Space-time structure may be topological and not geometrical



Gabriele Carcassi and Christine Aidala
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Assumptions of Physics

- This talk is part of a broader project called Assumptions of Physics (see http://assumptionsofphysics.org/)
- The aim of the project is to find a handful of physical principles and assumptions from which the basic laws of physics can be derived
- To do that we want to develop a general mathematical theory of experimental science: the theory that studies scientific theories
 - A formal framework that forces us to clarify our assumptions
 - From those assumptions the mathematical objects are derived
 - Each mathematical object has a clear physical meaning and no object is unphysical
 - Gives us concepts and tools that span across different disciplines
 - Allows us to explore what happens when the assumptions fail, possibly leading to new physics ideas

General mathematical theory of experimental science

Experimental verifiability

leads to topological spaces, sigma-algebras, ...

State-level assumptions

Infinitesimal reducibility

leads to classical phase space

Process-level assumptions

Deterministic and reversible evolution

leads to isomorphism on state space

Non-reversible evolution

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Hamilton's equations

$$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$$

Irreducibility

leads to quantum state space

Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi$$

Thermodynamics

Euler-Lagrange equations

$$\delta \int L(q,\dot{q},t) = 0$$

Kinematic equivalence

leads to massive particles

Mathematical structure for space-time

- Riemannian manifold
- Differentiable manifold + inner product
- Topological manifold + differentiable structure
- Ordered topological space + locally \mathbb{R}^n
- Topological space + order topology

• If we want to understand why (i.e. under what conditions) space-time has the structure it has, we first need to understand why (i.e. under what conditions) it is a topological space, it has an order topology, ...

Mathematical structure for space-time

- Riemannian manifold
- Differentiable manifold + inner product

Geometry (lengths and angles) starts here: most fundamental structures are not geometrical

- Topological manifold + differentiable structure
- Ordered topological space + locally \mathbb{R}^n
- Topological space + order topology

• If we want to understand why (i.e. under what conditions) space-time has the structure it has, we first need to understand why (i.e. under what conditions) it is a topological space, it has an order topology, ...

Simple things first

- A similar hierarchy is present for other mathematical structures used in physics
 - Hilbert space Inner product space + closure under Cauchy sequences –
 Vector space + inner product ...
- If we want true understanding, then we need to understand the simpler structure first
 - This is what our project, Assumptions of Physics, is about

Outline

- In this talk we will focus on topology and order. We will:
 - Show that topologies naturally emerge from requiring experimental verifiability
 - Show that an order topology corresponds to experimental verifiability of quantities: outcomes than can be smaller, greater or equal to others
 - Then we need to understand how quantities are constructed from experimental verifiability
 - That is, find a set of necessary and sufficient conditions under which experimental verifiability gives us an order topology
 - Argue that, in the end, those conditions are untenable at Planck scale, and that ordering cannot be experimentally defined
 - Conclude that all that is built on top of an order topology (manifolds, differentiable structures, inner product) fails to be well defined at Planck scale

Verifiable statements

- The most fundamental math structures are from logic and set theory
 - All other structures are based on that
- For science, we want to extend these with experimental verifiability
- Our fundamental object will be a verifiable statement: an assertion for which we have (in principle) an experimental test that, if the statement is true, terminates successfully in a finite amount of time
- Verifiable statements do not follow standard Boolean logic:
 - We may verify "there is extra-terrestrial life" but not its negation "there is no extra-terrestrial life"
 - No negation in general, finite conjunction, countable (infinite) disjunction

What is a topology?

- Given a set X, a topology $T \subseteq 2^X$ is a collection of subsets of X that:
 - It contains X and Ø
 - In general, not closed under complement
 - It is closed under finite intersection and arbitrary (infinite) union
- How do we get to this in physics?

Basis \mathcal{B} $e_1 \quad e_2 \quad e_3 \quad ...$

Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Basis ${\cal B}$			Verifiable statements \mathcal{D}_X		
e_1	e_2	e_3	 $s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain

Basis ${\cal B}$				Verifiable statements \mathcal{D}_X		
e_1	e_2	e_3		$s_1 = e_1 \lor e_2$	$s_2 = e_1 \wedge e_3$	•••
F	F	F		F	F	•••
		•••	•••	•••	•••	•••
F	Т	F	•••	Т	F	
Т	Т	F		Т	F	•••
•••	•••	•••	•••	•••		•••

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Consider all truth assignments: it is sufficient to assign the basis

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F	F	F		F	F	•••
•••				•••	•••	•••
F	Т	F	•••	Т	F	
T	T	F	111		F	
•••	•••		•••	•••		

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Remove truth assignments that are impossible (e.g. the distance is more than 5 and less than 3)

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•••		•••			•••	
F	Т	F		Т	F	
T	T	F			F	
•••	•••	•••	***			

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Basis ${\cal B}$				Verifiable statements \mathcal{D}_X		
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F	F	F		F	F	
		•••	•••			•••
F	Т	F	•••	Т	F	•••
T	T	F			F	
•••	•••	•••	•••			•••

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Each verifiable statement corresponds to a set of possibilities in which the statement is true.

