have a standard representation important for us



Takeaways

- Semantics define which assignments are possible on a given context
- The possible assignments define the logical relationships and operations
- The possible assignments describe "what could happen", which is inherently tied to the model
 - Certainty, equivalence, narrowness, etc... are all metaconcepts about the theory
- TODOs
 - Statement equivalence should be defined before functions on statements (technically, they should be operations on equivalence classes)



Axioms of verifiability



Axiom 1.27 (Axiom of verifiability). A verifiable statement is a statement that, if true, can be shown to be so experimentally. Formally, each logical context S contains a set of statements $S_v \subseteq S$ whose elements are said to be verifiable. Moreover, we have the following properties:

- every certainty $\top \in S$ is verifiable
- every impossibility $\bot \in S$ is verifiable
- a statement equivalent to a verifiable statement is verifiable

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



All tests must succeed

Axiom 1.32 (Axiom of countable disjunction verifiability). The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^{\infty} \subseteq S_v$ be a countable collection of verifiable statements. Then the disjunction $\bigvee_{i=1}^{\infty} s_i \in S_v$ is a verifiable statement.



Axiom 1.31 (Axiom of finite conjunction verifiability). The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^n \subseteq S_v$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge_{i=1}^n s_i \in S_v$ is a verifiable statement.

 s_i s_1 s_2 s_3 \dots

One successful test is sufficient

 ∞

i=1



Axiom 1.27 (Axiom of verifiability). A verifiable statement is a statement that, if true, can be shown to be so experimentally. Formally, each logical context S contains a set of statements $S_v \subseteq S$ whose elements are said to be verifiable. Moreover, we have the following properties:

- every certainty $\top \in S$ is verifiable
- every impossibility $\perp \in S$ is verifiable
- a statement equivalent to a verifiable statement is verifiable

New axiom to bring in the idea that some statements are experimentally verifiable



Tests may or may not terminate

Statements are verifiable if there is a test that always terminates successfully if the statement is true





Axiom 1.27 (Axiom of verifiability). A verifiable statement is a statement that, if true, can be shown to be so experimentally. Formally, each logical context S contains a set of statements $S_v \subseteq S$ whose elements are said to be verifiable. Moreover, we have the following properties:

- every certainty $\top \in S$ is verifiable
- every impossibility $\perp \in S$ is verifiable
- a statement equivalent to a verifiable statement is verifiable

The tests are not objects in the mathematical framework

Defining tests formally is cumbersome Capturing which statements are verifiable is enough

Formally we are only "tagging" which statements are verifiable

Only need to tag the verifiable statements: all other tests can be constructed from those



Axiom 1.27 (Axiom of verifiability). A verifiable statement is a statement that, if true, can be shown to be so experimentally. Formally, each logical context S contains a set of statements $S_v \subseteq S$ whose elements are said to be verifiable. Moreover, we have the following properties:

- every certainty $\top \in S$ is verifiable
- every impossibility $\perp \in S$ is verifiable
- a statement equivalent to a verifiable statement is verifiable

Certainties and impossibilities are defined to be true and false, therefore a trivial test that always succeeds or fails is adequate

If two statements are equivalent, the termination conditions on the tests are the same \Rightarrow we can use the same test



Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



 $e_{\neg}(e)$:

- 1. Run test *e*
- 2. If *e* fails, return SUCCESS
- 3. If *e* succeeds, return FAILURE

From *e*, we can construct the test $e_{\neg}(e)$ that switches SUCCESS with FAILURE, but the non-termination remains

⇒ the logic of verifiable statements does not include negation!



Definition 1.28. A falsifiable statement is a statement that, if false, can be shown to be so experimentally. Formally, a statement s is falsifiable if its negation $\neg s \in S_v$ is a verifiable statement.



Statements are falsifiable if there is a test that always terminates with failure if the statement is true

Note that formally falsifiable is defined to be the negation of verifiable statements

Reduces the number of primitive concepts

 \Rightarrow The justification must show these definitions to be equivalent



Definition 1.28. A falsifiable statement is a statement that, if false, can be shown to be so experimentally. Formally, a statement s is falsifiable if its negation $\neg s \in S_v$ is a verifiable statement.



⇒ If the negation of a statement is verifiable, then the statement is falsifiable



Definition 1.29. A decidable statement is a statement that can be shown to be either true or false experimentally. Formally, a statement s is decidable if $s \in S_v$ and $\neg s \in S_v$. We denote $S_d \subseteq S_v$ the set of all decidable statements.



Statements are decidable if there is a test that always terminates

Note that formally decidable statements are verifiable statements whose negation is verifiable

 \Rightarrow The justification must show these definitions to be equivalent



Definition 1.29. A decidable statement is a statement that can be shown to be either true or false experimentally. Formally, a statement s is decidable if $s \in S_v$ and $\neg s \in S_v$. We denote $S_d \subseteq S_v$ the set of all decidable statements.



Physics

Definition 1.29. A decidable statement is a statement that can be shown to be either true or false experimentally. Formally, a statement s is decidable if $s \in S_v$ and $\neg s \in S_v$. We denote $S_d \subseteq S_v$ the set of all decidable statements.

Corollary 1.30. Certainties and impossibilities are decidable statements.

Certainties and impossibilities are true and false by definition. Yet, we can make trivial tests for them.





Axiom 1.31 (Axiom of finite conjunction verifiability). The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^n \subseteq S_v$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge_{i=1}^n s_i \in S_v$ is a verifiable statement.

Conjunction (AND) of verifiable statements: check that all tests terminate successfully

 $\wedge (e_i)$:

- 1. Run all e_i
- 2. If all succeed, return SUCCESS
- 3. Return FAILURE

⇒ Only finite conjunction is guaranteed to terminate



Axiom 1.32 (Axiom of countable disjunction verifiability). The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^{\infty} \subseteq S_v$ be a countable collection of verifiable statements. Then the disjunction $\bigvee_{i=1}^{\infty} s_i \in S_v$ is a verifiable statement.

Disjunction (OR) of verifiable statements: check that ONE test terminates successfully

watch out for non-termination!

V (*e*_{*i*}):

- L. Initialize *n* to 1
- 2. For each i = 1 ... n
 - a) Run e_i for n seconds
 - b) If e_i succeeds, return SUCCESS
- 3. Increment *n* and go to 2

\Rightarrow Only countable disjunction can reach all tests



Proposition 1.33. The conjunction and disjunction of a finite collection of decidable statements are decidable. Formally, let $\{s_i\}_{i=1}^n \subseteq S_d$ be a finite collection of decidable statements. Then the conjunction $\bigwedge_{i=1}^n s_i \in S_d$ and the disjuction $\bigvee_{i=1}^n s_i \in S_d$ are decidable statements.

For decidable statements, we need both the statement and its negation to be verifiable

$$\neg \wedge e_i = \vee \neg e_i \qquad \neg \vee e_i = \wedge \neg e_i$$

Using De Morgan properties, we can construct tests using test for negation



Takeaways

- Adding the notion of verifiability only requires tagging which statements are verifiable
- We are essentially modeling procedures that output success/failure (i.e. one bit) and may not terminate
- These are the only axioms at this point
 - Everything else is a construction on top of this



Topology and the logic of experimental verifiability



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Topology and σ -algebra



				int(A) corresponds to the verifiable	(foundation of geometry,
	S_1	Test Result		part of a statement	prosability,,
		SUCCESS (in finite time)			Perfect map
	Т	UNDEFINED		∂A corresponds to the undecidable	between math and
	_	UNDEFINED		part of a statement	physics
	F	FAILURE (in finite time)		ext(A) corresponds to the falsifiable	
				part of a statement	NB: in physics, topology and
Open se	et (50	09.5, 510.5) ⇔ Verifiable	"the ma	ass of the electron is 510 \pm 0.5 KeV"	σ -algebra are parts of the
Closed s	set [5	$510] \Leftrightarrow$ Falsifiable "the m	ass of t	he electron is exactly 510 KeV"	Same logic structure

Borel set \mathbb{Q} ($int(\mathbb{Q}) \cup ext(\mathbb{Q}) = \emptyset$) \Leftrightarrow Theoretical "the mass of the electron in KeV is a rational number" (undecidable)

Experimental verifiability \Rightarrow

topology and σ -algebras



What is the largest set of verifiable statements it makes sense to consider?

$$S_1, S_2, S_3, \dots, S_n, \dots$$

Note: even assuming an indeterminate amount of time, we can only run up to countably many tests

$$S_1 \vee S_2, S_1 \wedge S_2, \dots$$

However, testing those statements implicitly tests all other statements that depend on those

⇒ Set of verifiable statements whose truth can be verified by running countably many tests



Definition 1.34. Given a set \mathcal{D} of verifiable statements, $\mathcal{B} \subseteq \mathcal{D}$ is a **basis** if the truth values of \mathcal{B} are enough to deduce the truth values of the set. Formally, all elements of \mathcal{D} can be generated from \mathcal{B} using finite conjunction and countable disjunction.

