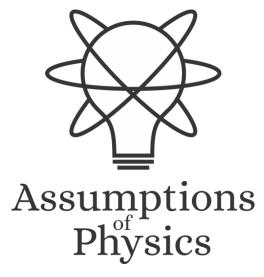
# Time as an operational definition

Gabriele Carcassi

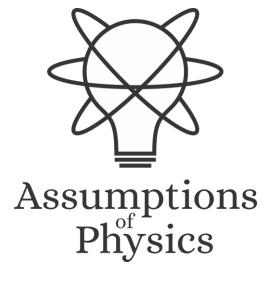
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## Introduction

- Together with Prof. Christine Aidala, I lead a project called Assumptions of Physics -<u>https://assumptionsofphysics.org</u> – that aims to find a minimal set of starting points from which the laws can be rederived
- It consists of two main efforts:
  - Reverse physics starts from the laws and finds physical assumptions that provide equivalent formulations
  - Physical mathematics starts from physical ideas, carefully encodes them in formal definitions, rederives the familiar mathematical structures



The mathematical structure of time (i.e. real numbers with the standard topology) can be understood as coming from an idealized operational model of clocks

As time resolution increases, this model must fail and needs to be replaced with a more realistic account

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## Outline

- What is time?
  - What it means to define something; the role of operational definitions in science; the topological structure imposed by experimental verification; the role of clock synchronization in defining time
- A metrological model of time
  - The metrological and logical structure of clocks; the necessary and sufficient conditions for continuous time (i.e. real numbers).
- Inevitable failure of time ordering
  - The untenability of the conditions for continuous time; how time ordering itself must break down.

## What is time?



#### Hard question because time is elusive

# What is time?

#### Then defining anything else should be easy!



#### table

#### () tā′bəl

#### noun

 An article of furniture supported by one or more vertical legs and having a flat horizontal surface.







# What is a table<sup>2</sup>



Assumptions Physics





#### Hard question because "what is" is elusive

# What is time?



#### Before giving a definition, we need to say:

#### What is the purpose of the definition?

# What are the "primitive elements" that are allowed to be in the definition?



In math, we want a formal definition: some set with some properties

#### E.g. a variable that can be used as a parameter for the evolution

I really don't know: I still haven't figured out what philosophers consider well-defined

In **philosophy**, we may look for an "ontological definition": some intrinsic feature of reality

E.g. A nonspatial continuum in which events occur

### These types of answers do not help us in a lab

# In a lab, an operational definition is necessary and sufficient

operational: it tells me what to do

This is what physicists consider well-defined!

Time is what is measured by a clock

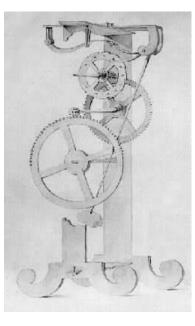
#### What is a clock?

A clock is what measures time

not operational: it does not tell me what to do



The sun



#### What is a clock?



The heart (pulse)



The seasons

#### A bucket of water with a hole

You can get a set of instructions on how to build these: they have operational definitions



#### But these are just examples of clocks!!!

A pendulum

Assumptions

Ph⁰vsics

#### What is a clock?

# A clock is anything that can be synchronized to other clocks

Clock synchronization is operationally defined

The idea of time comes out of our ability to synchronize our clocks; examples serve to "jumpstart" the process



We solved the "purpose" question (i.e. operational definition) We need to answer the "primitive elements" question

**Verifiable statements**: assertions that can be experimentally verified in a finite time

The mass of the photon is less than  $10^{-13} \text{ eV} \rightarrow \text{Verifiable}$ The mass of the photon is exactly  $0 \text{ eV} \rightarrow \text{Not verifiable due to infinite precision,}$ but falsifiable T SUCCESS (in finite time) F FAILURE (in finite time)

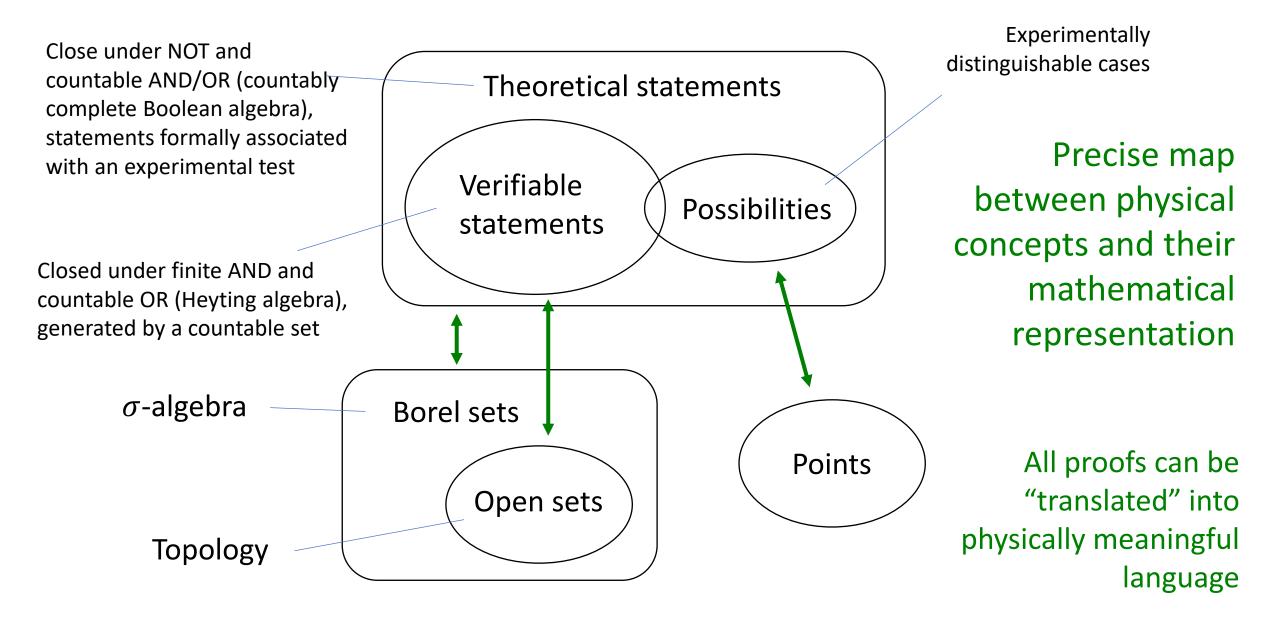
The syntax/semantics/structure of these statements cannot be formally specified further

For example, whether a specific statement is experimentally verifiable or even well-defined may depend on context (e.g. premises, idealization, theory, etc...)

The mass of the electron is 511  $\pm$  0.5 KeV

When measuring the mass, it is a verifiable hypothesis

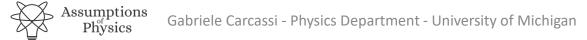
When performing particle identification, it is assumed to be true



# A physical theory is fully specified by a countable set of verifiable statements and their logical relationships

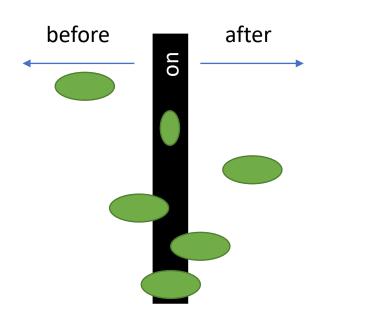
All mathematical objects we use in physics (symplectic manifolds, tensors, Hilbert spaces, Lie groups, ...) are ultimately identifying statements and their relationships

# A metrological model of time



### How do we formally model a clock?

A **reference** (i.e. a tick of a clock) is something that allows us to distinguish between a before and an after



Mathematically, it is a triple (b, o, a) such that:

- *b* and *a* are verifiable
- The reference has an extent ( $o \not\equiv \bot$ )
- If it's not before or after, it is on  $(\neg b \land \neg a \leq o)$
- If it's before and after, it is on  $(b \land a \leq o)$

Before	On	After
Т	F	F
F	Т	F
F	F	Т
Т	Т	F
F	Т	Т
Т	Т	Т

#### A clock is a collection of references



Imagine collecting the references of all possible clocks into a single logical structure. What are the necessary and sufficient conditions such that they identify a point on the real line?

