Space-time structure may be topological and not geometrical

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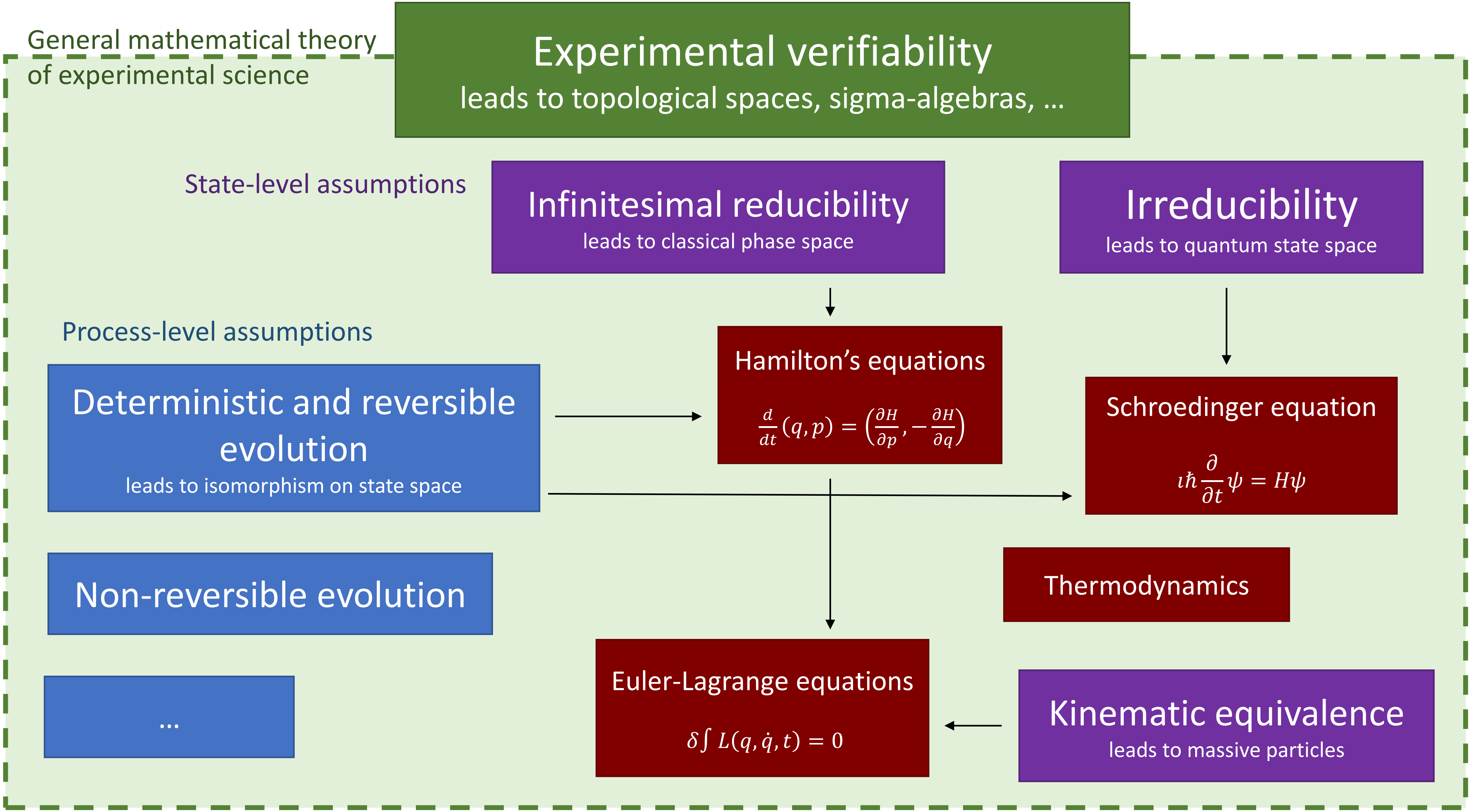
Assumptions of Physics

- This talk is part of a broader project called Assumptions of Physics (see http://assumptionsofphysics.org/)
- The aim of the project is to find a handful of physical principles and assumptions from which the basic laws of physics can be derived
- To do that we want to develop a general mathematical theory of experimental science: the theory that studies scientific theories
  - A formal framework that forces us to clarify our assumptions
  - From those assumptions the mathematical objects are derived
  - Each mathematical object has a clear physical meaning and no object is unphysical
  - Gives us concepts and tools that span across different disciplines
  - Allows us to explore what happens when the assumptions fail, possibly leading to new physics ideas


General mathematical theory of experimental science
Experimental verifiability
leads to topological spaces, sigma-algebras, …
State-level assumptions
Infinitesimal reducibility
leads to classical phase space
Irreducibility
leads to quantum state space
Process-level assumptions
Deterministic and reversible evolution
leads to isomorphism on state space
Hamilton's equations
$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$
Schroedinger equation
$\imath\hbar\frac{\partial}{\partial t}\psi = H\psi$
Non-reversible evolution
Thermodynamics
…
Euler-Lagrange equations
$\delta\int L(q,\dot{q},t) = 0$
Kinematic equivalence
leads to massive particles

Mathematical structure for space-time

- Riemannian manifold
- Differentiable manifold + inner product
- Topological manifold + differentiable structure
- Ordered topological space + locally $\mathbb{R}^n$
- Topological space + order topology

- If we want to understand why (i.e. under what conditions) space-time has the structure it has, we first need to understand why (i.e. under what conditions) it is a topological space, it has an order topology, …

Mathematical structure for space-time

- Riemannian manifold
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Geometry (lengths and angles) starts here: most fundamental structures are not geometrical

- If we want to understand why (i.e. under what conditions) space-time has the structure it has, we first need to understand why (i.e. under what conditions) it is a topological space, it has an order topology, …