Definition 1.35. An experimental domain \mathcal{D} represents a set of verifiable statements that can be tested and possibly verified in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the certainty, the impossibility, and a set of verifiable statements that can be generated from a countable basis.



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Every physical theory must be fully characterized by an experimental domain

All its content must be expressible in terms of verifiable statements The theory must be fully explorable with a countable set of tests



Definition 1.36. The theoretical domain $\overline{\mathcal{D}}$ of an experimental domain \mathcal{D} is the set of statements constructed from \mathcal{D} to which we can associate a test regardless of termination. We call theoretical statement a statement that is part of a theoretical domain. More formally, $\overline{\mathcal{D}}$ is the set of all statements generated from \mathcal{D} using negation, finite conjunction and countable disjunction.

Extend the domain to include all statements that are associated with a test, regardless of termination.

All statements depend on the verifiable statements (which depend on the basis)

No new information is captured



Definition 1.39. Let $\bar{s} \in \overline{D}$ be a theoretical statement. We call the **verifiable part** $ver(\bar{s}) = \bigvee_{s \in D \mid s \preccurlyeq \bar{s}} s$ the broadest verifiable statement that is narrower than \bar{s} . We call the **falsifiable part** $fal(\bar{s}) = \bigvee_{s \in D \mid s \neq \bar{s}} s$ the broadest verifiable statement that is incompatible with \bar{s} . We call the **undecidable part** $und(\bar{s}) = \neg ver(\bar{s}) \land \neg fal(\bar{s})$ the broadest statement incompatible with both the verifiable and the falsifiable part.

Formalizing successful termination is indeed enough to characterize termination



Definition 1.47. A possibility for an experimental domain \mathcal{D} is a statement $x \in \overline{\mathcal{D}}$ that, when true, determines the truth value for all statements in the theoretical domain. Formally, $x \not\equiv \bot$ and for each $s \in \overline{\mathcal{D}}$, either $x \leq s$ or $x \notin s$. The **full possibilities**, or simply the **possibilities**, X for \mathcal{D} are the collection of all possibilities.

A possibility of a domain is a statement that picks one assignment

Possibilities: experimentally defined alternative cases defined by the verifiable cases

Proposition 1.48. Let \mathcal{D} be an experimental domain. A possibility for \mathcal{D} is any minterm of a basis that is not impossible.





Start with a countable set of verifiable statements Add all dependent verifiable statements (close under finite AND countable OR)

Add all statements with tests / (close under negation as well)

> The points of the space (the possibilities, the distinguishable cases) are not given a priori but are constructed from the chosen verifiable statements

• $x = \neg e_1 \land e_2 \land \neg e_3 \land \cdots$

Fill in all possible assignments

For each possible assignment we have a theoretical statement that is true only in that case (minterm). We call these statements possibilities of the domain.



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Each column (statement)

 $s = \bigvee_{x \in U} x$

is also a set of possibilities

The experimental domain \mathcal{D}_X induces a topology on the possibilities X.

The theoretical domain $\overline{\mathcal{D}}_X$ induces a (Borel) σ -algebra

Topologies (needed for manifold/geometric spaces) and σ -algebras (needed for integration and probability spaces) naturally arise from requiring experimental verifiability

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Negation and countable AND become complement and countable union

	Basis <i>B</i>		Experimental domain \mathcal{D}_X Theoretical domain $\overline{\mathcal{D}}_X$					
<i>e</i> ₁	<i>e</i> ₂	<i>e</i> ₃	 $s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$		$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	
F	F	F	 F	F		т	т	
F	т	F	 т	F		F	т	
Т	Т	F	 T	F		T	F	

Finite AND and countable OR become

finite intersection and countable union

Topologies and σ -algebras

All definitions and all proofs about these structures have precise physical meaning in this context



If $U \subseteq X$ is an open set then "x is in U" is a verifiable statement (e.g. "the mass of the electron is 511 ± 0.5 KeV")

If $V \subseteq X$ is a closed set then "x is in V" is a falsifiable statement (e.g. "the mass of the electron is exactly 511 KeV")

If $A \subseteq X$ is a Borel set then "x is in A" is a theoretical statement: a test can be created, though we have no guarantee of termination (e.g. "the mass of the electron in KeV is a rational number" is undecidable, the test will never terminate)

Topologies and σ -algebras each capture part of the formal structure

For us, they are part of a single unified structure



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Examples



Standard topology on integers

Decidable domain (all statements are decidable) Discrete topology (every set is clopen); topology and σ -algebra both coincide with the power set

Standard topology on the reals

Finite precision measurements (open intervals are verifiable) Topology generated by open intervals (coincides with order and metric topology); separable, complete, connected (no clopen sets except full and empty set); σ -algebra is the Borel algebra (strict subset of power set)





Examples

Does extra-terrestrial life exist?

Semi-decidable question Topology $\{\emptyset, \{Y\}, \{Y, N\}\}$ is strictly T_0 ; σ -algebra is the power set





How many leptons (electron-like particles) are there? (through direct observation) Can only measure lower bound (e.g. "there are at least i") Topology contains empty set and $\{i, i + 1, i + 2, ...\}$ for all *i*; strictly T_0 ; σ -algebra is the power set

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Physical meaning of separation axioms

- All topologies are Kolmogorov (i.e. T_0)
 - Possibilities are experimentally well-defined i.e. possibilities constructible from a base by countable AND/OR and NOT (singletons in the σ -algebra)
- The topology is T_1 if all possibilities are approximately verifiable
 - Possibilities are the limit of a sequence of verifiable statements i.e. possibilities are the countable conjunction of verifiable statements
- The topology is Hausdorff (i.e. T_2) if all possibilities are pairwise experimentally distinguishable
 - Given two possibilities, we can find a test that confirms one and excludes the other
 - i.e. for any $x_1, x_2 \in X$ there is a statement $s \in \overline{\mathcal{D}}_X$ such that $x_1 \leq ver(s)$ and $x_2 \leq fal(s)$

_	S	Test Result	x_1	<i>x</i> ₂
	т	SUCCESS (in finite time)	Т	
		UNDEFINED		F
		UNDEFINED	F	
	F	FAILURE (in finite time)		т







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An inference relationship is a map $\mathscr{V}: \mathscr{D}_Y \to \mathscr{D}_X$ such that $\mathscr{V}(s) \equiv s$



Functions in physics must be "well-behaved"



We can verify we are in the triple point \Rightarrow topologically isolated point

Topologically continuous function can be analytically discontinuous at a topologically isolated point

Second countable space ⇒ up to countably many isolated points ⇒ up to countably many discontinuity ⇒ "well-behaved" Internal energy can change discontinuously through phase transitions



Phase transition \Leftrightarrow Topologically isolated regions



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Takeaway

- The most fundamental mathematical structures (topology and σ -algebra) are there to capture the logic of experimental verifiability
 - Precise science/math dictionary
 - "Well-behaved" mathematical objects are really "well-defined" physical objects
- Experimental verifiability is the basis for scientifically well-defined objects
- TODOs:
 - Space of possible composite experimental domains
 - Approximations of domains
 - Projections to domains



Quantities and ordering



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Quantities and ordering

Phys. Scr. 95 084003 (2020)

Goal: deriving the notion of quantities and numbers (i.e. integers, reals, ...) from an operational (metrological) model

A **reference** (i.e. a tick of a clock, notch on a ruler, sample weight with a scale) is something that allows us to distinguish between a before and an after



Mathematically, it is a triple (b, o, a) such that:

- b and a are verifiable
- The reference has an extent ($o \not\equiv \bot$)
- If it's not before or after, it is on $(\neg b \land \neg a \leq o)$
- If it's before and after, it is on $(b \land a \leq o)$

Numbers defined by metrological assumptions, NOT by ontological assumptions

The hard part is to

recover ordering. After

that, recovering reals

and integers is simple.