Intuitively, we would need clocks at higher and higher resolutions, all perfectly synchronized, ...



## 1. Strict references

#### A reference is strict if before/on/after are mutually exclusive

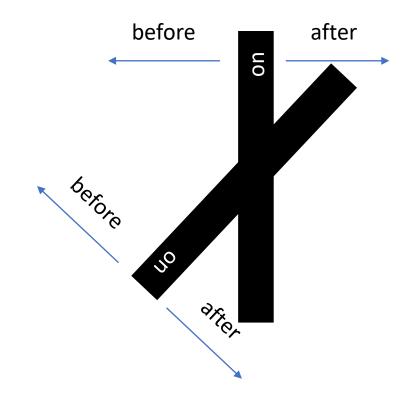
Before	On	After
Т	F	F
F	т	F
F	F	Т

Physically, this means assuming that the extent of what we measure is smaller than the extent of our reference

## Multiple references

Without further constraints, references would not lead to a linear order

	<b>b</b> <sub>2</sub>	<i>0</i> <sub>2</sub>	<i>a</i> <sub>2</sub>
<b>b</b> <sub>1</sub>	$\checkmark$	$\checkmark$	$\checkmark$
<i>o</i> <sub>1</sub>	$\checkmark$	$\checkmark$	$\checkmark$
<i>a</i> <sub>1</sub>	$\checkmark$	$\checkmark$	$\checkmark$

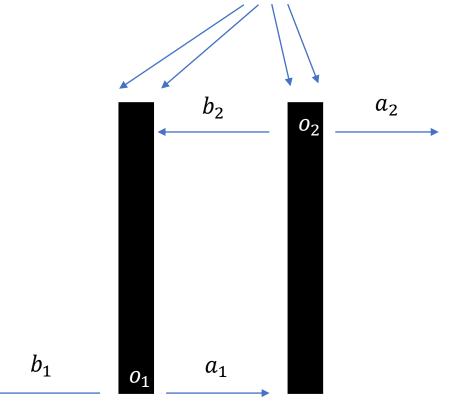


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## Multiple references

Note: the "boundaries" are ordered

The fact that a reference is "before" or "after" another is captured by the statements' logical relationship



# $b_2$ $o_2$ $a_2$ $b_1$ $\checkmark$ $\times$ $\times$ $o_1$ $\checkmark$ $\times$ $\times$ $a_1$ $\checkmark$ $\checkmark$ $\checkmark$

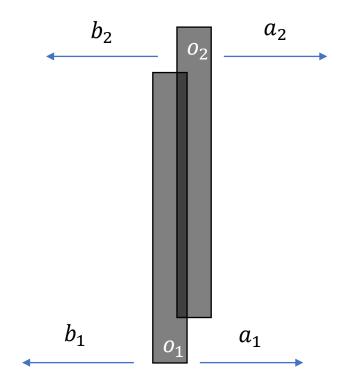
#### But order relationship between references is too restrictive

## 2. Aligned references

Two references are aligned if the before and not-after statement can be ordered by narrowness/implication

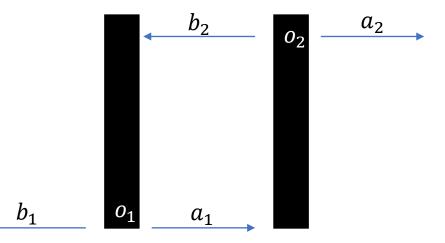
For example,  $b_1 \leq b_2 \leq \neg a_1 \leq \neg a_2$ 

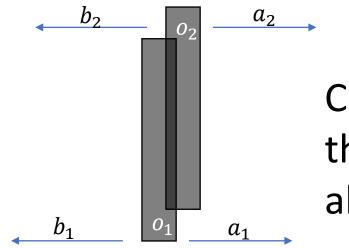
≼ Means that if the first statement is true then the second statement will be true as well That is, the first statement is narrower, more specific



## Filling the whole region

If two different references overlap, we can't say one is before the other: we can't fully resolve the linear order





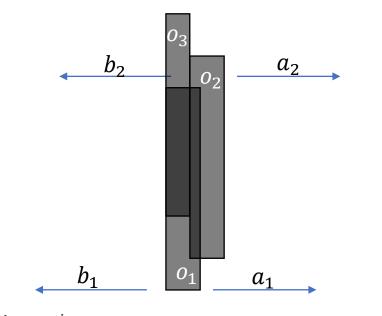
Conversely, if two references don't overlap and there can be something in between, we must be able to put a reference there

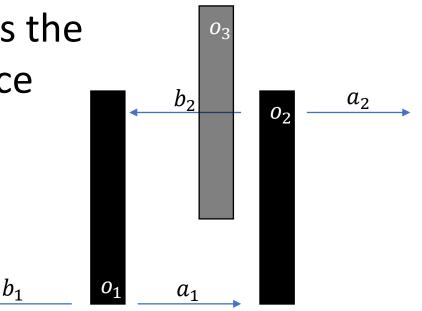
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## 3. Refinable references

A set of references is refinable if we can address the previous two problems and resolve the full space

If two references overlap, we can find a references that refines the overlap





If something can be found between two references, then there must be another reference in between

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#### CHAPTER 7 PROPERTIES AND OUANTITIES

the possibilities themselves can be ordered, and how this ordering, in the end, is uniquely

the positurities themserves can be ordered, and now this ordering, in the end, is uniquely characterized by statement narrowness: 10 is less than 42 because "the quantity is less then  $10^6$  is narrower than "the quantity is less then  $10^6$ . As the defining characteristic for a quantity is the ability to compare its values, then the values must be ordered in some fashion from smaller to greater. Therefore, given two different values, one must be before the other. Mathematically, we call linear order an order with such a characteristic as we can imagine the elements positioned along a line. Note that vector are not linearly ordered; no direction is greater than the other. Therefore, in this context, a

are not linearly ordered; no direction is greater than the other. Therefore, not this coveras, a vector will not articular by an quarry by an collection of quarrantic burnt. The line is that we should, a long, by a black to collect and of quarrantic burnt. The black is that we should, a long, by able to verify that the value of a grown quarranty is before or after a vector without the value of a grown quarranty is before on the structure of quarrantic dimension of the structure multicer as the structure of the has one natural satellite" is equivalent to the "the earth has more than zero natural satel and fewer than two?. Therefore we will define the order topology as the one generated by set of the type  $(a, \infty)$  and  $(-\infty, b)$ . A quantity, then, is an ordered property with the order topology.

Definition 3.4. A linear order on a set Q is a relationship  $\leq Q \times Q \rightarrow B$  such that ametry) if  $a_1 < a_2$  and  $a_2 < a_1$  then  $a_1 = a_2$ 

2. (trensitivity) if  $q_1 \leq q_2$  and  $q_2 \leq q_1$  been  $q_1 \leq q_2$ 3. (total) at least  $q_1 \leq q_2$  or  $q_2 \leq q_1$ A set together with a linear order is called a linearly ordered set.

Definition 3.5. Let  $(Q, \leq)$  be a linearly ordered set. The order topology is the topol

#### $(a, \infty) = \{q \in Q | a < q\}, (-\infty, b) = \{q \in Q | q < b\}.$

Definition 3.6. A mantitu for an emerimental domain Dy is a linearly ord Deminion 3.5.  $\times$  quantify jie to experimental bound  $\mathbb{P}_{\Delta}^{c}$  is a storing ordered property formally, it is a tuple  $(Q, \leq, q)$  where (Q, q) is a property,  $\leq Q \times Q \to \mathbb{B}$  is a linear or and Q is a topological space with the order topology with respect to  $\leq$ .