Simple things first

- A similar hierarchy is present for other mathematical structures used in physics
  - Hilbert space – Inner product space + closure under Cauchy sequences – Vector space + inner product – …
- If we want true understanding, then we need to understand the simpler structure first
  - This is what our project, Assumptions of Physics, is about

Outline

- In this talk we will focus on topology and order. We will:
  - Show that topologies naturally emerge from requiring experimental verifiability
  - Show that an order topology corresponds to experimental verifiability of quantities: outcomes than can be smaller, greater or equal to others
  - Then we need to understand how quantities are constructed from experimental verifiability
    - That is, find a set of necessary and sufficient conditions under which experimental verifiability gives us an order topology
  - Argue that, in the end, those conditions are untenable at Planck scale, and that ordering cannot be experimentally defined
  - Conclude that all that is built on top of an order topology (manifolds, differentiable structures, inner product) fails to be well defined at Planck scale

Verifiable statements

- The most fundamental math structures are from logic and set theory
    - All other structures are based on that
- For science, we want to extend these with experimental verifiability
- Our fundamental object will be a verifiable statement: an assertion for which we have (in principle) an experimental test that, if the statement is true, terminates successfully in a finite amount of time
- Verifiable statements do not follow standard Boolean logic:
    - We may verify “there is extra-terrestrial life” but not its negation “there is no extra-terrestrial life”
    - No negation in general, finite conjunction, countable (infinite) disjunction

What is a topology?

- Given a set X, a topology $T \subseteq 2^X$ is a collection of subsets of X that:
  - It contains X and $\emptyset$
  - In general, not closed under complement
  - It is closed under finite intersection and arbitrary (infinite) union
- How do we get to this in physics?

Basis $\mathcal{B}$			
e_1	e_2	e_3	…

Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…

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Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain

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e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…
F	F	F	…	F	F	…
…	…	…	…	…	…	…
F	T	F	…	T	F	…
T	T	F	…	T	F	…
…	…	…	…	…	…	…

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F	F	F	…	F	F	…
…	…	…	…	…	…	…
F	T	F	…	T	F	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…

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F	F	F	…	F	F	…
…	…	…	…	…	…	…
F	T	F	…	T	F	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…

Possibilities X

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…	…	…	…	…	…	…
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Each verifiable statement corresponds to a set of possibilities in which the statement is true.

The experimental domain $\mathcal{D}_X$ induces a natural topology on the set of possibilities X

The role of logic (and math) in science is to capture what is consistent (i.e. the possibilities) and what is verifiable (i.e. the verifiable statements)

Examples

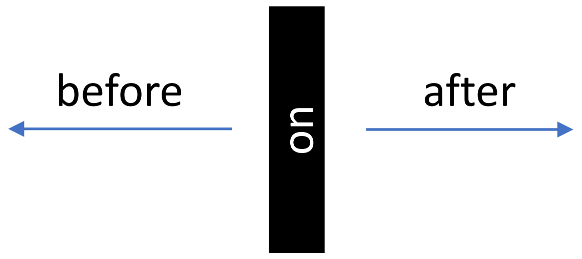
- “the mass of the photon is less than 10^{-13}eV” is verifiable and corresponds to an open set (a set in the topology)
- “the mass of the photon is exactly 0 eV” is not verifiable and is not an open set (not a set in the topology)
  - However, it is falsifiable and corresponds to a closed set (the complement is in the topology)
- Topological concepts (second countability, Hausdorff spaces, interior/exterior/boundary, …) can be better understood in terms of experimental verification
  - They are not some abstract mathematical thing: they are physically meaningful

Quantities

- We can define a quantity as a measurable property of a system that has a magnitude: can be compared to another of the same kind and found to be greater or smaller
- Mathematically a quantity is formed by:
  - a set Q
  - a linear (total) ordering $\leq: Q \times Q \rightarrow \mathbb{B}$
  - the order topology generated by the linear ordering, whose basis elements are of the form $(-\infty, q)$ and $(q, +\infty)$; that is, we can always tell experimentally whether something is more or less than something else
    - equality, in general, is not experimentally testable: for continuous quantities corresponds to infinite precision measurements

Constructing quantities and references

- The question is: how do we operationally construct quantities? How can we model that appropriately?
- We start with the idea of a reference: something physical that partitions our range into a before, on, and after
  - E.g. a line on a ruler, the tick of a clock, a standard weight for a balance scale, a threshold on an A/D converter

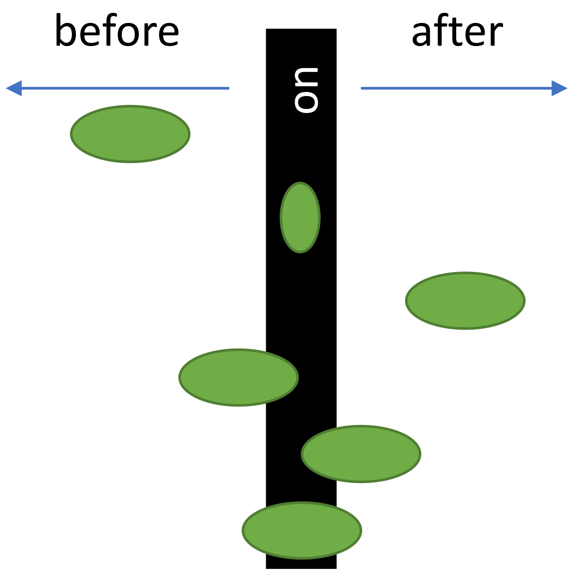


- Mathematically, a reference is a tuple of three statements b/o/a; only before and after are required to be experimentally verifiable