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The experimental domain \mathcal{D}_X induces a natural topology on the set of possibilities X

 $s_1 = e_1 \vee e_2$

Verifiable statements \mathcal{D}_X

 $s_2 = e_1 \wedge e_3$

F

F

The role of logic (and math) in science is to capture what is consistent (i.e. the possibilities) and what is verifiable (i.e. the verifiable statements)

Basis \mathcal{B}

 e_3

F

F

 e_2

F

 e_1

F

Start with a countable set of verifiable statements (the most we can test experimentally). We call this a basis.

Construct all verifiable statements that can be verified from the basis (close under finite conjunction and countable disjunction). We call this an experimental domain

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Examples

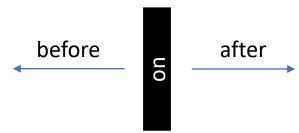
- "the mass of the photon is less than $10^{-13} \, \mathrm{eV}$ " is verifiable and corresponds to an open set (a set in the topology)
- "the mass of the photon is exactly 0 eV" is not verifiable and is not an open set (not a set in the topology)
 - However, it is falsifiable and corresponds to a closed set (the complement is in the topology)
- Topological concepts (second countability, Hausdorff spaces, interior/exterior/boundary, ...) can be better understood in terms of experimental verification
 - They are not some abstract mathematical thing: they are physically meaningful

Quantities

- We can define a quantity as a measurable property of a system that has a magnitude: can be compared to another of the same kind and found to be greater or smaller
- Mathematically a quantity is formed by:
 - a set Q
 - a linear (total) ordering $\leq: Q \times Q \to \mathbb{B}$
 - the order topology generated by the linear ordering, whose basis elements are of the form $(-\infty, q)$ and $(q, +\infty)$; that is, we can always tell experimentally whether something is more or less than something else
 - equality, in general, is not experimentally testable: for continuous quantities corresponds to infinite precision measurements

Constructing quantities and references

- The question is: how do we operationally construct quantities? How can we model that appropriately?
- We start with the idea of a reference: something physical that partitions our range into a before, on, and after
 - E.g. a line on a ruler, the tick of a clock, a standard weight for a balance scale, a threshold on an A/D converter

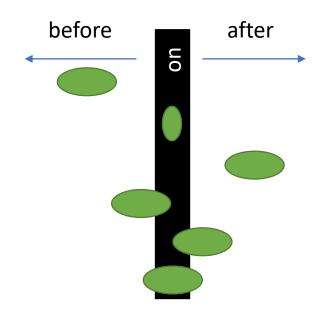


 Mathematically, a reference is a tuple of three statements b/o/a; only before and after are required to be experimentally verifiable

Constructing quantities and references

• Problem 1 - In general, before/on/after are not mutually exclusive

Before	On	After
Т	F	F
F	Т	F
F	F	Т
Т	Т	F
F	Т	T
Т	Т	T

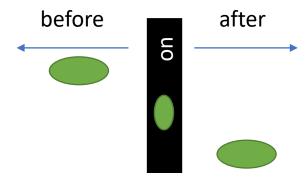


In this case, the possibilities of the domain cannot correspond to distinct values

Strict references

• We say a reference is strict if before/on/after are mutually exclusive

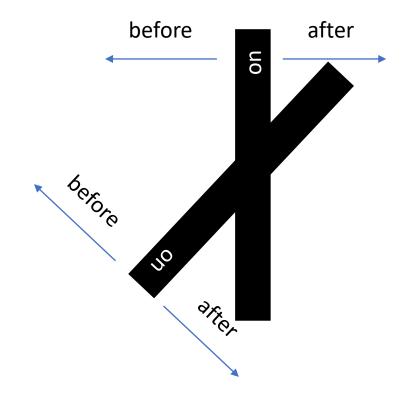
Before	On	After
Т	F	F
F	T	F
F	F	Т



• If the extent of what we measure is smaller than the extent of our reference, then we can always assume our references are strict