As for properties, the quantity values are just symbols used to label the different cases set Q may correspond to the integers, real numbers or a set of words ordered alphabetic. The units are not captured by the numbers themselves: they are captured by the funct

<sup>9</sup>In other innguages, there are two words to differentiate quantity as in "physical quantity" (e.g. grand-tosse, grandeur) and as in "amount" (e.g. quantith, Menge, quantité). It is the second meaning of qua-

at is imptured here. <sup>5</sup>The sentence: "the mass of the electron is 511+0.5 keV" could instead be referring to statistical unce The section: "We make of the electron us 31:a0 keV" could instead be referring to statistical unset: Instead of an accury bound and would constitute a different statistical distribution of the first statistical instances when the different manipular is atta We will be treading these types of statistical statements have in the book, but suffice it to say that they or be default bodies matematists that is donn'tly bounds. <sup>4</sup>When consulting the discinstry, we use the first that we can experimentally tail whether the word w looking for b holes or adult to a say randomly advance.

#### CHAPTER 3 PROPERTIES AND QUANTITIES

which returns elements of the original set and therefore reduces to countable conjunctions. Therefore, when forming  $D_b$  the only new elements will be the countable disjunctions

Consider two countable sets  $B_1, B_2 \subseteq B_6$ . Their disjunctions  $\mathbf{b}_1 = \bigvee_{i} \mathbf{b}_i$  and  $\mathbf{b}_2 = \bigvee_{i} \mathbf{b}_1$ represent the narrowest statement that is broader than all elements of the respective set. Suppose that for each element of  $B_1$  we can find a broader element in  $B_2$ . Then  $\mathbf{b}_2$ , being proader than all elements of  $B_2$ , will be broader than all elements of  $B_1$ . But since b<sub>1</sub> is to arrow the element has is broader than all elements in  $B_1$ , we have  $\mathbf{b}_2 \in \mathbf{b}_1$ . Conversely uppose there is some element in  $B_1$  for which there is no broader element in  $B_2$ . Since he initial set is fully ordered, it means that that element of  $B_1$  is broader than all the onts in  $R_2$ . This means that element is broader than  $b_2$  and since  $b_1$  is broader that Il elements in  $B_1$  we have  $b_1 \ge b_2$ . Therefore the domain  $D_b$  generated by  $B_b$  is linearly rdered by narroy

Now we show that  $(D, \geq)$  is linearly ordered. The basis  $R_{\tau}$  is linearly ordered by ness because the negation of its elements are part of B and are ordered by narrowness te that broadness is the opposite order of narrowness and therefore a set linearly ordered w one is linearly ordered by the other. Therefore  $B_{\alpha}$  is also linearly ordered by narrowness To show that  $D = D_b \cup \neg(D_a)$  is linearly ordered by narrowness, we only need to show

that the countable disjunctions of elements of  $B_h$  are either narrower or broader countable conjunctions of the negations of elements of  $B_a$ . Let  $B_1 \subset B_b$  and  $A_2$  ( lisjunction  $b_1 = \bigvee_{b \in B_1} b$  prepresents the narrowest statement that is broader than all  $b \in B_1$ .

 $biB_1$ of  $B_1$  while the conjunction  $\neg a_2 = \neg \bigvee_{i \neq j} a = \bigwedge_{i \neq j} \neg a$  represents the broadest state is narrower than all elements of  $\neg (A_2)$ . Suppose that for one element of  $\neg (A_1 - A_2)$ find a broader statement in  $B_1$ . Then  $b_1$ , being broader than all elements in Ereader than that one element in  $\neg(A_2)$ . But since  $\neg a_2$  is narrower than all el broader than that one equation in  $\neg \langle n_2 \rangle$ , but since  $\neg q_2$  is narrower than an e  $\neg \langle n_2 \rangle$ , we have  $\neg q_2 \neq b_1$ . Conversely, suppose that for no element of  $\neg \langle n_2 \rangle$  we broader statement in  $B_1$ . As B is linearly ordered, it means that all elements in roader than all elements in  $B_1$ . This means that all elements in  $\neg(A_2)$  are bro h and therefore  $h_1 \neq \neg a_2$ . Therefore D is linearly ordered by narr

Theorem 3.16 (Domain ordering theorem). An experimental domain  $D_X$  is referred if and only if it is the combination of two experimental domains  $D_X = D_g$ 

(i)  $D = D_b \cup \neg(D_a)$  is linearly ordered by narrowness (ii) all elements of D are part of a pair (s<sub>b</sub>, −s<sub>a</sub>) such that s<sub>b</sub> ∈ D<sub>b</sub>, s<sub>a</sub> ∈ D<sub>a</sub> ∈ either the immediate successor of  $s_k$  in D or  $s_k \equiv -s_n$ 

(iii) if  $s \in D$  has an immediate successor, then  $s \in D_b$ Proof. Let  $D_Y$  be a naturally ordered experimental domain. Let  $B_h$  and  $B_n$ 

Proof. Let  $D_X$  be a matriary ordered experimental domain. Let  $D_2$  and  $D_a$ as in 3.12 which means  $B = B_0 \cup B_a$  is the basis that generates the order topo  $D_b$  be the domain generated by  $B_5$  and  $D_a$  be the domain generated by  $B_a$ . T erated from  $D_b$  and  $D_a$  by finite conjunction and countable disjunction and  $D_X = D_b \times D_a$ .



#### 3.2. OUANTITIES AND ORDERING

that allows us to map statements to numbers and vice-versa. As we want to understand quantities better, we concentrate on those experimental domains that are fully characterized by a quantity. For example, the domain for the mass of a system will be fully characterized by a ranking and number greater than or equal to zero. Each possibility will be identified by a number which will correspond to the mass expressed in a particular distribution. unit, say in Kg. As the values of the mass are ordered, we can also say that the possibilitie themselves are ordered. That is, "the mass of the sustem is 1 Ka" proceedes "the mass of the ustem is 2 Ke<sup>\*</sup>. This ordering of the possibilities will be linked to the natural topology a he mass of the system is less than # Ke", the distuction of all possibilities that come befor a make by her spectra is not working it we dispute the dispute of an possibility, is a verifiable statement. We call a natural order for the possibility a linear order on them such that the order

topology is the matural topology. An experimental domain is fully characterized by a quantity if and only if it is naturally ordered and that quantity is ordered in the same way: it is order isomorphic. In other works, we can only assign a quantity to an experimental domain if it already has a natural ordering of the same type.

3.2. QUANTITIES AND ORDERING

Definition 3.12. Let  $D_X$  be a naturally ordered experimental domain and X its possil-ties. Define  $B_b = \{ {}^{a}x < x_1^n | x_1 \in X \}$ ,  $B_a = \{ {}^{a}x > x_1^n | x_1 \in X \}$  and  $B = B_b \cup \neg (B_a)$ .

Definition 3.13. Let  $(O, \leq)$  be an ordered set. Let  $a_1, a_2 \in O$ . Then  $a_2$  is an immediate

tween them in the ordering. That is,  $q_1 < q_2$  and there is no  $q \in Q$  such that  $q_1 < q < \infty$ so elements are **consecutive** if one is the immediate successor of the other.