To define an **ordered** sequence of possibilities, the references must be (nec/suff conditions):





Sparse



 \Rightarrow $(X, \leq) \cong (\mathbb{Z}, \leq)$





Assumptions untenable at Planck scale: no consistent ordering: no "objective" "before" and "after"

How do we formally model a quantity?

A **reference** (e.g. a tick of a clock) is something that allows us to distinguish between a before and an after Mathematically, it is a triple (b, o, a) such that:

•

•

•

b and a are verifiable



The experimental domain for a quantity is a collection of references

After **Before** On F Т F F Т F F Т F Т Т F Т F Т Т Т Т

If it's before and after, it is on $(b \land a \leq o)$

The reference has an extent ($o \not\equiv \bot$)

If it's not before or after, it is on $(\neg b \land \neg a \leq o)$



Imagine collecting the references of all possible clocks into a single logical structure. What are the necessary and sufficient conditions such that they identify a point on the real line?

Intuitively, we would need clocks at higher and higher resolutions, all perfectly synchronized, ...



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1. Strict references

A reference is strict if before/on/after are mutually exclusive

Before	On	After
Т	F	F
F	Т	F
F	F	т

Physically, the extent of what we measure is assumed to be smaller than the extent of our reference

before

after

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on

Multiple references

Without further constraints, references would not lead to a linear order

	b ₂	<i>0</i> ₂	<i>a</i> ₂
b ₁	\checkmark	\checkmark	\checkmark
<i>o</i> ₁	\checkmark	\checkmark	\checkmark
<i>a</i> ₁	\checkmark	\checkmark	\checkmark



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Multiple references

The fact that a reference is "before" or "after" another is captured by the statements' logical relationship

	b ₂	<i>0</i> ₂	<i>a</i> ₂
b ₁	\checkmark	×	×
<i>o</i> ₁	\checkmark	×	×
<i>a</i> ₁	\checkmark	\checkmark	\checkmark

Order relationship between references is too restrictive



Note: the "boundaries" are ordered

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2. Aligned references

Two references are aligned if the before and not-after statement can be ordered by narrowness/implication

For example, $b_1 \leq b_2 \leq \neg a_1 \leq \neg a_2$

 ≼ Means that if the first statement is true then the second statement will be true as well
 That is, the first statement is narrower, more specific



Filling the whole region

If two different references overlap, we can't say one is before the other: we can't fully resolve the linear order

 a_2



Conversely, if two references don't overlap and there can be something in between, we must be able to put a reference there



D.

 a_2

 a_1

3. Refinable references

A set of references is refinable if we can address the previous two problems and resolve the full space

If two references overlap, we can find a reference that refines the overlap



If something can be found between two references, then there must be another reference in between



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CHAPTER 7 PROPERTIES AND OUANTITIES

the possibilities themselves can be ordered, and how this ordering, in the end, is uniquely

the positumes themselves can be ordered, and now this ordering, in the end, is uniquely characterized by statement narrowness (1 b) is less than 42 because "the quantity is less than 10^6 is narrower than "the quantity is less than 10^6 . As the defining characteristic for a quantity is the ability to compare its values, then the values must be ordered in some fashion from smaller to greater. Therefore, given two different values, one must be before the other. Mathematically, we call linear order an order with such a characteristic as we can imagine the elements positioned along a line. Note that vectors are not linearly ordered; no direction is greater than the other. Therefore, in this context, a octor will not strictly be a quantity but a collection of quantities.²

vector will not strictly be a quantity but a collection of quantities². We also have to define how this coder can be experimentally verified. The idea is that we should, at issue, be able to verify that the values of a given quantity is before or after a set value. This allows us to construct bounds suith as "the mass of the electron is foll as 0.5 keV but less than 51.15 keV.² for imagers, this also allows us to verify particular numbers as "the entry". has one natural satellite" is equivalent to the "the earth has more than zero natural sate and fever then two?. Therefore we will define the order topolory as the one senerated by set and puter users in a reserve we must use of the type (a, ∞) and (-∞, b). A quantity, then, is an ordered property with the order topology.

Definition 3.4. A linear order on a set Q is a relationship $\leq Q \times Q \rightarrow B$ such that mmetry) if $q_1 \leq q_2$ and $q_2 \leq q_1$ then $q_1 = q_2$ 2. (transitivity) if $a_1 \le a_2$ and $a_2 \le a_1$ then $a_1 \le a_2$

I. (total) at least $q_1 \leq q_2$ or $q_2 \leq q_1$ A set together with a linear order is called a linearly ordered set.

Definition 3.5. Let $(O \leq)$ be a linearly ordered set. The order topology is the topol

$(a, \infty) = \{a \in Q \mid a < q\}, (-\infty, b) = \{a \in Q \mid q < b\},\$

Definition 3.6. A quantity for an experimental domain D_X is a linear Formally, it is a tuple $(Q, \leq q)$ where (Q, q) is a property, $\leq Q \times Q \rightarrow \mathbb{B}$ is a linear or and Q is a topological space with the order topology with respect to \leq .

As for properties, the quantity values are just symbols used to label the different cases set Q may correspond to the integers, real numbers or a set of words ordered alphabetica The units are not captured by the numbers themselves: they are captured by the functi-

⁹In other languages, there are two words to differentiate quantity as in "physical quantity" (e.g. grand Srösse, grandeur) and as in "amount" (e.g. quantité, Menge, quantité). It is the second meaning of qua

area nore. stence "the mass of the electron is 511 ± 0.5 keV" could instead be referring to statistical unus The measurem rule mass of the desires (\$31.03.2 keV could instead be referring to statistical unsert instead of an accessity bound and would constitute a different statistical different maniput is stats We will be treating these types of statistical statements have in the book, but suffine it to say that they or be defined before unsertiments that is donn'tly bound. ⁴When consulting the discinstry, we use the fact that we can experimentally tail whether the word w looking for h bubber or adm the one are randomly selector.

CHAPTER 3 PROPERTIES AND QUANTITIES

which returns elements of the original set and therefore reduces to countable conjunctions. Therefore, when forming D_b the only new elements will be the countable disjunctions Consider two countable sets $B_1, B_2 \subseteq B_b$. Their disjunctions $b_1 = \bigvee_{b \in B_1} b$ and $b_2 = \bigvee_{b \in B_2} b$ present the narrowest statement that is broader than all elements of the respective set.

uppose that for each element of B_1 we can find a broader element in B_2 . Then b_2 , being proader than all elements of B_2 , will be broader than all elements of B_1 . But since b₁ is to arrow the element has is broader than all elements in B_1 , we have $\mathbf{b}_2 \in \mathbf{b}_1$. Conversely uppose there is some element in B_1 for which there is no broader element in B_2 . Since he initial set is fully ordered, it means that that element of B_1 is broader than all the onte in R_{τ} . This means that element is breader than b_{τ} and since b_{τ} is breader the all elements in B_1 we have $b_1 \ge b_2$. Therefore the domain D_b generated by B_b is linearly rdered by narroy

Now we show that (\mathcal{D}, \geq) is linearly ordered. The basis \mathcal{B}_{r} is linearly ordered by because the negation of its elements are part of B and are ordered by narrownes te that broadness is the opposite order of narrowness and therefore a set linearly ordered w one is linearly ordered by the other. Therefore B_{α} is also linearly ordered by narrowness To show that $D = D_b \cup \neg(D_a)$ is linearly ordered by narrowness, we only need to show

that the countable disjunctions of elements of B_h are either narrower or broader countable conjunctions of the negations of elements of B_a . Let $B_1 \subset B_b$ and A_2 (lisjunction $b_1 = \bigvee_{b \in B_1} b$ prepresents the narrowest statement that is broader than all $b \in B_1$.