Multiple references

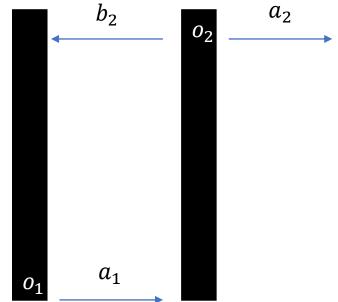
- Problem 2 To construct a reference scale we need multiple references, but in general these would not construct a linear order
- We need to define what it means for references to be aligned purely on the logical relationship between statements



Ordered references

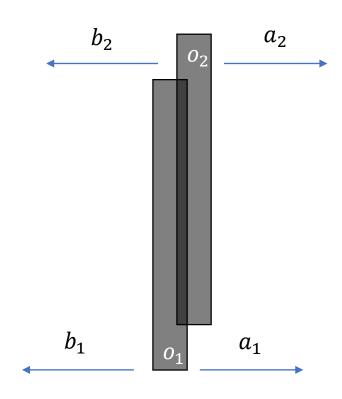
 We can say that reference 1 is before reference 2 if whenever we find something before or on the other, it must be before the second

- More precisely, if $b_1 \vee o_1 \not= o_2 \vee a_2$
 - Means the statements are incompatible, they can't be true at the same time
- Note how $b_1 \leqslant \neg a_1 \leqslant b_2 \leqslant \neg a_2$
 - Where $a \le b$ (a is narrower than b) means that if a then b must be true as well



Aligned references

- More in general, we can say that two references are aligned if the before and not-after statement can be ordered by narrowness
- For example, $b_1 \leq b_2 \leq \neg a_1 \leq a_2$
 - • Means that if the first statement is true
 then the second statement will be true as well
 - That is, the first statement is narrower, more specific
- Here we see how the ordering of references is related to the logical ordering defined by the specificity (narrowness) of the statements
- We need our references to be aligned if we want to construct a linear ordering



Resolving the overlaps

- Problem 3a If two different references overlap, we can't say one is before the other: we can't fully resolve the linear order
- Problem 3b Conversely, if two reference don't overlap and there can be something in between, we must be able to put a reference there
- We always need a way, then, to find (possibly finer) references to explore the full space

Refinable references

- Conceptually, a set of references is refinable if we can solve the previous problems:
 - if two references overlap we can always refine them to two that do not overlap
 - if two ordered references are not consecutive (there can be something in between) we can always construct a reference in the middle
- Mathematically is not complicated, but is tedious and not so interesting
- With these definitions and some work...

the possibilities themselves can be ordered, and how this ordering, in the end, is uniquely

the positionizes the discontinuous the discontinuous can be ordered, and now this ordering, in the ear, is unique the characterized by statement narrowness: 0 is less than 42 because "the quentity is less than 10^{α} is narrower than "the quantity is less then 12^{α} . As the defining characteristic for a quantity is the ability to compare its values, then the values must be ordered in some fashion from smaller to greater. Therefore, given two different values, one must be before the other. Mathematically, we call linear order an order with sucl a characteristic as we can imagine the elements positioned along a line. Note that vectors are not linearly ordered: no direction is greater than the other. Therefore, in this context, a ector will not strictly be a quantity but a collection of quantities.2

vector will not strictly be a quantity but a collection of quantities. We also have to define how this coder can be experimentally verified. The idea is that we should, at least, be able to verify that the value of a given quantity is before or after a set value. This allows us to construct someth such as "the mass of the electron is full as 5.6×10^{-1} which we take to be equivalent to "the mass of the electron is more than $5.0.6 \times 10^{-1}$ but less than $5.1.6 \times 10^{-1}$ by in tenses, this also allows us to verify particular numbers as "the earth." has one natural satellite" is equivalent to the "the earth has more than zero natural sate and fewer than two." Therefore we will define the order topology as the one generated by set of the type (a, ∞) and $(-\infty, b)$.

A quantity, then, is an ordered property with the order topology.

Definition 3.4. A linear order on a set Q is a relationship $\leq: Q \times Q \rightarrow \mathbb{B}$ such that

- mmetry) if $q_1 \le q_2$ and $q_2 \le q_1$ then $q_1 = q_2$ 2. (transitivity) if a₁ < a₂ and a₂ < a₃ then a₁ < a₂
- I. (total) at least $q_1 \le q_2$ or $q_2 \le q_1$

set together with a linear order is called a linearly ordered set. Definition 3.5. Let $(O \le)$ be a linearly ordered set. The order topology is the topol

 $(a, \infty) = \{a \in Q \mid a < q\}, (-\infty, b) = \{a \in Q \mid q < b\},$

Definition 3.6. A quantity for an experimental domain D_V is a linear Formally, it is a tuple (Q, \leq, q) where (Q, q) is a property, $\leq Q \times Q \rightarrow \mathbb{R}$ is a linear or and Q is a topological space with the order topology with respect to \leq .