Proposition 3.14. Let  $D_X$  be a naturally ordered experimental domain. Then  $(B_n, z)$  and (B, z) are incaring ordered sets. Moreover  $(B_n, z)$ ,  $(B_n, z)$  are order isomorp

Proof. Let  $f:X \to \mathcal{B}_{\mathrm{b}}$  be defined such that  $f(x_1) = {}^{\mathrm{s}} x < x_1{}^{\mathrm{s}}.$  As there is one

Free rate j,  $k \in [k]$ ,  $k \in [k]$  is the definition of an k in [j(1) - k < 1]. As there k only down stationant  $K < 2^{-1}$ , for each j -  $K \in [k]$ . Suppose  $z_1 \le z_2$ , wave  $f(z_2) = \sum_{i=1}^{k} V_i = x_i^{-1} = x_i^{-1} + x_i^{-$ 

we  $g(x_1) \equiv \bigvee_{\{x \in X \mid x > x_1\}} x \equiv \left(\bigvee_{\{x \in X \mid x > x_1\}} x\right) \vee \left(\bigvee_{\{x \in X \mid x > x_2\}} x\right) \equiv g(x_1) \vee g(x_2)$  and therefore

 $\begin{array}{ll} & \left| \kappa (x_{0}) - \left\{ \left| \kappa (x_{0}) - \left| \left| \kappa (x_{0}) - \right| \right\rangle \right\rangle \right| \left| \kappa (x_{0}) - \left| \kappa (x_{0}) - \right| \right\rangle \right\rangle \\ & \left| \kappa (x_{0}) - \left| \kappa (x_{0}) - \left| \kappa (x_{0}) - \kappa (x_{0}) - \left| \kappa (x_{0}) - \kappa (x_{0}) - \kappa (x_{0}) - \kappa (x_{0}) \right\rangle \right\rangle \right\rangle \\ & \left| \kappa (x_{0}) - \kappa (x_{0}) \right\rangle \\ & \left| \kappa (x_{0}) - \kappa (x_{0}) -$ 

n 3.15. Let  $B_b$  and  $B_a$  be two sets of verifiable statements such that ls linearly ordered by narrowness. Let  $D_b$  and  $D_a$  be the experimental doma wely generate and  $D = D_b \cup \neg (D_a)$ . Then  $(D_b, z)$ ,  $(D_a, z)$  and (D, z) are linearly

rst we show that  $(\mathcal{D}, z)$  is linearly ordered. We have that  $\mathcal{B}_i$  is linearly ordered.

rat we show that  $\{L_{n,k}\}$  is insuring ordered. We mass that  $\Phi_k$  is instarty ordered so because it is a subset of R which is insuring ordered by marrowness. Note insure that  $\Phi_k$  is a subset of statements handy ordered by marrowness. We instantiate the solution of the state of the statement is basely ordered by marrowness will start be constable disjunction of a finite solution. The commable disjunction, instand, can we deduced. That thus, these distances may also dispute the state of the statement is a state of combable disjunctions is the constable disjunction is the finite origunations.

Note that determining whether the quantity is exactly equal to the reference is not as ea

e mark on the ruler has a width, the balance has friction, the tick of our clock will last a

finite amount of time. That is, the reference itself can only be compared up to a finite leve

of precision. This may be a problem when constructing the references themselves: how do we

to precision: a not may be a process when donast uctual in a reasonates unanserves: now us we know that the marks on our rules are equally propared, or that the weights are equally propared, or that ticks of our clock are equally timed? It is a circular problem in the sense that, in a way, we need instruments of measurement to be able to create instruments of measurement.

Yet, even if our references can't be perfectly compared and are not perfectly equal, we can

(4) even is due detendent care be perfectly token and her not perfectly equals, we can still say whether the value is well before our will after any of them. To make matters worse, the object we are measuring may itself have an extent. If we are measuring the position of a tiny ball, it may be clearly before or clearly after the nearest

are measuring use periods of a tury ban, and be calarly before to cavity increasing the periods of the second seco

that part can only be below, or or are the reference. It may be a reasonable assumption in many cases but we have to be mindful that we made that assumption: our general definition will have to be able to work in the less ideal cases.

In our general maximum can usery or experimental scenare, we can compute the above discussion with the following definitions. A reference is represented by a set of three state-ments: they tell us whether the object is before, on or after a specific reference. To make sense, these have to satisfy the following minimal requirements. The before and the after statements must be verifiable, as otherwise they would not be usable as references. As the

reference must be somewhere, the on statement cannot be a contradiction. If the object is

not before and not after the reference, then it must be on the reference. If the object is before and after the reference, then it must also be on the reference. These requirements recognize

that, in general, a restructure and all extensions and so useds the UQEX relations in meritance. We can compare the extent of two references and say that one is finser than the other if the on statement is narrower than the other, and the before and after statements are wider. This corresponds to a finer tick of a rule or a finer pulse in our timing system. We say that

a reference is strict if the before, on and after statements are incompatible. That is, the three

Definition 3.17. A reference defines a before, an on and an after relationship between

iself and another object. Formally a reference  $\mathbf{r} = (\mathbf{b}, \mathbf{o}, \mathbf{a})$  is a tuple of three statements

1. we can verify whether the object is before or after the reference: b and a are verifiable

A beginning reference has nothing before it. That is,  $b \equiv \pm$ . An ending reference has nothing after it. That is,  $a \equiv \pm$ . A terminal reference is either beginning or ending.

that, in general, a reference has an extent and so does the object being measured

ses are distinct and can't be true at the same time

2. the object can be on the reference:  $0 \neq 1$ 

if it's not before or after, it's on the reference: ¬b∧ ¬a ≤ 0
if it's before and after, it's also on the reference: b∧ a ≤ 0

uch that:

In our general mathematical theory of experimental science, we can capture the above

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3.3. REFERENCES AND EXPERIMENTAL ORDERING

essor of a and a is an immediate predecessor of a if there is no

 $*x < x_1^n \wedge *x \ge x_1^n \equiv \bigvee_{x \in \Omega} x \equiv \bot.$ 

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#### CHAPTER 3. PROPERTIES AND QUANTITIES

mass of the system is more than  $q_1 \text{ Kg}^+$  is also ordered by narrowness but with the reverse ordering of the possibilities/values. These are the very statements whose verifiable sets define ordering of the possibilities/values. These are the very statements whose verifiable sets define the order topology and therefore pointly constitute a basis for the experimental domain. Now consider the statement  $y_1 = the mas of the system is less than or optil to t <math>A_{22}^{00}$ , with  $y_2 = the mas of the system to less than 1 Kg<sup>0</sup>. We have <math>y_2 \le y_1$ . Instead if we use a value the value in  $y_2$  with anything less than 1 Kg we'll still have  $y_2 \le y_1$ . Instead if we use a value greater than 1 Kg we'd have  $s_1 \neq s_2$ . In other words, if we call B the set that includes both the less-than-or-equal and less-than statements this is also linearly ordered by narrowness But "the mass of the system is less than or equal to 1 Ka" is equivalent to - "the mass of th But the mass of the system is low datas regard is  $I K_{2}^{\mu}$  is experiment to . We mass of the system is proton for  $I K_{2}^{\mu}$ . In our second,  $I = K_{2}^{\mu}$ ,  $I = K_{2}$  seconds such that the state is a second structure of the system is more then  $K_{2}^{\mu}$  and then such all innearly ordered by narrownaw. The ordering of R is no be further duration that is structure. It is not the struct embedding the system is for the system is low them e equal to  $I = K_{2}^{\mu}$ . In the immediate success of  $S_{2}$  , the mass of the system is low them e equal to  $I = K_{2}^{\mu}$  is the immediate success of  $S_{2}$ . The mess of the system is low them e equal to  $I = K_{2}^{\mu}$ . In the immediate success of  $S_{2}$ . The mess of the system is low them  $I = K_{2}^{\mu}$ . The system is  $I = K_{2}^{\mu}$  is the immediate success of  $S_{2}$ . The mess of the system is  $I = K_{2}^{\mu}$ .

is broader than s<sub>2</sub> but narrower than s<sub>2</sub> since they differ for a single case. This will happen is broader than is a bit moreover than is, since they differ for a single case. This will happen from a more subset. So if is composed of two must caption is the moreins of X. Yubern and a start of the single case of the single case of the single case. This is a single case statement in D has an immediate necessary, there must be only one case that separate the wave of the single case of the single case. The single case is a single case of the s ted with  $q_1$ . Therefore statements in B that have an immediate successor must be in B<sub>k</sub> as well. The main result is that the above characterization of the basis of the domain is necessary

and sufficient to order the possibilities. If an emerimental domain has a basis composed of

