On the reality of the quantum state once again

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Assumptions of Physics

The paper

On the reality of the quantum state once again

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Under review

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Thesis

ψ-ontic models are not compatible with quantum mechanics

Combined with PBR ⇒ Harrigan-Spekkens categorization is essentially “empty”

More in general: measure theory cannot reproduce quantum probability and quantum information theory

Plan

- Ontological model review
  - Ontological vs epistemic states, implicit use of standard (Kolmogorov) probability, PBR rules out epistemic models (w/ independent preparation)
- ψ-ontic models violate quantum information theory
  - Epistemic states and quantum mixed states form a real vector space partially ordered by information entropy (which quantifies the epistemic content), the linear map between them is not an order isomorphism (cannot map the epistemic content in any meaningful way)
- Failure of classical measure theory and quantum contextuality
  - "Counting" is different in measure theory and quantum mechanics (i.e. $1 + 1 \leq 2$), quantum contextuality is linked to this difference

Disclaimer

- We are not going to talk about specific interpretations
  - ONLY about Harrigan-Spekkens categorization
- We are not going to talk about epistemic and ontological models in general
  - ONLY the narrow definition given by the Harrigan-Spekkens categorization
- We are not going to talk about the philosophical implications
  - ONLY the mathematical and physical implications

Ontological model review

Start with the operational settings for Quantum Mechanics

Preparation protocol P

Density operator ρ

Measurement protocol M

Measurement values $\{k\}$ POVM $\{E_k\}$

$$p(k|P, M) = \mathrm{tr}(\rho E_k)$$

Ontological model puts an intermediate step

Ontological states Λ

$$\int_\Lambda d\lambda \, p(k|\lambda, M) \, p(\lambda|P) = \mathrm{tr}(\rho E_k)$$

Epistemic state

Epistemic states mix according to standard (Kolmogorov) rules

Mixture of pure states $\{\psi_i\}$ with probabilities $\{w_i\}$

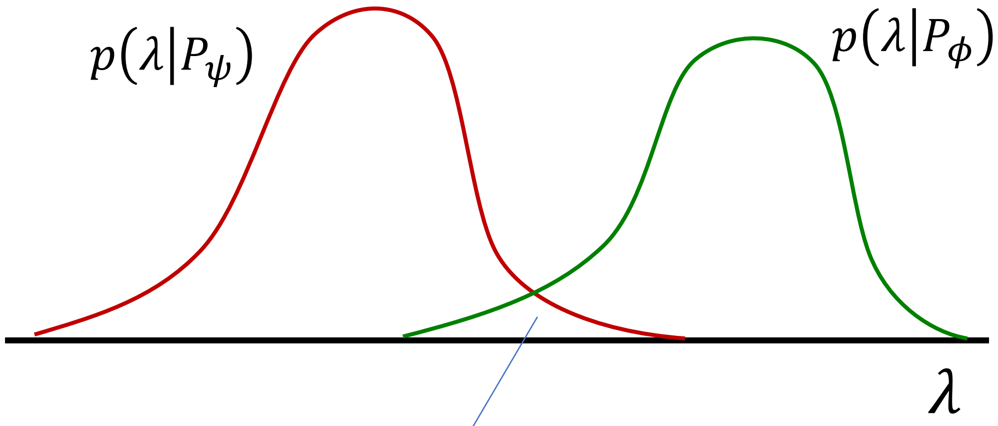
$$p(\lambda|P) = \sum_i w_i p\left(\lambda \middle| P_{\psi_i}\right)$$

Each ontological model is therefore characterized simply by the epistemic states that correspond to pure states

We will gloss over some technical imprecisions
(e.g. probability and probability densities are different objects)