Constructing quantities and references

- Problem 1 - In general, before/on/after are not mutually exclusive

Before	On	After
T	F	F
F	T	F
F	F	T
T	T	F
F	T	T
T	T	T

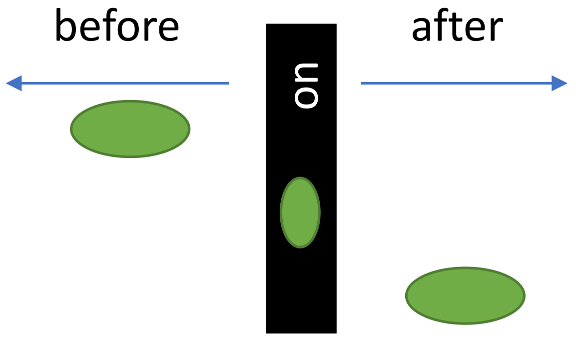


In this case, the possibilities of the domain cannot correspond to distinct values

Strict references

- We say a reference is strict if before/on/after are mutually exclusive

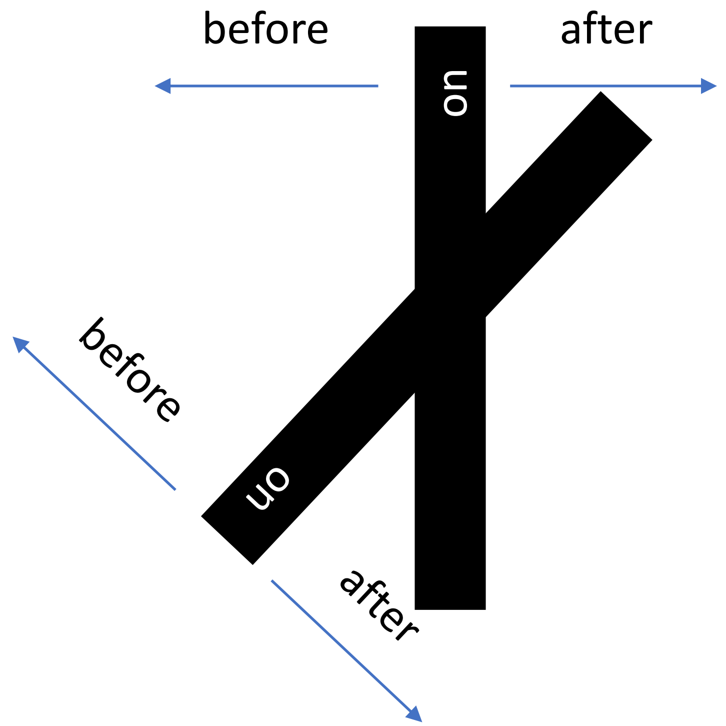
Before	On	After
T	F	F
F	T	F
F	F	T



- If the extent of what we measure is smaller than the extent of our reference, then we can always assume our references are strict

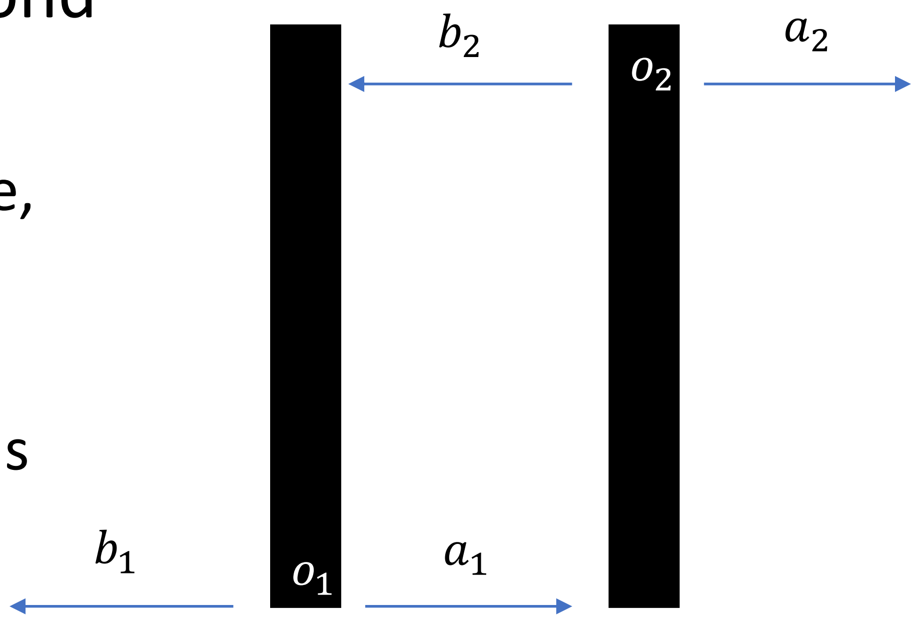
Multiple references

- Problem 2 - To construct a reference scale we need multiple references, but in general these would not construct a linear order
- We need to define what it means for references to be aligned purely on the logical relationship between statements