#### 3.2 OUANTITIES AND ORDERING

To prove (1), we have that  $B_b$  and  $B_a$  are linearly ordered by 3.14. We need to show that e linear ordering holds across the sets. Let  $x_1, x_2 \in X$  and consider the two statements

 $= D_b \cup \neg (D_a)$  is also linearly ordered.  $D = U_0 \supset (U_0)$  is also integral ordered. To prove (1), let  $s_0 \in D_0$ . Take  $s_n \in D_a$  such that  $\neg s_n$  is the narrowest statement in  $\neg (D_a)$  that is broader than  $s_0$ . This exists because  $D_a$  is closed by infinite disjunction. As

 $s_1 \ge s_4$ , let X<sub>1</sub> be the set of possibilities compatible with  $\neg s_2$  but not compatible with  $s_4$ . he set cannot have more than one element, or we could find an element  $\tau_1 \in X_1$  such that he see called the set of the non-set one dense is of we could induce a dense  $x_1 \in X_1$  but that  $x_1 \in X_1 = X_1$  is the interval  $x_1 \in X_1$  contains one possibility, then  $-s_0$  is the immediate successor. If  $x_1$  is empty then  $s_0 = -s_0$ . Similarly, we can start with  $s_0 \in D_0$  and find  $s_0 \in D_0$  such that  $s_0$ .  $i_1$  is simply then  $i_2 \subset \neg u_{n-1}$  multiply, we can stark with  $v_n \in v_n$  and multiply  $i_2 \subset v_{n-1}$  solution in  $i_2$ , but it is the broadest statement in  $D_k$  that is narrower than  $\neg u_n$ . Let  $X_1$  be the set of possibilities compatible with  $u_n$  but to compatible with  $u_n$ . If  $X_1$  contains one possibility, then  $\neg u_n$  is the immediate successor and if  $X_1$  is empty then  $s_0 \equiv \neg u_n$ . To prove (10), let  $s_1, s_2 \in D$  such that  $s_2$  is the immediate successor of  $s_1$ . This means can write  $s_2 \equiv s_1 \lor x_1$  for some  $x_1 \in X$ . This means  $s_1 \equiv "x < x_1"$  while  $s_2 \equiv "x \le x_1$ " and

herefore  $s_1 \in B_b$ . 

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of the immediate successors. Let  $(\cdot)^{++}: \mathcal{B}_h \to \mathcal{B}_n$  be the function such that  $\neg(b^{++}) = \neg b^{++}$  is the immediate successor of b. Let b:  $X \to B_b$  be the function such that  $x = b(x) - b(x)^{++}$ . On X define the ordering  $\leq$  such that  $x_1 \leq x_2$  if and only if  $b(x_1) \leq b(x_2)$ . Since  $(B_b, \leq)$ s linearly ordered so is (X, <). To show that the ordering is natural, suppose  $x_1 < x_2$ hen  $b(x_1) \leq -b(x_2)^{++} \leq b(x_2)$  and therefore  $x_1 \leq b(x_2)$ . We also have  $-b(x_2)^{++} \leq b(x_2)$ . and  $o_{11}(x \to o_{12}) \to o_{12}(x)$  and therefore  $x_1 \in o_{12}(x)$ . We also have  $\neg o_{11}(x) \to o_{12}(x) \to o_{12}(x) \to o_{12}(x)$ . We also have  $\neg o_{11}(x) \to o_{12}(x) \to o_{12}(x)$ . We have  $a_1(x) \to a_2(x) \to o_{12}(x)$ . This means that given a possibilities  $x_1 \in X$ , all and only the possibilities lower than  $x_1$  are compatible with  $b(x_1)$  and therefore  $b(x_1) \equiv "x < b_{12}(x)$ .  $x_1^n$ , while all and only the possibilities greater than  $x_1$  are compatible with  $b(x_1)^{++}$  and herefore  $b(x_1)^{*+} \equiv "x > x_1"$ . The topology is the order topology and the domain has a

#### 3.3 References and experimental ordering

In the previous section we have characterized what a quantity is and how it relates to an experimental domain. But as we saw in the first chapters, the possibilities of a domain are not objects that exist a priori: they are defined based on what can be verified experimentally. Therefore simply assigning an ordering to the possibilities of a domain does not answer the more fundamental question: how are quantities actually constructed? How do we, in practice, create a system of references that allows us to measure a quantity at a given level of precision? What are the assumptions we make in that process?

In this section we construct ordering from the idea of a reference that physically defines a boundary between a before and an after. In general, a reference has an extent and may overlap with others. We define ordering in terms of references that are clearly before and overapy want Others, we denote the possibilities have as on references unat and covery neuron and after others. We see that the possibilities have a natural ordering only if they are generated from a set of references that is refinable (we can always find finer ones that do not overlap) and for which before/on/after are mutually exclusive cases. The possibilities, then, are the finest references possible.

We are by now so used of the ideas of real numbers, negative numbers and the number zero that it is difficult to realize that these are mental constructs that are, in the end, somewhat recent in the history of humankind. Yet geometry itself started four thousand years ago as an experimentally discovered collection of rules concerning lengths, areas and angles. That is, human beings were measuring quantities well before the real numbers were invented. So, how does one construct instruments that measure values?

To measure position, we can use a ruler, which is a series of equally spaced marks. We give a label to each mark (e.g. a number) and note which two marks are closest to the targe osition (e.g. between 1.2 and 1.3 cm). To measure weight, we can use a balance and a set of pually prepared reference weights. The balance can clearly tell us whether one side is heavier than the other, so we use it to compare the target with a number of reference weights and note the two closest (e.g. between 300 and 400 grams). A clock gives us a series of events to note two caseses (e.g. networks not and nos (and nos (and nos)). A cake gives us a sense or versits or compare to (e.g. earth's roution on its such that is, the takes of a cock). We can pour water from a reference container into another as many times as are needed to measure its volume. In all these cases what actually happens is similar: we have a reference (e.g. a mark on a ruler, these cases what actually happens is similar: we have a reference (e.g. a mark on a ruler, these cases what actually happens is similar: we have a reference (e.g. a mark on a ruler, these cases we have a set of the comparison of the compa a set of equally prepared weights, a number of ticks of a clock) and it is fairly easy to tel

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*Proof.* By definition, we have  $\neg b \land \neg a \leq o$  and by 1.23  $\neg (\neg b \land \neg a) \lor o \equiv T \equiv b \lor a \lor o$ . Definition 3.19. A reference  $r_1 = (b_1, o_1, a_1)$  is finer than another reference  $r_2 = (b_2, o_2, a_2)$ if  $b_1 \ge b_2$ ,  $o_1 \le o_2$  and  $a_1 \ge a_2$ .

Corollary 3.20. The finer relationship between references is a partial order

Proof. As the finer relationship is directly based on narrowness, it inherits its reflexivity. tisymmetry and transitivity properties and is therefore a partial order.

Definition 3.21. A reference is strict if its before, on and after statements are incom with a tible. Formally, r = (b, o, a) is such that  $b \neq a$  and  $o \equiv \neg b \land \neg a$ . A reference is loose if it e not strict

Remark. In general, we can't turn a loose reference into a strict one. The on statement an be made strict by replacing it with  $\neg b \land \neg a$ . This is possible because o is not required to e verifiable. The before (and after) statements would need to be replaced with statements like  $b \wedge \neg a$ , which are not in general verifiable because of the negation.