Classify models based on $p(\lambda|P_{\psi_i})$

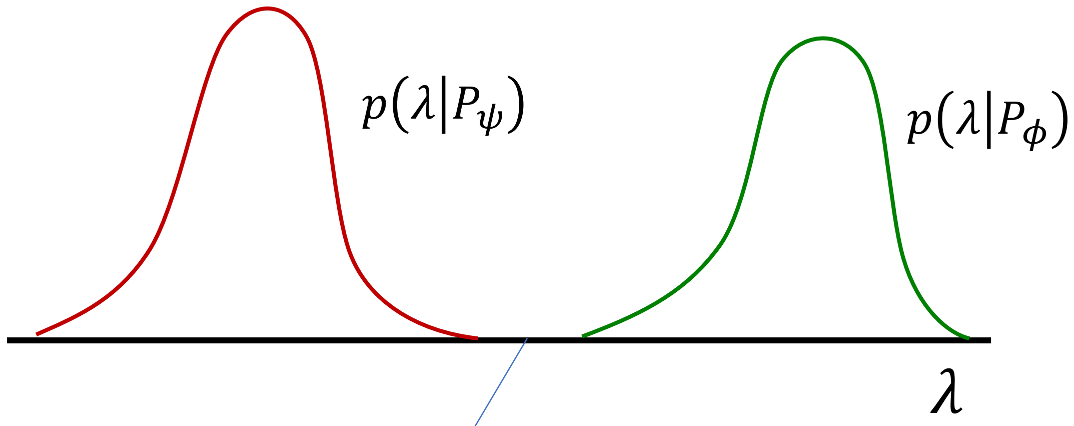
Epistemic models



Can overlap: same ontological state can be prepared in different ways

PBR claims these are ruled out (w/ independent preparation)

Ontological models



No overlap: ontological state retains information about preparation

Complete if $p(\lambda_\psi|P_\psi) = 1$ for all ψ

What about these?

ψ-ontic models violate quantum information theory

$$\mu(U)$$

Measure theory

ρ uniform over U

Standard probability

Information theory

$$\rho(x) = \frac{1}{\mu(U)}$$

$$H(\rho) = \log \mu(U)$$

Violation of probability ≡ violation of information theory ≡ measure theory does not reproduce Hilbert spaces

$$\langle\psi|\phi\rangle$$

Inner product

$$\rho = \frac{1}{2}\rho_\psi + \frac{1}{2}\rho_\phi$$

*quotient the phase

Quantum probability

Quantum information theory

$$p(\psi|\phi) = |\langle\psi|\phi\rangle|^2$$

$$H(\rho) = H\left(\frac{1+\sqrt{p}}{2}, \frac{1-\sqrt{p}}{2}\right)$$

Since in a ψ-ontic model the distributions are not overlapping, and since the entropy of each pure state must be zero, we have:

$$H\left(\frac{1}{2}p(\lambda|\psi)+\frac{1}{2}p(\lambda|\phi)\right)=1+\frac{1}{2}H\big(p(\lambda|\psi)\big)+\frac{1}{2}H\big(p(\lambda|\phi)\big)=1$$

In a ψ-ontic model, the equal mixture of any two distinct pure states has unitary entropy

In quantum mechanics, the entropy of the equal mixture of two pure states is given by:

$$H\left(\frac{1}{2}\rho_\psi+\frac{1}{2}\rho_\phi\right)=-\frac{1+\sqrt{p}}{2}\log\frac{1+\sqrt{p}}{2}-\frac{1-\sqrt{p}}{2}\log\frac{1-\sqrt{p}}{2} \qquad p=|\langle\psi|\phi\rangle|^2$$

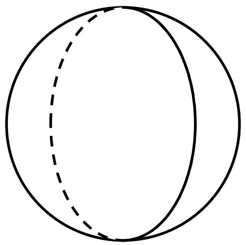
This is equal to one if and only if $p=0$, the states are orthogonal

In a ψ-ontic model, all pure states must be orthogonal ⇒ NO GO!

The degree of overlap between epistemic states is constrained by the entropic structure of the mixed states

To dig deeper, consider a complete ontological model of a qubit

Epistemic states $E_{HS} : p(\lambda|P)$



Quantum mixed states $E_{QM} : \rho$

Epistemic states and density matrices one-to-one for pure states

Ordered by entropy: $H(\rho)$

Real vector space: $a\rho_a + b\rho_b$

Pure states are one-to-one but not the mixed states!!!