 biB_1 of B_1 while the conjunction $\neg a_2 = \neg \bigvee_{i \neq j} a = \bigwedge_{i \neq j} \neg a$ represents the broadest state is narrower than all elements of $\neg (A_2)$. Suppose that for one element of $\neg (A_1 - A_2)$ find a broader statement in B_1 . Then b_1 , being broader than all elements in Ereader than that one element in $\neg(A_2)$. But since $\neg a_2$ is narrower than all el broader than that one equations in $\gamma_{(A_2)}$, but suce \neg_{A_2} is narrower than an equivalence γ_{A_2} , have $\neg_{A_2} \ge 0$. Conversely, suppose that for no element of $\neg(A_2)$ we broader statement in B_1 . As B is linearly ordered, it means that all elements in proader than all elements in B_1 . This means that all elements in $\neg(A_2)$ are bro h and therefore $h_1 \neq \neg a_2$. Therefore D is linearly ordered by narr

Theorem 3.16 (Domain ordering theorem). An experimental domain D_X is referred if and only if it is the combination of two experimental domains $D_X = D_g$

(i) $D = \mathcal{D}_b \cup \neg (\mathcal{D}_a)$ is linearly ordered by narrowness (ii) all elements of D are part of a pair $(s_b, \neg s_a)$ such that $s_b \in \mathcal{D}_b$, $s_a \in \mathcal{D}_a$ of either the immediate successor of s_k in D or $s_k \equiv -s_n$

(iii) if $s \in D$ has an immediate successor, then $s \in D_b$ Proof. Let D_Y be a naturally ordered experimental domain. Let B_h and B_n

Proof. Let D_X be a matriary ordered experimental domain. Let D_2 and D_a as in 3.12 which means $B = B_0 \cup B_a$ is the basis that generates the order topo D_b be the domain generated by B_5 and D_a be the domain generated by B_a . The erated from D_b and D_a by finite conjunction and countable disjunction and $D_X = D_b \times D_a$.

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3.2 OUANTITIES AND ORDERING

that allows us to map statements to numbers and vice-versa. As we want to understand quantities better, we concentrate on those experimental domains that are fully characterized by a quantity. For example, the domain for the mass of a system will be fully characterized by a rank and number graterize than or equal to zero. Each possibility will be identified by a number which will correspond to the mass expressed in a particular the second for the system of the sys a those experimental doma unit, say in Kg. As the values of the mass are ordered, we can also say that the possibilitie themselves are ordered. That is, "the mass of the system is 1 Ka" proceedes "the mass of the sustem is 2 Ke". This ordering of the possibilities will be linked to the natural topology as he mass of the system is less than # Ke", the distuction of all possibilities that come befor ular possibility, is a verifiable st

We call a natural order for the possibility a linear order on them such that the order we can it mainth order to the possibility a most order on them such that to order topology is the natural topology. An experimental domain is fully characterized by a quantity if and only if it is naturally ordered and that quantity is ordered in the same way: it is order isomorphic. In other words, we can only assign a quantity to an experimental domain if it already has a natural ordering of the same type.

3.2. QUANTITIES AND ORDERING

Definition 3.12. Let D_X be a naturally ordered experimental domain and X its possil-ties. Define $B_b = \{ ^{\mu}x < x_1^n | x_1 \in X \}$, $B_a = \{ ^{\mu}x > x_1^n | x_1 \in X \}$ and $B = B_b \cup \neg (B_a)$.

Definition 3.13. Let (O, <) be an ordered set. Let $a_1, a_2 \in O$. Then a_2 is an immediate

tween them in the ordering. That is, $q_1 < q_2$ and there is no $q \in Q$ such that $q_1 < q < \infty$ so elements are **consecutive** if one is the immediate successor of the other.

Proposition 3.14. Let D_X be a naturally ordered experimental domain. Then (B_n, z) and (B, z) are incaring ordered sets. Moreover (B_n, z) , (B_n, z) are order isomorp

Proof. Let $f:X \to \mathcal{B}_{\mathrm{b}}$ be defined such that $f(x_1) = {}^{\mathrm{s}} x < x_1{}^{\mathrm{s}}.$ As there is one

Free rate j, $n \in Q_0$ is the first second state $|q_{11}| = |x_{11}| = |x_{11}|$. As there is done by one statement $x_{11} < x_{11}^2$ for each $|x_{11}| < x_{11}^2$ is a hybrical support $|z_{11}| < x_{11}^2$, $|z_{11}| < x_{11}^2$, $|z_{11}| < |z_{11}| < |z$

we $g(x_1) \equiv \bigvee_{\{x \in X \mid x > x_1\}} x \equiv \left(\bigvee_{\{x \in X \mid x > x_1\}} x\right) \lor \left(\bigvee_{\{x \in X \mid x > x_2\}} x\right) \equiv g(x_1) \lor g(x_2)$ and therefore

n 3.15. Let B_b and B_a be two sets of verifiable statements such that ls linearly ordered by narrowness. Let D_b and D_a be the experimental doma wely generate and $D = D_b \cup \neg (D_a)$. Then (D_b, z) , (D_a, z) and (D, z) are linearly

rst we show that (\mathcal{D}, z) is linearly ordered. We have that \mathcal{B}_{i} is linearly ordered.

ray we how that $(D_{n,k})$ is linearly ordered. We have that B_k is linearly ordered so because it is a subset of d which is linearly ordered by marrowness. We is a start of a finite set of statements linearly ordered by marrowness will return to its evolution of the disputch of a finite set of statements linearly ordered by marrowness will study be discussed as the model, can we obtain the study in the disputch of a finite set of statements linearly ordered or contable disputches spin will still be a contable disputch set in the finite compatibility of contable disputches in the finite disputch of the finite order disputch set of the statements of the statement of the statements of the statement of the s

Note that determining whether the quantity is exactly equal to the reference is not as ea

finite amount of time. That is, the reference itself can only be compared up to a finite leve

of precision. This may be a problem when constructing the references themselves: how do we

to precision: a not may be a process when donast uctual in a reasonates unanserves: now us we know that the marks on our rules are equally propared, or that the weights are equally propared, or that ticks of our clock are equally timed? It is a circular problem in the sense that, in a way, we need instruments of measurement to be able to create instruments of measurement.

Yet, even if our references can't be perfectly compared and are not perfectly equal, we can

(4) even is due detendent care be perfectly token and her not perfectly sequent, we can still say whether the value is well before our well after any of them. To make matters worse, the object we are measuring may itself have an extent. If we are measuring the position of a tiny ball, it may be clearly before or clearly after the nearest

are measuring use periods of a tary ban, and be calarly before to cavity increasing the periods of the second seco

that part can only be below, on or are the restriction in may be a reasonable assumption in many cases but we have to be mindful that we made that assumption: our general definition will have to be able to work in the less ideal cases.

discussion with the following definitions. A reference is represented by a set of three state-

uscussed with the bolowing dominations. A thereton is represented by a fee on time scale ments: they fell as whether the object is before, on or after a specific reference. To make sense, these have to satisfy the following minimal requirements. The before and the after statements must be verifiable, as otherwise they would not be usable as references. As the

reference must be somewhere, the on statement cannot be a contradiction. If the object is

not before and not after the reference, then it must be on the reference. If the object is before and after the reference, then it must also be on the reference. These requirements recognize

that, in general, a restructure and all extensions and so useds the UQEX relations in meritance. We can compare the extent of two references and say that one is finear than the other if the on statement is narrower than the other, and the before and after statements are wider. This corresponds to a finer tick of a rule or a finer pulse in our timing system. We say that

a reference is strict if the before, on and after statements are incompatible. That is, the three

Definition 3.17. A reference defines a before, an on and an after relationship between

iself and another object. Formally a reference $\mathbf{r} = (\mathbf{b}, \mathbf{o}, \mathbf{a})$ is a tuple of three statements