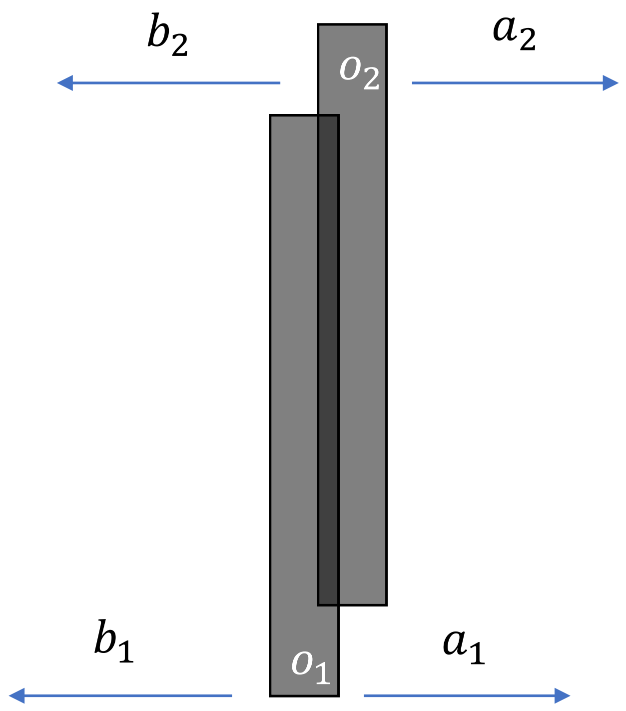
Ordered references

- We can say that reference 1 is before reference 2 if whenever we find something before or on the other, it must be before the second
- More precisely, if $b_1 \vee o_1 \not\top o_2 \vee a_2$
  - $\not\top$ Means the statements are incompatible, they can't be true at the same time
- Note how $b_1 \preccurlyeq \neg a_1 \preccurlyeq b_2 \preccurlyeq \neg a_2$
  - Where $a \preccurlyeq b$ (a is narrower than b) means that if a then b must be true as well



Aligned references

- More in general, we can say that two references are aligned if the before and not-after statement can be ordered by narrowness
- For example, $b_1 \preccurlyeq b_2 \preccurlyeq \neg a_1 \preccurlyeq a_2$
  - $\preccurlyeq$ Means that if the first statement is true then the second statement will be true as well
  - That is, the first statement is narrower, more specific
- Here we see how the ordering of references is related to the logical ordering defined by the specificity (narrowness) of the statements
- We need our references to be aligned if we want to construct a linear ordering



Resolving the overlaps

- Problem 3a - If two different references overlap, we can’t say one is before the other: we can’t fully resolve the linear order
- Problem 3b - Conversely, if two reference don’t overlap and there can be something in between, we must be able to put a reference there
- We always need a way, then, to find (possibly finer) references to explore the full space

Refinable references

- Conceptually, a set of references is refinable if we can solve the previous problems:
  - if two references overlap we can always refine them to two that do not overlap
  - if two ordered references are not consecutive (there can be something in between) we can always construct a reference in the middle
- Mathematically is not complicated, but is tedious and not so interesting
- With these definitions and some work…

Proof. By definition, we have $\neg b \wedge \neg a \preceq o$ and by 1.23 $\neg(\neg b \wedge \neg a) \vee o \equiv \top \equiv b \vee a \vee o$. ∎

Definition 3.19. *A reference* $r_1 = (b_1, o_1, a_1)$ *is **finer** than another reference* $r_2 = (b_2, o_2, a_2)$ *if* $b_1 \succcurlyeq b_2$, $o_1 \preccurlyeq o_2$ *and* $a_1 \succcurlyeq a_2$.

Corollary 3.20. *The finer relationship between references is a partial order.*

Proof. As the finer relationship is directly based on narrowness, it inherits its reflexivity, antisymmetry and transitivity properties and is therefore a partial order. ∎

Definition 3.21. *A reference is **strict** if its before, on and after statements are incompatible. Formally,* $r = (b, o, a)$ *is such that* $b \nparallel a$ *and* $o \equiv \neg b \wedge \neg a$. *A reference is **loose** if it is not strict.*

Remark. In general, we can't turn a loose reference into a strict one. The on statement can be made strict by replacing it with $\neg b \wedge \neg a$. This is possible because o is not required to be verifiable. The before (and after) statements would need to be replaced with statements like $b \wedge \neg a$, which are not in general verifiable because of the negation.

To measure a quantity we will have many references one after the other: a ruler will have many marks, a scale will have many reference weights, a clock will keep ticking. What does it mean that a reference comes after another in terms of the before/on/after statements?

If reference r_1 is before reference r_2 we expect that if the value measured is before the first it will also be before the second, and if it is after the second it will also be after the first. Note that this is not enough, though, because as references have an extent they may overlap. And if they overlap one can't be after the other. To have an ordering properly defined we must have that the first reference is entirely before the second. That is, if the value measured is on the first it will be before the se...

For transitivity, if $r_1 < r_2$, we have $b_1 \vee o_1 \nparallel o_2 \vee a_2$ and therefore $\neg(b_1 \vee o_1) \succcurlyeq o_2 \vee a_2$ by 1.23. Since $b_1 \vee o_1 \vee a_1 \equiv \top$, we have $a_1 \succcurlyeq \neg(b_1 \vee o_1)$. Similarly if $r_2 < r_3$ we'll have $a_2 \succcurlyeq \neg(b_2 \vee o_2) \succcurlyeq o_3 \vee a_3$. Putting it all together $\neg(b_1 \vee o_1) \succcurlyeq o_2 \vee a_2 \succcurlyeq a_2 \succcurlyeq \neg(b_2 \vee o_2) \succcurlyeq o_3 \vee a_3$, which means $b_1 \vee o_1 \nparallel o_3 \vee a_3$. ∎

Corollary 3.24. *The relationship* $r_1 \le r_2$, *defined to be true if* $r_1 < r_2$ *or* $r_1 = r_2$, *is a partial order.*

As we saw, two references may overlap and therefore an ordering between them cannot be defined. But references can overlap in different ways.

Suppose we have a vertical line one millimeter thick and call the left side the part before the line and the right side the part after. We can have another vertical line of the same thickness overlapping but we can also have a horizontal line which will also, at some point, overlap. The case of the two vertical lines is something that, through finding finer references, can be given a linear order. The case of the vertical and horizontal line, instead, cannot. Intuitively, the vertical lines are aligned while the horizontal and vertical are not.

Conceptually, the overlapping vertical lines are aligned because we can imagine narrower lines around the borders, and those lines will be ordered references in the above sense: each line would be completely before or after, without intersection. This means that the before and not-after statements of one reference are either narrower or broader than the before and not-after statements of the other. That is, alignment can also be defined in terms of narrowness of statements.