To measure a quantity we will have many references one after the other: a ruler will have many marks, a scale will have many reference weights, a clock will keep ticking. What does it mean that a reference comes after another in terms of the before/on/after statements? If reference  $\mathbf{r}_1$  is before reference  $\mathbf{r}_2$  we expect that if the value measured is before the first it will also be before the second, and if it is after the second it will also be after the first Note that this is not enough, though, because as references have an extent they may overlap.

And if they overlap one can't be after the other. To have an ordering properly defined we must have that the first reference is entirely before the second. That is, if the value measured is on the first it will be before the Mathematically, this type of orde

before and strictly after. It does n One may be tempted to define the roquires refining the references and refined references, not the original or

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Definition 3.22. A reference is be he first it cannot be on or after the Proposition 3.23. Reference ord irreflexivity; not r < r</li> transitivity: if r<sub>1</sub> < r<sub>2</sub> and r<sub>2</sub>

Proof. For irreflexivity, since th d therefore by o = o v a. Therefo

and is therefore a strict partial reflexive.

Proposition 3.28. Let  $r_1 = (b_1, o_1, a_1)$  and  $r_2 = (b_2, o_2, a_2)$  be two strict references. Then  $\textit{Proof. Let } \mathbf{r}_1 < \mathbf{r}_2. \textit{ By 3.27, we have } \neg \mathbf{a}_1 \preccurlyeq \mathbf{b}_2. \textit{ Conversely, let } \neg \mathbf{a}_1 \preccurlyeq \mathbf{b}_2. \textit{ Then } \neg \mathbf{a}_1 \ne \neg \mathbf{b}_2.$ because the references are strict,  $\neg a_1 \equiv b_1 \lor o_1$  and  $\neg b_2 \equiv o_2 \lor a_2$ . Therefore  $b_1 \lor o_1 \neq o_2 \lor a_2$ and  $\mathbf{r}_1 < \mathbf{r}_2$  by definition.

Proof. We have  $b_1 \vee o_1 \equiv (b_1 \vee o_1) \land \top \equiv (b_1 \vee o_1) \land (b_2 \vee o_2 \vee a_2) \equiv ((b_1 \vee o_1) \land b_2) \lor$ 

 $(b_1 \vee o_1) \land (o_2 \vee a_2) \equiv ((b_1 \vee o_1) \land b_2) \lor \perp \equiv (b_1 \vee o_1) \land b_2$ . Therefore  $b_1 \lor o_1 \preccurlyeq b_2$ . And

Similarly, we have  $o_2 \vee a_2 \equiv (o_2 \vee a_2) \wedge T \equiv (o_2 \vee a_2) \wedge (b_1 \vee o_1 \vee a_1) = ((o_2 \vee a_2) \wedge (b_1 \vee a_2))$ 

Since  $b_1 \lor o_1 \lor a_1 \equiv \top$ , we have  $\neg a_1 \preccurlyeq b_1 \lor o_1$ . Similarly  $\neg b_2 \preccurlyeq o_2 \lor a_2$ . Since  $b_1 \lor o_1 \end{cases} \preccurlyeq o_2 \lor a_2$ .

Since  $b_1 \leq b_2$ ,  $a_2 \leq a_1$ ,  $b_1 \leq \neg a_2$  and  $\neg a_1 \leq b_2$ , the two references are aligned.

Since  $b_1 \vee o_1 \neq o_2 \vee a_2,$  we have  $b_1 \neq a_2$  which means  $b_1 \preccurlyeq \neg a_2.$ 

 $(o_1 \lor a_2) \land a_1 ) \equiv \bot \lor ((o_2 \lor a_2) \land a_1) \equiv (o_2 \lor a_2) \land a_1$ . Therefore  $a_2 \lor o_2 \lessdot a_2) \land (o_1 \lor a_1) \equiv (o_2 \lor a_2) \land a_1$ .

ince  $b_1 \leq b_1 \vee o_1$ , we have  $b_1 \leq b_2$ .

 $\leq o_2 \lor a_2$ , we have  $a_2 \leq a_1$ .

 $a_1 \neq \neg b_2$  and therefore  $\neg a_1 \leq b_2$ .

 $1 < r_2$  if and only if  $\neg a_1 \leq b_2$ .

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Definition 3.29. A reference is the immediate predecessor of another if nothing can be bound before the second and after the first. Formally,  $r_1 < r_2$  and  $a_1 * b_2$ . Two references cutive if one is the immediate successor of the oth

Proposition 3.30. Let  $r_1 = (b_1, o_1, a_1)$  and  $r_2 = (b_2, o_2, a_2)$  be two references. If  $r_1$  is nmediately before  $\mathbf{r}_2$  then  $\mathbf{b}_2 \equiv \neg \mathbf{a}_1$ .

*Proof.* Let  $r_1$  be immediately before  $r_2$ . Then  $a_1 \neq b_2$  which means  $b_2 \preccurlyeq \neg a_1$ . By 3.27 e also have  $\neg a_1 \leq b_2$ . Therefore  $b_2 \equiv \neg a_1$ .

Proposition 3.31. Let  $r_1 = (b_1, o_1, a_1)$  and  $r_2 = (b_2, o_2, a_2)$  be two strict references. Then is immediately before  $\mathbf{r}_2$  if and only if  $\mathbf{b}_2 \equiv -\mathbf{a}_1$ 

Proof. Let  $r_1$  be immediately before  $r_2$ . Then  $b_2 \equiv \neg a_1$  by 3.30. Conversely, let  $b_2 \equiv \neg a_1$ . Then  $r_1 < r_2$  by 3.28. We also have  $a_1 \neq \neg a_1$ , therefore  $a_1 \neq b_2$  and  $r_1$  is immediately before r<sub>2</sub> by definition.

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neans we can find  $\mathbf{r}_3 = (\mathbf{b}, \neg \mathbf{b} \land \neg \mathbf{a}_2, \mathbf{a}_2)$  for some  $\mathbf{b} \in \mathcal{D}_k$  such that  $\mathbf{r}_3 < \mathbf{r}_2$  and therefore ¬a1 ≤ b ≤ ¬a2.

For the third, suppose  $a_1 \in D_a$  and  $b_2 \in D_b$  such that  $\neg a_1 \prec b_2$ . Then  $r_1 = (\bot, \neg a_1, a_1)$ and  $\mathbf{r}_2 = (\mathbf{b}_2, \neg \mathbf{b}_2, \bot)$  are strict references aligned with the domain such that  $\mathbf{r}_1 < \mathbf{r}_2$  but  $\mathbf{r}_2$ fiate successor of  $\mathbf{r}_1$ . This means we can find  $\mathbf{r}_3 = (\mathbf{b}, \neg \mathbf{b} \land \neg \mathbf{a}, \mathbf{a})$  such that  $\mathbf{r}_1 < \mathbf{r}_3 < \mathbf{r}_2$  and therefore  $\neg a_1 \leq b < \neg a \leq \neg b_2$ .

Proposition 3.37. Let D be an experimental domain generated by a set of refinable aligned references. Then all elements of D are part of a pair  $(s_b, \neg s_a)$  such that  $s_b \in D_b$ ,  $s_a \in D_a$  and  $\neg s_a$  is the immediate successor of  $s_b$  in D or  $s_b \equiv \neg s_a$ . Moreover if  $s \in D$  has liate successor, then  $s \in D_b$ .