1. we can verify whether the object is before or after the reference: b and a are verifiable

A beginning reference has nothing before it. That is, $b \equiv \bot$. An ending reference has solving after it. That is, $a \equiv \bot$. A terminal reference is either beginning or ending.

that, in general, a reference has an extent and so does the object being measured

ses are distinct and can't be true at the same time

2. the object can be on the reference: $0 \neq 1$

if it's not before or after, it's on the reference: ¬b∧ ¬a ≤ 0
 if it's before and after, it's also on the reference: b∧ a ≤ 0

uch that:

In our general mathematical theory of experimental science, we can capture the above

e mark on the ruler has a width, the balance has friction, the tick of our clock will last a

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cessor of q, and q, is an immediate predecessor of q, if there is no

 $*x < x_1^n \wedge *x \ge x_1^n \equiv \bigvee_{x \in \Omega} x \equiv \bot.$

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mass of the system is more than $q_1 \text{ Kg}^+$ is also ordered by narrowness but with the reverse ordering of the possibilities/values. These are the very statements whose verifiable sets define ordering of the possibilities/values. These are the very statements whose verifiable sets define the order topologic and therefore polarity constitute a basis for the experimental domain. Now consider the statement $u_i = the mass of the system is less than or equal to 1: <math>K_i^{a}$ with $u_2 = the mass of the system is less than 1: <math>K_0^{a}$. We have $u_2 < u_1$, infact, if we replace the value $u_2 > u$ with anything bese than 1: K_0^{a} will still have $u_2 < u_1$. Infact, if we replace greater than 1 Kg we'd have $s_1 \neq s_2$. In other words, if we call B the set that includes both the loss-than-or-equal and less-than statements this is also linearly ordered by narrowness But "the mass of the system is less than or equal to 1 Ka" is equivalent to - "the mass of th But the mass of the system is low datas regard is $I K_{2}^{\mu}$ is experiment to . We mass of the system is proton for $I K_{2}^{\mu}$. In our second, $I = K_{2}^{\mu}$, $I = K_{2}$ seconds such that the state is a second structure of the system is more then K_{2}^{μ} and then such all innearly ordered by narrownaw. The ordering of R is no be further duration that is structure. It is not the struct embedding the system is for the system is low them e equal to $I = K_{2}^{\mu}$. In the immediate success of S_{2} , the mass of the system is low them e equal to $I = K_{2}^{\mu}$ is the immediate success of S_{2} . The mess of the system is low them e equal to $I = K_{2}^{\mu}$. In the immediate success of S_{2} . The mess of the system is low them $I = K_{2}^{\mu}$. The system is $I = K_{2}^{\mu}$ is the immediate success of S_{2} . The mess of the system is low them $I = K_{2}^{\mu}$. The system is $I = K_{2}^{\mu}$ is the system and there explicitly any order statement in I H that

is broader than s₂ but narrower than s₂ since they differ for a single case. This will happen is broader than is a bit moreover than is, since they differ for a single case. This will happen from a more subset. So if is composed of two must caption is the moreins of X. Yubern and a start of the single case of the single case of the single case. This is a single case statement in D has an immediate necessary, there must be only one case that separate the wave of the single case of the single case. The single case is a single case of the s ted with q_1 . Therefore statements in B that have an immediate successor must be in B_k as well. The main result is that the above characterization of the basis of the domain is necessary

and sufficient to order the possibilities. If an emerimental domain has a basis composed of

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To prove (i), we have that \mathcal{B}_b and \mathcal{B}_a are linearly ordered by 3.14. We need to show that e linear ordering holds across the sets. Let $x_1, x_2 \in X$ and consider the two statements $= D_b \cup \neg (D_a)$ is also linearly ordered.

 $D = U_0 \supset (U_0)$ is also integral ordered. To prove (1), let $s_0 \in D_0$. Take $s_n \in D_a$ such that $\neg s_n$ is the narrowest statement in $\neg (D_a)$ that is broader than s_0 . This exists because D_a is closed by infinite disjunction. As $-s_* \ge s_*$, let X_1 be the set of possibilities compatible with $-s_*$ but not compatible with s_* . he set cannot have more than one element, or we could find an element $\pi_1 \in X_2$ such that he see called the set of the non-set one dense is of we could induce a dense $x_1 \in X_1$ but that $x_1 \in X_1 = X_1$ is the interval $x_1 \in X_1$ contains one possibility, then $-s_0$ is the immediate successor. If x_1 is empty then $s_0 = -s_0$. Similarly, we can start with $s_0 \in D_0$ and find $s_0 \in D_0$ such that s_0 . A 1 is emply then $q_0 = \gamma_{q_0}$. Similarly, we can state with $\gamma_{q_0} \in \omega_{q_0}$ and may $q_0 \in \omega_{q_0}$ solution as q_0 . It is the broadest statement in D_0 that is narrower than $-q_{q_0}$. Let X_1 be the set of possibilities compatible with q_{q_0} into a compatible with q_{q_0} if X_1 notatins one possibility, then $-q_{q_0}$ is the immediate successor and if X_1 is empty then $s_0 \equiv -q_{q_0}$. To prove (iii), let $s_1, s_2 \in D$ such that s_2 is the immediate successor of s_1 . This means can write $s_2 \equiv s_1 \lor x_1$ for some $x_1 \in X$. This means $s_1 \equiv "x < x_1"$ while $s_2 \equiv "x \le x_1$ " and

herefore $s_1 \in B_b$.

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of the immediate successors. Let $(\cdot)^{++}: \mathcal{B}_h \to \mathcal{B}_n$ be the function such that $\neg(b^{++}) = \neg b^{++}$ is the immediate successor of b. Let b : $X \to B_b$ be the function such that $x = b(x) - b(x)^{++}$. On X define the ordering \leq such that $x_1 \leq x_2$ if and only if $b(x_1) \leq b(x_2)$. Since (B_b, \leq) s linearly ordered so is (X, \leq) . To show that the ordering is natural, suppose $x_1 < x_2$ hen $b(x_1) \leq -b(x_2)^{++} \leq b(x_2)$ and therefore $x_1 \leq b(x_2)$. We also have $-b(x_2)^{++} \leq b(x_2)$. and $o_{11}(x \to o_{12}) \to o_{12}(x)$ and therefore $x_1 \in o_{12}(x)$. We also have $\neg o_{11}(x) \to o_{12}(x) \to o_{12}(x) \to o_{12}(x)$. We also have $\neg o_{11}(x) \to o_{12}(x) \to o_{12}(x)$. We have $a_1(x) \to a_2(x) \to o_{12}(x)$. This means that given a possibilities $x_1 \in X$, all and only the possibilities lower than x_1 are compatible with $b(x_1)$ and therefore $b(x_1) \equiv "x < b_{12}(x)$. x_1^n , while all and only the possibilities greater than x_1 are compatible with $b(x_1)^{++}$ and herefore $b(x_1)^{*+} \equiv "x > x_1"$. The topology is the order topology and the domain has a

3.3 References and experimental ordering

In the previous section we have characterized what a quantity is and how it relates to an experimental domain. But as we saw in the first chapters, the possibilities of a domain are not objects that exist a priori: they are defined based on what can be verified experimentally. Therefore simply assigning an ordering to the possibilities of a domain does not answer the more fundamental question: how are quantities actually constructed? How do we, in practice create a system of references that allows us to measure a quantity at a given level of precision? What are the assumptions we make in that process?

In this section we construct ordering from the idea of a reference that physically defines a boundary between a before and an after. In general, a reference has an extent and may overlap with others. We define ordering in terms of references that are clearly before and overapy want Others, we denote the possibilities have as on references unat and covery neuron and after others. We see that the possibilities have a natural ordering only if they are generated from a set of references that is refinable (we can always find finer ones that do not overlap) and for which before/on/after are mutually exclusive coses. The possibilities, then, are the finest references possible.