Note that if a reference is strict, before and after statements are not compatible and therefore the before statement is narrower than the not-after statement. This means that, given a set of aligned strict references, the set of all before and not-after statements is linearly ordered by narrowness. As we saw in the previous section, this was a necessary condition ...

Proof. We have $b_1 \vee o_1 \equiv (b_1 \vee o_1) \wedge \top \equiv (b_1 \vee o_1) \wedge (b_2 \vee o_2 \vee a_2) \equiv ((b_1 \vee o_1) \wedge b_2) \vee ((b_1 \vee o_1) \wedge (o_2 \vee a_2)) \equiv ((b_1 \vee o_1) \wedge b_2) \vee \bot \equiv (b_1 \vee o_1) \wedge b_2$. Therefore $b_1 \vee o_1 \preccurlyeq b_2$. And since $b_1 \preccurlyeq b_1 \vee o_1$, we have $b_1 \preccurlyeq b_2$.

Similarly, we have $o_2 \vee a_2 \equiv (o_2 \vee a_2) \wedge \top \equiv (o_2 \vee a_2) \wedge (b_1 \vee o_1 \vee a_1) \equiv ((o_2 \vee a_2) \wedge (b_1 \vee o_1)) \vee ((o_2 \vee a_2) \wedge a_1) \equiv \bot \vee ((o_2 \vee a_2) \wedge a_1) \equiv (o_2 \vee a_2) \wedge a_1$. Therefore $a_2 \vee o_2 \preccurlyeq a_1$. And since $a_2 \preccurlyeq o_2 \vee a_2$, we have $a_2 \preccurlyeq a_1$.

Since $b_1 \vee o_1 \nparallel o_2 \vee a_2$, we have $b_1 \nparallel a_2$ which means $b_1 \preccurlyeq \neg a_2$.

Since $b_1 \vee o_1 \vee a_1 \equiv \top$, we have $\neg a_1 \preccurlyeq b_1 \vee o_1$. Similarly $\neg b_2 \preccurlyeq o_2 \vee a_2$. Since $b_1 \vee o_1 \nparallel o_2 \vee a_2$, $\neg a_1 \nparallel \neg b_2$ and therefore $\neg a_1 \preccurlyeq b_2$.

Since $b_1 \preccurlyeq b_2$, $a_2 \preccurlyeq a_1$, $b_1 \preccurlyeq \neg a_2$ and $\neg a_1 \preccurlyeq b_2$, the two references are aligned. ∎

Proposition 3.28. *Let* $r_1 = (b_1, o_1, a_1)$ *and* $r_2 = (b_2, o_2, a_2)$ *be two strict references. Then* $r_1 < r_2$ *if and only if* $\neg a_1 \preccurlyeq b_2$.

Proof. Let $r_1 < r_2$. By 3.27, we have $\neg a_1 \preccurlyeq b_2$. Conversely, let $\neg a_1 \preccurlyeq b_2$. Then $\neg a_1 \nparallel \neg b_2$. Because the references are strict, $\neg a_1 \equiv b_1 \vee o_1$ and $\neg b_2 \equiv o_2 \vee a_2$. Therefore $b_1 \vee o_1 \nparallel o_2 \vee a_2$ and $r_1 < r_2$ by definition. ∎

Definition 3.29. *A reference is the **immediate predecessor** of another if nothing can be found before the second and after the first. Formally,* $r_1 < r_2$ *and* $a_1 \nparallel b_2$. *Two references are **consecutive** if one is the immediate successor of the other.*

Proposition 3.30. *Let* $r_1 = (b_1, o_1, a_1)$ *and* $r_2 = (b_2, o_2, a_2)$ *be two references. If* r_1 *is immediately before* r_2 *then* $b_2 \equiv \neg a_1$.

Proof. Let r_1 be immediately before r_2. Then $a_1 \nparallel b_2$ which means $b_2 \preccurlyeq \neg a_1$. By 3.27 we also have $\neg a_1 \preccurlyeq b_2$. Therefore $b_2 \equiv \neg a_1$. ∎

Proposition 3.31. *Let* $r_1 = (b_1, o_1, a_1)$ *and* $r_2 = (b_2, o_2, a_2)$ *be two strict references. Then* r_1 *is immediately before* r_2 *if and only if* $b_2 \equiv \neg a_1$.

Proof. Let r_1 be immediately before r_2. Then $b_2 \equiv \neg a_1$ by 3.30. Conversely, let $b_2 \equiv \neg a_1$. Then $r_1 < r_2$ by 3.28. We also have $a_1 \nparallel \neg a_1$, therefore $a_1 \nparallel b_2$ and r_1 is immediately before r_2 by definition. ∎

Definition 3.33. *Let* $\mathcal{D}$ *be a domain generated by a set of references* R. *A reference* $r = (b, o, a)$ *is said to be aligned with* $\mathcal{D}$ *if* $b \in \mathcal{D}_b$ *and* $a \in \mathcal{D}_a$.

Proposition 3.34. *Let* $\mathcal{D}$ *be an experimental domain generated by a set of aligned strict references* R *and let* $D = \mathcal{D}_b \cup \neg(\mathcal{D}_a)$. *Then* $(D, \le)$ *is linearly ordered.*

Proof. By 3.26 we have that $B = \mathcal{B}_b \cup \neg(\mathcal{B}_a)$ is aligned by narrowness. By 3.15 the ordering extends to D. ∎

Having a set of aligned references is not necessarily enough to cover the whole space at all levels of precision. To do that we need to make sure that, for example, between two references that are not consecutive we can at least put a reference in between. Or that if we have two references that overlap, we can break them apart into finer ones that do not overlap and one is after the other.

We call a set of references refinable if the domain they generate has the above mentioned properties. This allows us to break up the whole domain into a sequence of references that do not overlap, are linearly ordered and that cover the whole space. As we go to the finest references, their before statements will be immediately followed by the negation of their after statements, since there can't be any reference in between. Conceptually, this will give us the second and the third condition of the domain ordering theorem 3.16.