*Proof.* Let  $\mathcal D$  be an experimental domain generated by a set of refinable aligned strict eferences. Let  $s_b \in D_b$ . Let  $A = \{a \in D_a | a \lor s_b \notin \top\}$ . Let  $s_a = \bigvee_{a \in A} a$ . First we show that  $s_b \leq \neg s_a$ . We have  $s_b \wedge \neg s_a \equiv s_b \wedge \neg \bigvee_{a \neq a} a \equiv s_b \wedge \bigwedge_{a \neq a} \neg a \equiv \bigwedge_{a \neq a} s_b \wedge \neg a$ . For all  $a \in A$  we have  $a \lor s_b \notin T$ ,  $\neg a \notin s_b$  which means  $s_b \notin \neg a$  because of the total order of D. This means that

 $\wedge \neg a \equiv s_b$  for all  $a \in A$ , therefore  $s_b \wedge \neg s_a \equiv s_b$  and  $s_b \preccurlyeq \neg s_a$ . Next we show that no statement  $s \in D$  is such that  $s_h < s < -s_a$ . Let  $a \in D_a$  such that  $s_{b} < -a$ . By construction  $a \in A$  and therefore  $-s_{a} \leq -a$ . Therefore we can't have  $s_{b} < a < -s_{a}$ . We also can't have  $b \in D_{b}$  such that  $s_{b} < b < -s_{a}$ : by 3.36 we'd find  $a \in D_{a}$  such that

 $s_b < a \le b < -s_0$  which was ruled out. So there are two cases. Either  $s_b \neq -s_0$  then  $s_b < -s_0$ :  $s_0$  is the immediate successor of b. Or  $s_b \equiv \neg s_0$ .

The same reasoning can be applied starting from  $s_a \in D_a$  to find a  $s_b \in D_b$  such that  $s_b$  is he immediate predecessor of  $\neg s_a$  or an equivalent statement. This shows that all elements of D are paired

To show that if a statement in D has a successor then it must be a before statement, let  $s_1, s_2 \in D$  such that  $s_2$  is the immediate successor of  $s_1$ . By 3.36, in all cases where  $s_1 \notin D_8$  and  $s_2 \notin D_a$  we can always find another statement between the two. Then we must have that  $s_1 \in D_k$  and  $s_2 \in D_n$ .

Theorem 3.38 (Reference ordering theorem). An experimental domain is naturally orlered if and only if it can be generated by a set of refinable aligned strict references

*Proof.* Suppose  $D_X$  is an experimental domain generated by a set of refinable aligned ences. Then by 3.34 and 3.37 the domain satisfies the requirement of theorem 16 and therefore is naturally ordered.

Now suppose  $D_X$  is naturally ordered. Define the set  $B_b$ ,  $B_a$  and D as in 3.12. Let  $R = \{(b, \neg b \land \neg a, a) | b \in B_b, a \in B_a, b \prec \neg a\}$  be the set of all references constructed from the basis. First let us verify they are references. The before and after statements are verifiable since they are part of the basis. The on statement  $\neg b \land \neg a$  is not a contradiction since b < ¬a means b  $\neq$  a and b  $\neq$  ¬a. The on statement is broader than ¬b ∧ ¬a as they are equivalent and it is broader than b ∧ a as that is a contradiction since b < ¬a. Therefore R s a set of references. Since the before and after statements of R coincide with the basis of the domain,  $D_X$  is generated by R.

3.3. REFERENCES AND EXPERIMENTAL ORDERING

For transitivity, if  $\mathbf{r}_1 < \mathbf{r}_2$ , we have  $\mathbf{b}_1 \vee \mathbf{o}_1 \neq \mathbf{o}_2 \vee \mathbf{a}_2$  and therefore  $\neg(\mathbf{b}_1 \vee \mathbf{o}_1) \geq \mathbf{o}_2 \vee \mathbf{a}_2$ by 1.23. Since  $b_1 \vee o_1 \vee a_1 \equiv \tau$ , we have  $a_1 \ge \neg(b_1 \vee o_1)$ . Similarly if  $r_2 < r_3$  we'll have  $a_2 \ge \neg (b_2 \lor o_2) \ge o_3 \lor a_3$ . Putting it all together  $\neg (b_1 \lor o_1) \ge o_2 \lor a_2 \ge a_2 \ge \neg (b_2 \lor o_2) \ge o_3 \lor a_3$ . which means  $b_1 \vee o_1 \neq o_3 \vee a_3$ . Corollary 3.24. The relationship  $r_1 \le r_2$ , defined to be true if  $r_1 < r_2$  or  $r_1 = r_2$ , is a

partial on

As we saw, two references may overlap and therefore an ordering between them cannot be defined. But references can overlap in different ways. Suppose we have a vertical line one millimeter thick and call the left side the part before

the line and the right side the part after. We can have another vertical line of the same thickness overlapping but we can also have a horizontal line which will also, at some point, overlap. The case of the two vertical lines is something that, through finding finer references can be given a linear order. The case of the vertical and horizontal line, instead, cannot Intuitively, the vertical lines are aligned while the horizontal and vertical are not. Conceptually, the overlapping vertical lines are aligned because we can imagine narrower

lines around the borders, and those lines will be ordered references in the above sense: each line would be completely before or after, without intersection. This means that the before and not-after statements of one reference are either narrower or broader than the before and notafter statements of the other. That is, alignment can also be defined in terms of narr of statements

Note that if a reference is strict, before and after statements are not compatible and therefore the before statement is narrower than the not-after statement. This means that given a set of aligned strict references, the set of all before and not-after statements is linearly

#### 3.3. REFERENCES AND EXPERIMENTAL ORDERING

Definition 3.33. Let D be a domain generated by a set of references R. A reference = (b, o, a) is said to be aligned with D if  $b \in D_b$  and  $a \in D_a$ .

Proposition 3.34. Let D be an experimental domain generated by a set of aligned stric ferences R and let  $D = D_b \cup \neg (D_a)$ . Then  $(D, \preccurlyeq)$  is linearly ordered.

Proof. By 3.26 we have that  $B = B_b \cup \neg(B_a)$  is aligned by narrowness. By 3.15 the rdering extends to D.

Having a set of aligned references is not necessarily enough to cover the whole space at all levels of precision. To do that we need to make sure that, for example, between two references that are not consecutive we can at least put a reference in between Or that if we have two references that overlap, we can break them apart into finer ones that do not overlap and one is after the other

We call a set of references refinable if the domain they generate has the above mentioned properties. This allows us to break up the whole domain into a sequence of references that o not overlap, are linearly ordered and that cover the whole space. As we get to the fines references, their before statements will be immediately followed by the negation of their after statements, since there can't be any reference in between. Conceptually, this will give us the second and the third condition of the domain ordering theorem 3.16.

Definition 3.35. Let D be an experimental domain generated by a set of aligned references 2. The set of references is refinable if, given two strict references  $r_1 = (b_1, o_1, a_1)$  and  $\mathbf{p}_2 = (\mathbf{b}_2, \mathbf{o}_2, \mathbf{a}_2)$  aligned with  $\mathcal{D}$ , we can always:

• find an intermediate one if they are not consecutive; that is, if  $r_1 < r_2$  but  $r_2$  is not the immediate successor of  $\mathbf{r}_1$ , then we can find a strict reference  $\mathbf{r}_2$  aligned with T such that  $r_1 < r_3 < r_2$ .

• refine overlapping references if one is finer than the other; that is, if  $o_2 < o_1$ , we can find a strict reference  $r_3$  aligned with D such that  $o_3 \le o_1$  and either  $b_3 \equiv b_1$  and  $r_1 < r_2$  or  $a_2 \equiv a_1$  and  $r_2 < r_1$ .

Proposition 3.36. Let D be an experimental domain generated by a set of refinable aligned

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#### 3.4. DISCRETE OUANTITIES

Now we show that R consists of aligned strict references. We already saw that b \* aand we also have  $\neg b \land \neg a$  is incompatible with both b and a. The references are strict. To show they are aligned, take two references. The before and not after statements are inearly ordered by 3.14 which means the references are aligned.