We are by now so used of the ideas of real numbers, negative numbers and the number zero that it is difficult to realize that these are mental constructs that are, in the end, somewhat recent in the history of humankind. Yet geometry itself started four thousand years ago as an experimentally discovered collection of rules concerning lengths, areas and angles. That is, human beings were measuring quantities well before the real numbers were invented. So, how does one construct instruments that measure values?

To measure position, we can use a ruler, which is a series of equally spaced marks. We give a label to each mark (e.g. a number) and note which two marks are closest to the targe sition (e.g. between 1.2 and 1.3 cm). To measure weight, we can use a balance and a set of ually prepared reference weights. The balance can clearly tell us whether one side is heavier than the other, so we use it to compare the target with a number of reference weights and note the two closest (e.g. between 300 and 400 grams). A clock gives us a series of events to more new cool cosess (e.g., ostewen; sool and 'soo gland). A cook gives us a sense or evends to compare to (e.g. earth's rotation on its axis, the ticks of a clock). We can pour water from a reference container into another as many times as are needed to measure its volume. In all these cases what actually happens is similar: we have a reference (e.g. a mark on a ruler, a set of equally prepared weights, a number of ticks of a clock) and it is fairly easy to tel

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Proof. By definition, we have $\neg b \land \neg a \leq o$ and by 1.23 $\neg (\neg b \land \neg a) \lor o \equiv T \equiv b \lor a \lor o$. Definition 3.19. A reference $r_1 = (b_1, o_1, a_1)$ is finer than another reference $r_2 = (b_2, o_2, a_2)$ if $b_1 \ge b_2$, $o_1 \le o_2$ and $a_1 \ge a_2$.

Corollary 3.20. The finer relationship between references is a partial order

Proof. As the finer relationship is directly based on narrowness, it inherits its reflexivity, tisymmetry and transitivity properties and is therefore a partial order.

Definition 3.21. A reference is strict if its before, on and after statements are incomatible. Formally, r = (b, o, a) is such that $b \neq a$ and $o \equiv \neg b \land \neg a$. A reference is loose if it not strict

Remark. In general, we can't turn a loose reference into a strict one. The on statement an be made strict by replacing it with $\neg b \land \neg a$. This is possible because o is not required to e verifiable. The before (and after) statements would need to be replaced with statements like $b \wedge \neg a$, which are not in general verifiable because of the negation.

To measure a quantity we will have many references one after the other: a ruler will have many marks, a scale will have many reference weights, a clock will keep ticking. What does it mean that a reference comes after another in terms of the before/on/after statements? If reference \mathbf{r}_1 is before reference \mathbf{r}_2 we expect that if the value measured is before the

first it will also be before the second, and if it is after the second it will also be after the first. Note that this is not enough, though, because as references have an extent they may overlap. And if they overlap one can't be after the other. To have an ordering properly defined we must have that the first reference is entirely before the second. That is, if the value measured is on the first it will be before the

Mathematically, this type of orde before and strictly after. It does n One may be tempted to define the es refining the references and refined references, not the original or

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Definition 3.22. A reference is be he first it cannot be on or after the Proposition 3.23. Reference ord irreflexivity: not r < r transitivity: if r₁ < r₂ and r₂

and is therefore a strict partial Proof. For irreflexivity, since th nd therefore b v o = o v a. Therefo

reflexive.

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 $\textit{Proof. Let } \mathbf{r}_1 < \mathbf{r}_2. \textit{ By 3.27, we have } \neg \mathbf{a}_1 \preccurlyeq \mathbf{b}_2. \textit{ Conversely, let } \neg \mathbf{a}_1 \preccurlyeq \mathbf{b}_2. \textit{ Then } \neg \mathbf{a}_1 \ne \neg \mathbf{b}_2.$ Because the references are strict, $\neg a_1 \equiv b_1 \lor o_1$ and $\neg b_2 \equiv o_2 \lor a_2$. Therefore $b_1 \lor o_1 \neq o_2 \lor a_2$ and $\mathbf{r}_1 < \mathbf{r}_2$ by definition.

Definition 3.29. A reference is the immediate predecessor of another if nothing can be bound before the second and after the first. Formally, $r_1 < r_2$ and $a_1 * b_2$. Two references ecutive if one is the immediate successor of the oth

Since $b_1 \vee o_1 \neq o_2 \vee a_2,$ we have $b_1 \neq a_2$ which means $b_1 \preccurlyeq \neg a_2.$

Proposition 3.30. Let $r_1 = (b_1, o_1, a_1)$ and $r_2 = (b_2, o_2, a_2)$ be two references. If r_1 is nmediately before \mathbf{r}_2 then $\mathbf{b}_2 \equiv \neg \mathbf{a}_1$.

Proof. We have $b_1 \lor o_1 \equiv (b_1 \lor o_1) \land \top \equiv (b_1 \lor o_1) \land (b_2 \lor o_2 \lor a_2) \equiv ((b_1 \lor o_1) \land b_2) \lor$

 $(b_1 \vee o_1) \land (o_2 \vee a_2) \equiv ((b_1 \vee o_1) \land b_2) \lor \perp \equiv (b_1 \vee o_1) \land b_2$. Therefore $b_1 \lor o_1 \preccurlyeq b_2$. And

Similarly, we have $o_2 \vee a_2 \equiv (o_2 \vee a_2) \wedge T \equiv (o_2 \vee a_2) \wedge (b_1 \vee o_1 \vee a_1) = ((o_2 \vee a_2) \wedge (b_1 \vee a_2))$

 $(o_1 \lor a_2) \land a_1) \equiv \bot \lor ((o_2 \lor a_2) \land a_1) \equiv (o_2 \lor a_2) \land a_1$. Therefore $a_2 \lor o_2 \lessdot a_2) \land (o_1 \lor a_1) \equiv (o_2 \lor a_2) \land a_1$.

Since $b_1 \vee o_1 \vee a_1 \equiv T$, we have $\neg a_1 \leq b_1 \vee o_1$. Similarly $\neg b_2 \leq o_2 \vee a_2$. Since $b_1 \vee o_1 \neq o_2 \vee a_2$,

Since $b_1 \leq b_2$, $a_2 \leq a_1$, $b_1 \leq \neg a_2$ and $\neg a_1 \leq b_2$, the two references are aligned.

Proposition 3.28. Let $\mathbf{r}_1 = (\mathbf{b}_1, \mathbf{o}_1, \mathbf{a}_1)$ and $\mathbf{r}_2 = (\mathbf{b}_2, \mathbf{o}_2, \mathbf{a}_2)$ be two strict references. Then

ince $b_1 \leq b_1 \vee o_1$, we have $b_1 \leq b_2$.

 $\leq o_2 \lor a_2$, we have $a_2 \leq a_1$.

 $a_1 \neq \neg b_2$ and therefore $\neg a_1 \neq b_2$.

 $1 < r_2$ if and only if $\neg a_1 \leq b_2$.

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Proof. Let \mathbf{r}_1 be immediately before $\mathbf{r}_2.$ Then $a_1 \neq b_2$ which means $b_2 \preccurlyeq \neg a_1.$ By 3.27 e also have $\neg a_1 \leq b_2$. Therefore $b_2 \equiv \neg a_1$.

Proposition 3.31. Let $r_1 = (b_1, o_1, a_1)$ and $r_2 = (b_2, o_2, a_2)$ be two strict references. Then is immediately before \mathbf{r}_2 if and only if $\mathbf{b}_2 \equiv -\mathbf{a}_1$

Proof. Let r_1 be immediately before r_2 . Then $b_2 \equiv \neg a_1$ by 3.30. Conversely, let $b_2 \equiv \neg a_1$. Then $r_1 < r_2$ by 3.28. We also have $a_1 \neq \neg a_1$, therefore $a_1 \neq b_2$ and r_1 is immediately before r₂ by definition.