Definition 3.35. *Let* $\mathcal{D}$ *be an experimental domain generated by a set of aligned references* R. *The set of references is **refinable** if, given two strict references* $r_1 = (b_1, o_1, a_1)$ *and* $r_2 = (b_2, o_2, a_2)$ *aligned with* $\mathcal{D}$, *we can always:*

- *find an intermediate one if they are not consecutive; that is, if* $r_1 < r_2$ *but* r_1 *is not the immediate successor of* r_1, *then we can find a strict reference* r_3 *aligned with* $\mathcal{D}$ *such that* $r_1 < r_3 < r_2$.
- *refine overlapping references if one is finer than the other; that is, if* $o_2 < o_1$, *we can find a strict reference* r_3 *aligned with* $\mathcal{D}$ *such that* $o_3 \le o_1$ *and either* $b_3 \equiv b_1$ *and* $r_3 < r_2$ *or* $a_3 \equiv a_1$ *and* $r_2 < r_3$.

Proposition 3.36. *Let* $\mathcal{D}$ *be an experimental domain generated by a set of refinable aligned strict references* R ...

means we can find $r_3 = (b, \neg b \wedge \neg a_2, a_2)$ for some $b \in \mathcal{D}_b$ such that $r_3 < r_2$ and therefore $\neg a_1 \preccurlyeq b \prec \neg a_2$.

For the third, suppose $a_1 \in \mathcal{D}_a$ and $b_2 \in \mathcal{D}_b$ such that $\neg a_1 < b_2$. Then $r_1 = (\bot, \neg a_1, a_1)$ and $r_2 = (b_2, \neg b_2, \bot)$ are strict references aligned with the domain such that $r_1 < r_2$ but r_2 is not an immediate successor of r_1. This means we can find $r_3 = (b, \neg b \wedge \neg a, a)$ such that $r_1 < r_3 < r_2$ and therefore $\neg a_1 \preccurlyeq b \prec \neg a \preccurlyeq b_2$. ∎

Proposition 3.37. *Let* $\mathcal{D}$ *be an experimental domain generated by a set of refinable aligned strict references. Then all elements of* D *are part of a pair* $(s_b, \neg s_a)$ *such that* $s_b \in \mathcal{D}_b$, $s_a \in \mathcal{D}_a$ *and* $\neg s_a$ *is the immediate successor of* s_b *in* D *or* $s_b \equiv \neg s_a$. *Moreover if* $s \in D$ *has an immediate successor, then* $s \in \mathcal{D}_b$.

Proof. Let $\mathcal{D}$ be an experimental domain generated by a set of refinable aligned strict references. Let $s_b \in \mathcal{D}_b$. Let $A = \{a \in \mathcal{D}_a \mid a \vee s_b \not\equiv \top\}$. Let $s_a = \bigvee_{a \in A} a$. First we show that $s_b \preccurlyeq \neg s_a$. We have $s_b \wedge \neg s_a \equiv s_b \wedge \neg \bigvee_{a \in A} a \equiv s_b \wedge \bigwedge_{a \in A} \neg a \equiv \bigwedge_{a \in A} s_b \wedge \neg a$. For all $a \in A$ we have $a \vee s_b \not\equiv \top$, $\neg a \not\preccurlyeq s_b$ which means $s_b \preccurlyeq \neg a$ because of the total order of D. This means that $s_b \wedge \neg a \equiv s_b$ for all $a \in A$, therefore $s_b \wedge \neg s_a \equiv s_b$ and $s_b \preccurlyeq \neg s_a$.

Next we show that no statement $s \in D$ is such that $s_b < s < \neg s_a$. Let $a \in \mathcal{D}_a$ such that $s_b < \neg a$. By construction $a \in A$ and therefore $\neg s_a \preccurlyeq \neg a$. Therefore we can't have $s_b < a < \neg s_a$. We also can't have $b \in \mathcal{D}_b$ such that $s_b < b < \neg s_a$: by 3.36 we'd find $a \in \mathcal{D}_a$ such that $s_b < a \preccurlyeq b < \neg s_a$ which was ruled out. So there are two cases. Either $s_b \not\equiv \neg s_a$ then $s_b < \neg s_a$; $\neg s_a$ is the immediate successor of s_b. Or $s_b \equiv \neg s_a$.

The same reasoning can be applied starting from $s_a \in \mathcal{D}_a$ to find a $s_b \in \mathcal{D}_b$ such that s_b is the immediate predecessor of $\neg s_a$ or an equivalent statement. This shows that all elements of D are paired.

To show that if a statement in D has a successor then it must be a before statement, let $s_1, s_2 \in D$ such that s_2 is the immediate successor of s_1. By 3.36, in all cases where $s_1 \notin \mathcal{D}_b$ and $s_2 \notin \mathcal{D}_a$ we can always find another statement between the two. Then we must have that $s_1 \in \mathcal{D}_b$ and $s_2 \in \mathcal{D}_a$. ∎

Theorem 3.38 (Reference ordering theorem). *An experimental domain is naturally ordered if and only if it can be generated by a set of refinable aligned strict references.*

Proof. Suppose $\mathcal{D}_X$ is an experimental domain generated by a set of refinable aligned strict references. Then by 3.34 and 3.37 the domain satisfies the requirement of theorem 3.16 and therefore is naturally ordered.

Now suppose $\mathcal{D}_X$ is naturally ordered. Define the set $\mathcal{B}_b$, $\mathcal{B}_a$ and D as in 3.12. Let $R = \{(b, \neg b \wedge \neg a, a) \mid b \in \mathcal{B}_b, a \in \mathcal{B}_a, b < \neg a\}$ be the set of all references constructed from the basis. First we note that all references are strict. The before and after statements are verifiable since they are part of the basis. The on statement $\neg b \wedge \neg a$ is not a contradiction since $b < \neg a$ means $b \nparallel a$ and $b \not\equiv \neg a$. The on statement is broader than $\neg b \wedge \neg a$ as they are equivalent and it is broader than $b \wedge a$ as that is a contradiction since $b < \neg a$. Therefore R is a set of references. Since the before and after statements of R coincide with the basis of the domain, $\mathcal{D}_X$ is generated by R.