$$\rho(\theta, \varphi)$$

$$x^2 + y^2 + z^2 \leq 1$$

Continuous distributions over the surface

Infinite dimensional function space

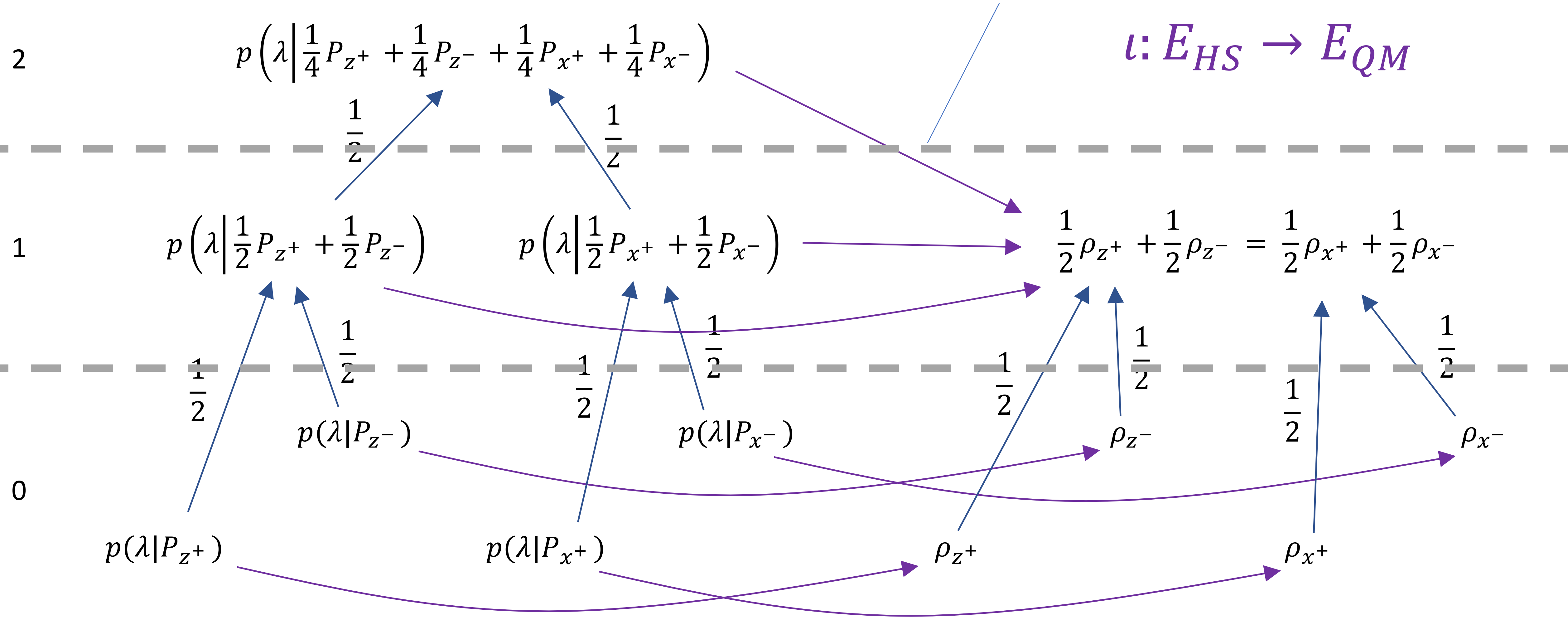
Closed three dimensional unit ball

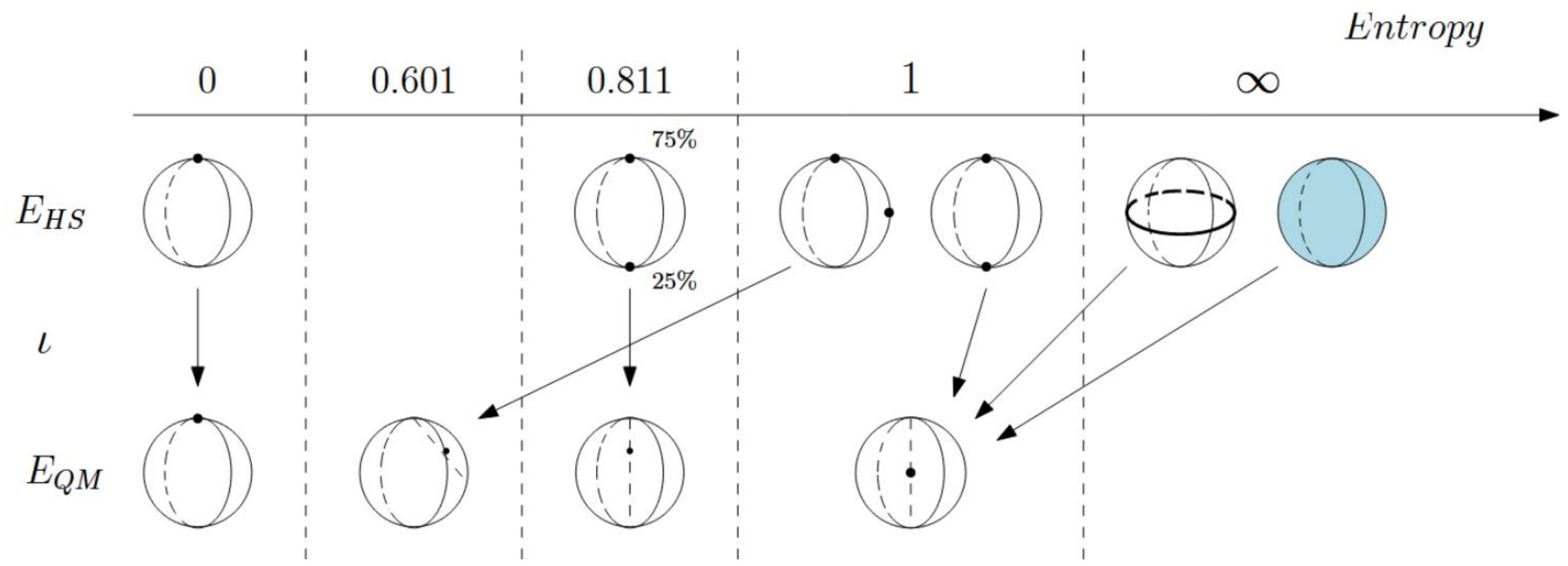
Three dimensional manifold

Not all is lost: clear way to go from E_{HS} to E_{QM}

Lost information about preparation?

Information is lost but entropy goes down!





Entropic order is lost when going from E_{HS} to E_{QM}

No ψ-ontic models

- The space of epistemic states of a ψ-ontic model is radically different from the space of mixed states of QM
- The map from epistemic states to mixed states does not preserve the order in terms of entropy
- Could we define a “new entropy” on the epistemic states?
  - It would be equivalent to redefining the full inner product, since the entropic structure is equivalent (minus unphysical phase) to the inner product
  - But you also make the rules of probability consistent with the new entropy (because a failure of classical information theory was equivalent to a failure of measure theory and Kolmogorov probability)