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means we can find $\mathbf{r}_3 = (\mathbf{b}, \neg \mathbf{b} \land \neg \mathbf{a}_2, \mathbf{a}_2)$ for some $\mathbf{b} \in \mathcal{D}_b$ such that $\mathbf{r}_3 < \mathbf{r}_2$ and therefore hai≼b≺ha₂

For the third, suppose $a_1 \in D_a$ and $b_2 \in D_b$ such that $\neg a_1 \prec b_2$. Then $\mathbf{r}_1 = (\bot, \neg a_1, a_1)$ and $\mathbf{r}_2 = (\mathbf{b}_2, \neg \mathbf{b}_2, \bot)$ are strict references aligned with the domain such that $\mathbf{r}_1 < \mathbf{r}_2$ but \mathbf{r}_2 is not an immediate successor of \mathbf{r}_1 . This means we can find $\mathbf{r}_3 = (\mathbf{b}, \neg \mathbf{b} \land \neg \mathbf{a}, \mathbf{a})$ such that $\mathbf{r}_1 < \mathbf{r}_3 < \mathbf{r}_2$ and therefore $\neg \mathbf{a}_1 \leq \mathbf{b} < \neg \mathbf{a} \leq \neg \mathbf{b}_2$.

Proposition 3.37. Let D be an emerimental domain generated by a set of refinable aligned trict references. Then all elements of D are part of a pair $(s_b, \neg s_a)$ such that $s_b \in D_b$, $s_a \in D_a$ and $\neg s_a$ is the immediate successor of s_b in D or $s_b \equiv \neg s_a$. Moreover if $s \in D$ has n immediate successor, then $s \in D_h$.

Proof. Let \mathcal{D} be an experimental domain generated by a set of refinable aligned strict eferences. Let $s_b \in D_b$. Let $A = \{a \in D_a \mid a \lor s_b \notin \top\}$. Let $s_a = \bigvee a$. First we show that $s_b \leq \neg s_a$. We have $s_b \wedge \neg s_a \equiv s_b \wedge \neg \lor a \equiv s_b \wedge \land \neg a \equiv \land s_b \wedge \neg a$. For all $a \in A$ we have $a \lor s_k \notin T$, $\neg a \preccurlyeq s_k$ which means $s_k \preccurlyeq \neg a$ because of the total order of D. This means that

 $\wedge \neg a \equiv s_b$ for all $a \in A$, therefore $s_b \wedge \neg s_a \equiv s_b$ and $s_b \preccurlyeq \neg s_a$. Next we show that no statement $s \in D$ is such that $s_k < s < -s_n$. Let $a \in D_n$ such that $< \neg_a$. By construction $a \in A$ and therefore $\neg s_a \preccurlyeq \neg_a$. Therefore we can't have $s_b < a < \neg s_a$.

We also can't have $b \in D_b$ such that $s_b \prec b \prec \neg s_a$: by 3.36 we'd find $a \in D_a$ such that $s_b < a \le b < \neg s_a \text{ which was ruled out. So there are two cases. Either \\ s_b \notin \neg s_a \text{ then } s_b < \neg s_a \text{:}$ $\neg s_n$ is the immediate successor of b. Or $s_b \equiv \neg s_n$.

The same reasoning can be applied starting from $s_\alpha\in\mathcal{D}_\alpha$ to find a $s_b\in\mathcal{D}_b$ such that s_b is e immediate predecessor of $\neg s_a$ or an equivalent statement. This shows that all elements of D are paired

To show that if a statement in D has a successor then it must be a before statement, let $s_1, s_2 \in D$ such that s_2 is the immediate successor of s_1 . By 3.36, in all cases where $s_1 \notin D_b$ and $s_2 \notin D_a$ we can always find another statement between the two. Then we must ave that $s_1 \in D_b$ and $s_2 \in D_a$.

Theorem 3.38 (Reference ordering theorem). An experimental domain is naturally orlered if and only if it can be generated by a set of refinable aligned strict references

Proof. Suppose D_X is an experimental domain generated by a set of refinable aligned ct references. Then by 3.34 and 3.37 the domain satisfies the requirement of theorem .16 and therefore is naturally ordered.

Now suppose D_X is naturally ordered. Define the set B_b , B_a and D as in 3.12. Let $R = \{(b, \neg b \land \neg a, a) | b \in B_b, a \in B_a, b \prec \neg a\}$ be the set of all references constructed from the basis. First let us verify they are references. The before and after statements are verifiable since they are part of the basis. The on statement $\neg b \land \neg a$ is not a contradiction since $b < \neg a$ means $b \neq a$ and $b \not\equiv \neg a$. The on statement is broader than $\neg b \land \neg a$ as they are equivalent and it is broader than $b \land a$ as that is a contradiction since $b < \neg a$. Therefore R is a set of references. Since the before and after statements of R coincide with the basis of the domain, D_X is generated by R.

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For transitivity, if $\mathbf{r}_1 < \mathbf{r}_2$, we have $\mathbf{b}_1 \vee \mathbf{o}_1 \neq \mathbf{o}_2 \vee \mathbf{a}_2$ and therefore $\neg(\mathbf{b}_1 \vee \mathbf{o}_1) \geq \mathbf{o}_2 \vee \mathbf{a}_2$ is transitively, if $\mathbf{r}_1 < \mathbf{r}_2$, we have $\mathbf{s}_1 \neq \mathbf{o}_2 \neq \mathbf{s}_2$ and therefore $\{\mathbf{o}_1 \lor \mathbf{o}_1\} \neq \mathbf{o}_2 \neq \mathbf{s}_2$ by 1.23. Since $\mathbf{b}_1 \lor \mathbf{o}_1 \lor \mathbf{s}_1 \equiv \top$, we have $\mathbf{s}_1 \ge \neg(\mathbf{b}_1 \lor \mathbf{o}_1)$. Similarly if $\mathbf{r}_2 < \mathbf{r}_3$ we'll have $a_2 \ge \neg (b_2 \lor o_2) \ge o_3 \lor a_3$. Putting it all together $\neg (b_1 \lor o_1) \ge o_2 \lor a_2 \ge a_2 \ge \neg (b_2 \lor o_2) \ge o_3 \lor a_3$ which means $b_1 \vee o_1 \neq o_3 \vee a_3$. Corollary 3.24. The relationship $r_1 \le r_2$, defined to be true if $r_1 < r_2$ or $r_1 = r_2$, is a

partial orde

As we saw, two references may overlap and therefore an ordering between them cannot be defined. But references can overlap in different ways. Suppose we have a vertical line one millimeter thick and call the left side the part before

the line and the right side the part after. We can have another vertical line of the same thickness overlapping but we can also have a horizontal line which will also, at some point, overlap. The case of the two vertical lines is something that, through finding finer references can be given a linear order. The case of the vertical and horizontal line, instead, cannot Intuitively, the vertical lines are aligned while the horizontal and vertical are not.

Conceptually, the overlapping vertical lines are aligned because we can imagine narrower lines around the borders, and those lines will be ordered references in the above sense: each line would be completely before or after, without intersection. This means that the before and not-after statements of one reference are either narrower or broader than the before and notafter statements of the other. That is, alignment can also be defined in terms of na of statements

Note that if a reference is strict, before and after statements are not compatible and therefore the before statement is narrower than the not-after statement. This means that given a set of aligned strict references, the set of all before and not-after statements is linearly

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Definition 3.33. Let D be a domain generated by a set of references R. A reference = (b, o, a) is said to be aligned with D if $b \in D_b$ and $a \in D_a$.

Proposition 3.34. Let D be an experimental domain generated by a set of aligned stric ferences R and let $D = D_b \cup \neg (D_a)$. Then (D, \preccurlyeq) is linearly ordered.

Proof. By 3.26 we have that $B = B_b \cup \neg(B_a)$ is aligned by narrowness. By 3.15 the rdering extends to D.

Having a set of aligned references is not necessarily enough to cover the whole space at all levels of precision. To do that we need to make sure that, for example, between two references that are not consecutive we can at least put a reference in between Or that if we have two references that overlap, we can break them apart into finer ones that do not overlap and one is after the other

We call a set of references refinable if the domain they generate has the above mentioned properties. This allows us to break up the whole domain into a sequence of references that o not overlap, are linearly ordered and that cover the whole space. As we get to the fines references, their before statements will be immediately followed by the negation of their after statements, since there can't be any reference in between. Conceptually, this will give us the second and the third condition of the domain ordering theorem 3.16.