Now we show that R consists of aligned strict references. We already saw that $b \nparallel a$ and we also have $\neg b \wedge \neg a$ is incompatible with both b and a. The references are strict. To show they are aligned, take two references. The before and not after statements are linearly ordered by 3.14 which means the references are aligned.

To show R is refinable, note that each reference can be expressed as ("$x < x_1$", "$x_1 \le x \le x_2$", "$x > x_2$") where $x_1, x_2 \in X$ and "$x_1 \le x \le x_2$" $\equiv$ "$x \ge x_1$" ∧ "$x \le x_2$". That is, every reference is identified by two possibilities x_1, x_2 such that $x_1 \le x_2$. Therefore take two references $r_1, r_2 \in R$ and let (x_1, x_2) and (x_3, x_4) be the respective pair of possibilities we can use to express the references as we have shown. Suppose $r_1 < r_2$ but they are not consecutive. Then "$x \le x_2$" < "$x < x_3$". That is, we can find $x_5 \in X$ such that $x_2 < x_5 < x_3$ which means "$x \le x_2$" < "$x < x_5$" and "$x \le x_5$" < "$x < x_3$". Therefore the reference $r_3 \in R$ identified by (x_5, x_5) is between the two references. On the other hand, assume the second reference is finer than the first. Then $x_1 \le x_3$ and $x_4 \le x_2$ with either $x_1 \ne x_3$ or $x_4 \ne x_2$. Consider the references $r_3, r_4 \in R$ identified by (x_1, x_3) and (x_2, x_4). Either $r_3 < r_2$ or $r_2 < r_4$. Also note that the before statements of r_1 and r_3 are the same and the after statements of r_1 and r_4 are the same. Therefore we satisfy all the requirements and the set R is refinable by definition. ∎

To recap, experimentally we construct ordering by placing references and being able to tell whether the object measured is before or after. We can define a linear order on the possibilities, and therefore a quantity, only when the set of references meets special conditions. The references must be strict, meaning that before, on and after are mutually exclusive. They must be aligned, meaning that the before and not-after statement must be ordered by narrowness. They must be refinable, meaning when they overlap we can always find finer references with well defined before/after relationships. If all these conditions apply, we have a linear order. If any of these conditions fail, a linear order cannot be defined.

The possibilities, then, correspond to the finest references we can construct within the domain. That is, given a value q_0, we have the possibility "*the value of the property is* q_0" and we have the reference ("*the value of the property is less than* q_0", "*the value of the property is* q_0", "*the value of the property is more than* q_0").

3.4 Discrete quantities

Now that we have seen the general conditions to have a naturally ordered experimental domain, we study common types of quantities and under what conditions they arise. We start with discrete ones: the number of chromosomes for a species, the number of inhabitants of a country or the atomic number for an element are all discrete quantities. These are quantities that are fully characterized by integers (positive or negative).

We will see that discrete quantities have a simple characterization: between two references there can only be a finite number of other references.

The first thing we want to do is characterize the ordering of the integers. That is, we want to find necessary and sufficient conditions for an ordered set of elements to be isomorphic to a subset of integers. First we note that between any two integers there are always finitely many elements. Let's call sparse an ordered set that has that property: between two elements there are only finitely many. This is enough to say that the order is isomorphic to the integers.

Reference ordering theorem

- An experimental domain is fully characterized by a quantity if and only if it can be generated by a set of refinable aligned strict references

Property of references	Meaning
Strict	The quantity is always only before/on/after the reference. This can be assumed if the extent of what we measure is smaller than the extent of the reference.
Aligned	The before/after statement have an ordering in term of narrowness (specificity). Necessary to have a coherent before and after over the whole range.
Refinable	If we have overlaps, we can always construct finer references. Necessary to create smallest mutually exclusive cases that correspond to the values.

Integers and reals

- If we assume that between two non-overlapping references we can only put finitely many references, then the ordering is the one of the integers
  - Equality can be tested as well
- If we assume that between two non-overlapping references we can always put another, then the ordering is the one of the reals
  - Equality cannot be tested in this case
- These are the only two orderings that are homogeneous, where all references have the same properties
  - And that is why they are the most fundamental in physics

Are these requirements tenable at Planck scale?

Property of references	Meaning	Problems
Strict	The quantity is always only before/on/after the reference. This can be assumed if the extent of what we measure is smaller than the reference.	Objects measured and references are ultimately of the same kind; their extent should be comparable
Aligned	The before/after statements have an ordering in term of narrowness (specificity). Necessary to have a coherent before and after over the whole range.	If indistinguishable particles are the smallest references and are placed very close to each other, it is not clear how can be sure they haven't switched
Refinable	If we have overlaps, we can always construct finer references. Necessary to create smallest mutually exclusive cases that correspond to the values.	The whole point of reaching Planck length is that we cannot further refine our references

Are these requirements tenable at Planck scale?