  - Essentially you are creating a new probability theory that is equivalent to that of quantum mechanics
- So, what’s going on? What is the origin of the problem?

Failure of measure theory and quantum contextuality

Measure theory defines "how big sets are", "how many elements are there in a set", "how we count"

$$\mu(U) \to [0, +\infty]$$

Countable additivity is a fundamental axiom of measure theory

$$\mu(\cup_i U_i) = \sum_i \mu(U_i)$$

Disjoint sets

	Single point		Finite continuous range	
	$\mu(U)$	$\log \mu(U)$	$\mu(U)$	$\log \mu(U)$
Counting measure $\mu(U) = \#U$ Number of points	1	0	$+\infty$	$+\infty$
Lebesgue measure $\mu([a,b]) = b - a$ Interval size	0	$-\infty$	$< \infty$	$< \infty$
"Quantum" measure $\mu(U) = 2^{H(\rho_U)}$ Entropy over uniform distribution	1	0	$< \infty$	$< \infty$

1. Single point is a single case (i.e. $\mu(\{\psi\}) = 1$)
2. Finite range carries finite information (i.e. $\mu(U) < \infty$)
3. Measure is additive for disjoint sets (i.e. $\mu(\cup U_i) = \sum \mu(U_i)$)

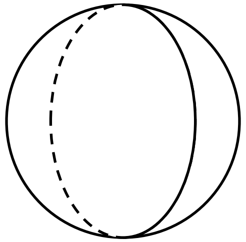
Pick two!