To show R is refinable, note that each reference can be expressed as  $(*x < x_1^n, *x_1 \le x \le x_2^n, *x > x_2^n)$  where  $x_1, x_2 \in X$  and  $*x_1 \le x \le x_2^n \equiv *x \ge x_1^n \land *x \le x_2^n$ . That is, very reference is identified by two possibilities  $x_1, x_2$  such that  $x_1 \leq x_2$ . Therefore take two references  $\mathbf{r}_1, \mathbf{r}_2 \in R$  and let  $(x_1, x_2)$  and  $(x_3, x_4)$  be the respective pair of possibilities we can use to express the references as we have shown. Suppose  $\mathbf{r}_1 < \mathbf{r}_2$  but they are not consecutive. Then " $x \le x_2$ " < " $x < x_3$ ". That is, we can find  $x_5 \in X$  such that  $x_2 < x_5 < x_3$ which means " $x \le x_2$ "  $\le$  " $x < x_5$ " and " $x \le x_5$ "  $\le$  " $x < x_3$ ". Therefore the reference  $\mathbf{r}_3 \in \mathbb{R}$ identified by  $(x_5, x_5)$  is between the two references. On the other hand, assume the second reference is finer than the first. Then  $x_1 \le x_3$  and  $x_4 \le x_2$  with either  $x_1 \ne x_3$  or  $x_4 \ne x_2$ . Consider the references  $\mathbf{r}_3, \mathbf{r}_4 \in R$  identified by  $(x_1, x_1)$  and  $(x_2, x_2)$ . Either  $\mathbf{r}_3 < \mathbf{r}_2$  or  $r_2 < r_4$ . Also note that the before statements of  $r_1$  and  $r_3$  are the same and the after statements of  $r_1$  and  $r_4$  are the same. Therefore we satisfy all the requirements and the set R is refinable by definition.

To recap, experimentally we construct ordering by placing references and being able to tell whether the object measured is before or after. We can define a linear order on the possibilities, and therefore a quantity, only when the set of references meets special conditions. The references must be strict, meaning that before, on and after are mutually exclusive. They must be aligned, meaning that the before and not-after statement must be ordered by narrowness. They must be refinable, meaning when they overlap we can always find finer references with well defined before/after relationships. If all these conditions apply, we have a linear order. If any of these conditions fail, a linear order cannot be defined.

The possibilities, then, correspond to the finest references we can construct within the domain. That is, given a value  $q_0$ , we have the possibility "the value of the property is  $q_0$ " and we have the reference ("the value of the property is less than  $q_0$ ", "the value of the property is  $q_0$ ", "the value of the property is more than  $q_0$ ").

#### 3.4 Discrete quantities

Now that we have seen the general conditions to have a naturally ordered experimental domain, we study common types of quantities and under what conditions they arise. We start with discrete ones: the number of chromosomes for a species, the number of inhabitants of a country or the atomic number for an element are all discrete quantities. These are quantities that are fully characterized by integers (positive or negative) We will see that discrete quantities have a simple characterization: between two references

there can only be a finite number of other references. The first thing we want to do is characterize the ordering of the integers. That is, we want

to find necessary and sufficient conditions for an ordered set of elements to be isomorphic to a subset of integers. First we note that between any two integers there are always finitely many elements. Let's call sparse an ordered set that has that property: that between two elements there are only finitely many. This is enough to say that the order is isomorphic to

### Reference ordering theorem

To define an **ordered** sequence of "instants", the references must be (nec/suff conditions):

- Strict an event is strictly before/on/after the reference (doesn't extend over the tick)
- Aligned shared notion of before and after (logical relationship between statements)
- Refinable overlaps can always be resolved

#### Additionally:

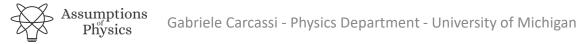
Between any two references we can always have another reference  $\Rightarrow$  real numbers

Only finitely many references between any two references  $\Rightarrow$  **integers** 

## For time, these conditions are idealizations



## Inevitable failure of time ordering



### How does this model of time break down?

The ticks of a clock have an extent and so do the events (references not strict) If clocks have jitter, they cannot achieve perfect synchronization (references not aligned) We cannot make clock ticks as narrow as we want (references not refinable)

### No consistent ordering: no "objective" "before" and "after"

In relativity, different observers measure time differently, but the order is the same. We should expect this to fail at "small" scales.

# A better understanding of space-time means creating a more realistic formal model that accounts for those failures



Hard to say, but we can argue from necessity

## What type of models should we use?

(N.B. this is a toy model, each point should have infinitely many neighbors) Lack of order at small scales, order at large enough scale

What we can distinguish experimentally (i.e. topology) seems to be linked to how precisely we want to distinguish (i.e. geometry)

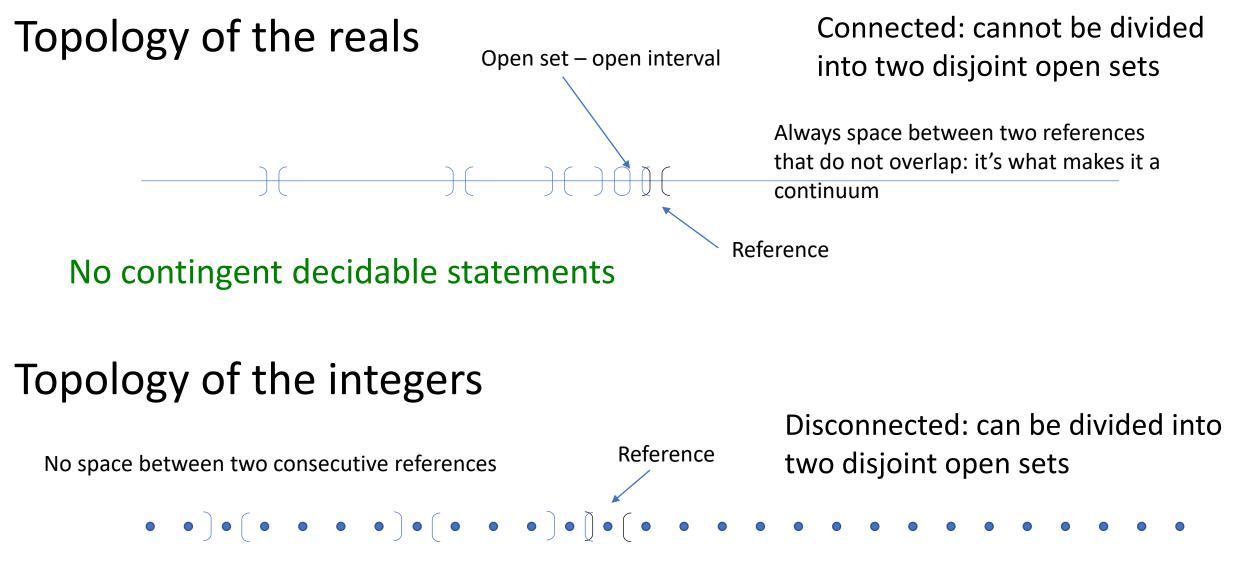
## Current mathematical tools have a hard division between topology and geometry

sumptions Bhysics Gabriele Carcassi - Physics Department - University of Michigan Our reasoning contradicts the expectations of many that time is simply "discrete" at the smallest scale

This intuition is based on the idea that the continuum is like the discrete but "with more points"

This idea (though extremely common in physics) is flawed

Assumptions Physics Gabriele Carcassi - Physics Department - University of Michigan



#### All contingent statements are decidable

Gabriele Carcassi - Physics Department - University of Michigan

## Conclusion

- Physically well-defined objects must be in terms of operational definitions
  - The primitive elements are the verifiable statements, which are typically idealized and left formally undefined
  - A physical theory is fully characterized by the logical relationships of countably many verifiable statements
- Time is what is measured by a clock
  - The main feature of clocks is that they can be synchronized with each other, which is operationally defined
- Clocks can be formally modeled by a set of ticks that experimentally define a before and after
  - We recover time as a continuum under suitable idealized conditions (i.e. all clocks can be perfectly synchronized, reach arbitrary resolution, ... )
- We should not expect the assumptions required by ordered (and continuous) time to hold at the smallest scale
  - A better understanding of space-time means creating a more realistic formal model that takes into account those failures



# Supplemental