- If we take the quantum nature of the references into consideration, all the requirements seem untenable
  - Note that all three are necessary: if even only one fails we have a problem
- What fails is ordering itself
  - Is not that the real numbers need to be changed to rationals or integers: we don’t have numbers to begin with

Failure of ordering

- Riemannian manifold
- Differentiable manifold + inner product
- Topological manifold + differentiable structure
- Ordered topological space + locally $\mathbb{R}^n$
- Topological space + order topology

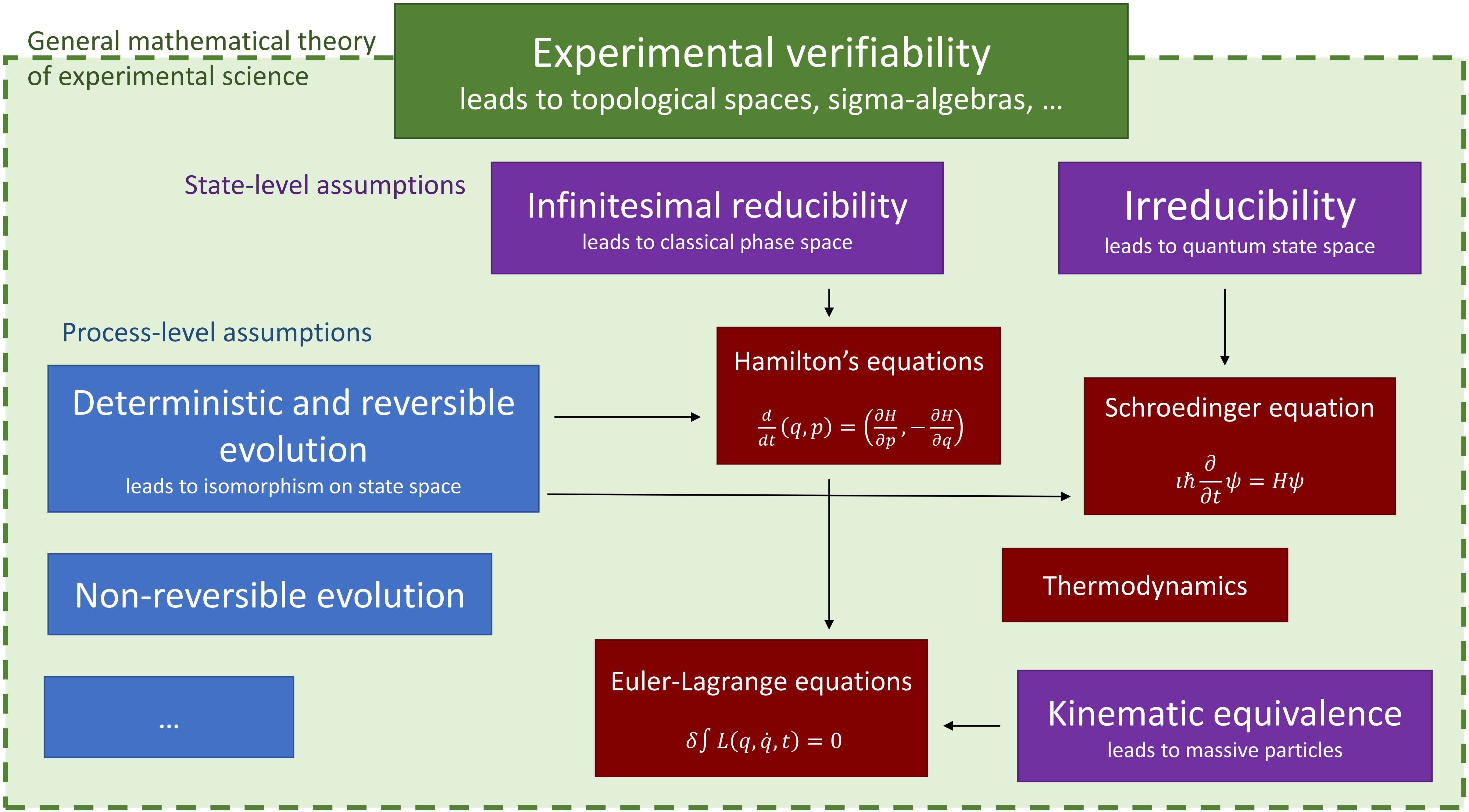
- If ordering fails, all the structures that are based on ordering fail as well. No manifold, no differentiability, no calculus, no inner product, no geometry. We need to develop a new chain of mathematical tools.

Conclusion

- Topology, the simplest mathematical structure needed for geometry, has a clear well-defined meaning in terms of experimental verifiability
  - This is appropriate as experimental verifiability is the foundation of science
- Order topology, the next required structure, formally captures the ability to experimentally compare quantities
  - The ordering is generated by logical relationships: if “x<8” then also “x<10”
- For real numbers, the requirements can only be satisfied ideally, most likely leading to a breakdown at Planck scale
  - The idea that our “measurement device” is “classical” is baked into the very nature of the order topology, which can’t then be undone up the stack

Conclusion

- The standard mathematical toolchain (i.e. manifolds, differentiability/integration, differential geometry, Riemannian geometry, …) needs to be rethought
  - The idea that we can take something and divide it into infinitesimal contributions is intrinsically classical
- In the same way that the geometry of space-time (i.e. the metric tensor) depends on the energy/mass distribution, the topology may depend on it as well
- The foundations of physics lie in understanding the most basic mathematical structures, their physical significance and how they can be generalized


General mathematical theory of experimental science
Experimental verifiability
leads to topological spaces, sigma-algebras, …
State-level assumptions
Infinitesimal reducibility
leads to classical phase space
Irreducibility
leads to quantum state space
Process-level assumptions
Hamilton's equations
$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$
Deterministic and reversible evolution
leads to isomorphism on state space
Schroedinger equation
$\imath\hbar\frac{\partial}{\partial t}\psi = H\psi$
Non-reversible evolution
Thermodynamics
…
Euler-Lagrange equations
$\delta\int L(q,\dot{q},t) = 0$
Kinematic equivalence
leads to massive particles

For more information

- Assumptions of Physics project website: http://assumptionsofphysics.org/
- Topology and Experimental Distinguishability
  Christine A. Aidala, Gabriele Carcassi, and Mark J. Greenfield, *Top. Proc.* **54** (2019) pp. 271-282