Counting measure picks 1 and 3
Lebesgue measure picks 2 and 3
Quantum mechanics picks 1 and 2

But QM is tested using standard probability!!!

and that is why measure theory (classical probability and information theory) cannot reproduce quantum mechanics

Let's further unpack the difference



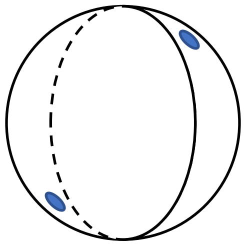
A probability space is made of three things:

- The sample space Ω of all possible cases
- A σ-algebra Σ_Ω of all statements of interest (events)
- A measure $\mu: \Sigma_\Omega \to [0, 1]$ that assigns a probability to each event

In standard probability, the measure is over the whole σ-algebra Σ_Ω

This is not what happens in quantum mechanics

In quantum mechanics…



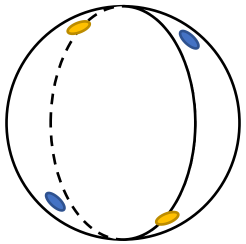
… only the first two elements are the same:

- The space $\mathcal{H}$ of all possible cases (state space)
- A σ-algebra $\Sigma_{\mathcal{H}}$ of all statements of interest
- A density operator $\rho: \mathcal{H} \to \mathcal{H}$

To retrieve a measure:

- Pick an observable O, which identifies a basis $\mathcal{B}_O \subset \mathcal{H}$
- Take the sub-algebra $\Sigma_O \subset \Sigma_{\mathcal{H}}$ defined on $\mathcal{B}_O$
- Calculate the measure $\mu_O: \Sigma_O \to [0, 1]$

No single measure: one for each sub-algebra



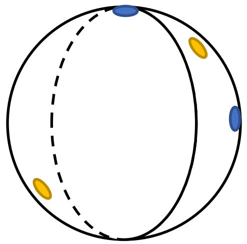
There are infinitely many sub-algebras (and measures)

The sub-algebras are "classical" in the sense that allow additive measures… (though not countably additive!)

Physically, each sub-algebra represents a "context"

Only once a context is selected, QM gives us a probability measure

But it also allows us to do something standard measure theory will never be able to do: mix contexts themselves



Spin up and spin left “live” in two different contexts

An equal mixture of spin up and spin left “live” in a “combined context” that is different from both

Physically, the “combined context” is the one associated to the measurement that extracts the most information

The HS model assumes every preparation (epistemic state) is represented by a single probability measure on all ontic states
This assumes that there is a single σ-algebra over which the whole distribution is defined, a single “context”

Failure of measure theory and quantum contextuality

- QM cannot be based on standard measure theory because entropy would require the measure to be non-additive
  - Non-additivity stems from requiring finite information/measure for both points and finite continuous intervals
- In QM, probability distributions are not defined on the full σ-algebra, but over a sub-algebra associated to a basis: a context
  - We cannot mix events from different contexts because they live in different σ-algebras
- Conversely, in HS the epistemic states are all defined on the same σ-algebra, the one of the ontic states
  - This means there is one "master context" that defines everything, and we can mix and match all probability events

Supplemental

Let $\rho_1(\lambda)$ and $\rho_2(\lambda)$ be two disjoint probability (or probability density) distributions over Λ with support U_1 and U_2 respectively

Let us calculate the entropy of $H\left(\frac{1}{2}\rho_1(\lambda) + \frac{1}{2}\rho_2(\lambda)\right)$

$$H\left(\frac{1}{2}\rho_1(\lambda) + \frac{1}{2}\rho_2(\lambda)\right) = -\int_\Lambda d\lambda \left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2\right)\log\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2\right)$$

$$= -\int_{U_1} d\lambda \frac{1}{2}\rho_1 \log\left(\frac{1}{2}\rho_1\right) - \int_{U_2} d\lambda \frac{1}{2}\rho_2 \log\left(\frac{1}{2}\rho_2\right)$$

$$= -\frac{1}{2}\log\frac{1}{2}\left(\int_{U_1} d\lambda\rho_1 + \int_{U_2} d\lambda\rho_2\right) - \frac{1}{2}\int_{U_1} d\lambda\rho_1 \log\rho_1 - \frac{1}{2}\int_{U_2} d\lambda\rho_2 \log\rho_2$$

$$= 1 + \frac{1}{2}H(\rho_1) + \frac{1}{2}H(\rho_2)$$