Definition 3.35. Let D be an experimental domain generated by a set of aligned references 2. The set of references is refinable if, given two strict references $r_1 = (b_1, o_1, a_1)$ and $\mathbf{p}_2 = (\mathbf{b}_2, \mathbf{o}_2, \mathbf{a}_2)$ aligned with \mathcal{D} , we can always:

• find an intermediate one if they are not consecutive; that is, if $r_1 < r_2$ but r_2 is not the immediate successor of \mathbf{r}_1 , then we can find a strict reference \mathbf{r}_2 aligned with \mathcal{D} such that $r_1 < r_3 < r_2$.

• refine overlapping references if one is finer than the other; that is, if $o_2 < o_1$, we can find a strict reference r_3 aligned with ${\mathcal D}$ such that $o_3 \preccurlyeq o_1$ and either $b_3 \equiv b_1$ and $r_1 < r_2$ or $a_2 \equiv a_1$ and $r_2 < r_1$.

Proposition 3.36. Let D be an experimental domain generated by a set of refinable aligned

3.4. DISCRETE QUANTITIES

Now we show that R consists of aligned strict references. We already saw that $b \neq a$ and we also have $\neg b \land \neg a$ is incompatible with both b and a. The references are strict. To show they are aligned, take two references. The before and not after statements are inearly ordered by 3.14 which means the references are aligned. To show R is refinable, note that each reference can be expressed as (" $x < x_1$ ", " $x_1 \le$

 $x \le x_2^n$, " $x > x_2^n$) where $x_1, x_2 \in X$ and " $x_1 \le x \le x_2^n \equiv$ " $x \ge x_1^n \land$ " $x \le x_2^n$. That is, every reference is identified by two possibilities x_1, x_2 such that $x_1 \le x_2$. Therefore take two references $\mathbf{r}_1, \mathbf{r}_2 \in R$ and let (x_1, x_2) and (x_3, x_4) be the respective pair of possibilities we can use to express the references as we have shown. Suppose $\mathbf{r}_1 < \mathbf{r}_2$ but they are not consecutive. Then " $x \le x_2$ " < " $x < x_3$ ". That is, we can find $x_5 \in X$ such that $x_2 < x_5 < x_3$ which means " $x \le x_2$ " \le " $x < x_5$ " and " $x \le x_5$ " \le " $x < x_3$ ". Therefore the reference $\mathbf{r}_3 \in R$ dentified by (x_5, x_5) is between the two references. On the other hand, assume the second efference is finer than the first. Then $x_1 \le x_3$ and $x_4 \le x_2$ with either $x_1 \ne x_3$ or $x_4 \ne x_2$. Consider the references $\mathbf{r}_3, \mathbf{r}_4 \in R$ identified by (x_1, x_1) and (x_2, x_2) . Either $\mathbf{r}_3 < \mathbf{r}_2$ or $_2 < \mathbf{r}_4$. Also note that the before statements of \mathbf{r}_1 and \mathbf{r}_3 are the same and the after atements of r_1 and r_4 are the same. Therefore we satisfy all the requirements and the set R is refinable by definition.

To recap, experimentally we construct ordering by placing references and being able to tell whether the object measured is before or after. We can define a linear order on the possibilities, and therefore a quantity, only when the set of references meets special conditions. The references must be strict, meaning that before, on and after are mutually exclusive. They must be aligned, meaning that the before and not-after statement must be ordered by narrowness. They must be refinable, meaning when they overlap we can always find finer a linear order. If any of these conditions fail, a linear order cannot be defined.

The possibilities, then, correspond to the finest references we can construct within the domain. That is, given a value a, we have the possibility "the value of the property is a," and we have the reference ("the value of the property is less than q0", "the value of the property is a,", "the value of the property is more than a,").

3.4 Discrete quantities

Now that we have seen the general conditions to have a naturally ordered experimental do main, we study common types of quantities and under what conditions they arise. We start, with discrete ones: the number of chromosomes for a species, the number of inhabitants of a country or the atomic number for an element are all discrete quantities. These are quantities that are fully characterized by integers (positive or negative). We will see that discrete quantities have a simple characterization: between two references

there can only be a finite number of other references. The first thing we want to do is characterize the ordering of the integers. That is, we want

to find necessary and sufficient conditions for an ordered set of elements to be isomorphic to a subset of integers. First we note that between any two integers there are always finitely many elements. Let's call sparse an ordered set that has that property: that between two elements there are only finitely many. This is enough to say that the order is isomorphic to

Reference ordering theorem

To define an **ordered** sequence (e.g. of "instants"), the references must be (nec/suff conditions):

- Strict an event is strictly before/on/after the reference (doesn't extend over the tick)
- Aligned shared notion of before and after (logical relationship between statements)
- Refinable overlaps can always be resolved

Additionally:

Between any two references we can always have another reference \Rightarrow real numbers

Only finitely many references between any two references \Rightarrow integers

For time/space, these conditions are idealizations

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Assumptions

Physics

How does this model break down?

The ticks of a clock have an extent and so do the events (references not strict) If clocks have jitter, they cannot achieve perfect synchronization (references not aligned) We cannot make clock ticks as narrow as we want (references not refinable)

No consistent ordering: no "objective" "before" and "after"

In relativity, different observers measure time differently, but the order is the same. We should expect this to fail at "small" scales.

A better understanding of space-time means creating a more realistic formal model that accounts for those failures



What type of models should we use? Hard to say can arg

Hard to say, but we can argue from necessity

(N.B. this is a toy Lack of order at small scales, model, each point should have order at large enough scale infinitely many neighbors) What we can distinguish experimentally (i.e. topology) seems to be linked to how precisely we want to distinguish (i.e. geometry)

Current mathematical tools have a hard division between topology and geometry

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Assumptions

Physics

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Our reasoning contradicts the expectations of many that time is simply "discrete" at the smallest scale

This intuition is based on the idea that the continuum is like the discrete but "with more points"

This idea (though extremely common in physics) is flawed



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Takeaway

- Ordering, the defining features of quantities, is a logical structure
 - $3 \le 5$ precisely because "there are less than 3 items" \le "there are less than 5 items"
- TODOs:
 - Find whether one can construct topological spaces that are not locally metrizable but are "sort of metrizable" on long "distances"



Space of the well-posed scientific theories



Physical theories

Specializations of the general theory under the different assumptions

Assumptions

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Assumptions Physics

Information granularity

Logical relationships \Leftrightarrow Topology/ σ -algebra

- "The position of the object is between 0 and 1 meters"
 ≼ "The position of the object is between 0 and 1 kilometers"
- "The fair die landed on $1" \leq$ "The fair die landed on 1 or 2"
- "The first bit is 0 and the second bit is $1^{"} \leq$ "The first bit is 0"

Granularity relationships ⇔ Geometry/Probability/Information

- "The position of the object is between 0 and 1 meters"
 ≤ "The position of the object is between 2 and 3 kilometers"
- "The fair die landed on 1" \leq "The fair die landed on 3 or 4"
- "The first bit is 0 and the second bit is $1'' \leq$ "The third bit is 0"

⇒ Measure theory, geometry, probability theory, information theory, ... all quantify the level of granularity of different statements

A partially ordered set allows us to compare size at different level of infinity and to keep track of incommensurable quantities (i.e. physical dimensions)





Once a "unit" is chosen, a measure quantifies the granularity of another statement with respect to the unit

$$\begin{split} \mu_u(u) &= 1\\ s_1 \leq s_2 \Rightarrow \mu_u(s_1) \leq \mu_u(s_2)\\ \mu_u(s_1 \lor s_2) &= \mu_u(s_1) + \mu_u(s_2) \text{ if } s_1 \text{ and } s_2 \text{ are incompatible} \end{split}$$



Takeaway

- Only rough ideas at this point
- TODOs:
 - Find "right" basic axioms by reverse engineering measure theory



Wrapping it up

- We have a good foundational layer done that recovers topological structures from requiring experimental verifiability
 - Though some elements can still be developed and better understood
- The layer to describe more quantitative elements (geometry, probability, ...) is still to be understood