As for properties, the quantity values are just symbols used to label the different cases set Q may correspond to the integers, real numbers or a set of words ordered alphabetica. The units are not captured by the numbers themselves: they are captured by the function

⁹In other languages, there are two words to differentiate quantity as in "physical quantity" (e.g. grand frôsse, grandeur) and as in "amount" (e.g. quantith, Menge, quantité). It is the second meaning of qua ntence. "the mass of the electron is 511±0.5 keV" could instead be referring to statistical unce

The sentence. The make of the electric to 1 stable ser cross statement be reterring to extention unever-lineared of an accurry bound and evolution establishes a different statement when the different meaning is attain the wild be treating these types of statistical statements have in the book, but suffice it to say that they or be defined before unecessate that delarity bounds.

"When consulting the dictionary, we use the fact that we can experimentally still whether the word w looking in a before or after the one we randomly selected.

CHAPTER 3 PROPERTIES AND QUANTITIES

which returns elements of the original set and therefore reduces to countable conjunctions. Therefore, when forming D_b the only new elements will be the countable disjunctions

Consider two countable sets $B_1, B_2 \subseteq B_b$. Their disjunctions $b_1 = \bigvee_{k \in B_1} b_k$ and $b_2 = \bigvee_{k \in B_2} b_k$ represent the narrowest statement that is broader than all elements of the respective set. Suppose that for each element of B_1 we can find a broader element in B_2 . Then b_2 , being proader than all elements of B_2 , will be broader than all elements of B_1 . But since b_1 is he narrows telement that is broader than all elements in B_1 , we have $b_2 > b_1$. Conversely uppose there is some element in B_1 for which there is no broader element in B_2 . Since he initial set is fully ordered, it means that that element of B_1 is broader than all the onts in R. This means that element is broader than b. and since b. is broader tha Ill elements in B_1 we have $b_1 \ge b_2$. Therefore the domain D_b generated by B_b is linearly rdered by narroy

Now we show that (D. >) is linearly ordered. The basis R. is linearly ordered by ness because the negation of its elements are part of B and are ordered by narrownes te that broadness is the opposite order of narrowness and therefore a set linearly ordered w one is linearly ordered by the other. Therefore B_a is also linearly ordered by narrowness and so is \mathcal{D}_a by the previous argument. Therefore \mathcal{D}_a is ordered by broadness. To show that $D = \mathcal{D}_b \cup \neg(\mathcal{D}_a)$ is linearly ordered by narrowness, we only need to show

that the countable disjunctions of elements of B_k are either narrower or broader countable conjunctions of the negations of elements of \mathcal{B}_a . Let $B_1 \subset \mathcal{B}_b$ and A_2 (disjunction $b_1 = \bigvee_{b \in B_1} b$ represents the narrowest statement that is broader than all

of B_1 while the conjunction $-\mathbf{a}_2 = -\bigvee_{i} \mathbf{a} = \bigvee_{s \in A_2} \mathbf{a} = \bigcap_{s \in A_2} -\mathbf{a}$ represents the broadest state is narrower than all elements of $-(A_2)$. Suppose that for one element of $-(A_2)$ find a broader statement in B_1 . Then \mathbf{b}_1 , being broader than all elements in Ebroader than that one element in $\neg(A_2)$. But since $\neg a_2$ is narrower than all el $\gamma(A_2)$, we have $\neg a_2 \le b_1$. Conversely, suppose that for no element of $\neg (A_2)$ we broader statement in B_1 . As B is linearly ordered, it means that all elements in proader than all elements in B_1 . This means that all elements in $\neg(A_2)$ are bro by and therefore by s -ao. Therefore D is linearly ordered by narr

Theorem 3.16 (Domain ordering theorem). An experimental domain D_X is ordered if and only if it is the combination of two experimental domains $D_X = D_a$

(i) $D = \mathcal{D}_b \cup \neg(\mathcal{D}_a)$ is linearly ordered by narrowness (ii) all elements of D are part of a pair $(s_b, \neg s_a)$ such that $s_b \in \mathcal{D}_b$, $s_a \in \mathcal{D}_a$ (either the immediate successor of St. in D or St. = -S. (iii) if $s \in D$ has an immediate successor, then $s \in D_b$

Proof. Let D_V be a naturally ordered experimental domain. Let B_b and B_a 1 as in 3.12 which means $\mathcal{B} = \mathcal{B}_b \cup \mathcal{B}_a$ is the basis that generates the order topo \mathcal{D}_b be the domain generated by \mathcal{B}_b and \mathcal{D}_a be the domain generated by \mathcal{B}_a . Ti erated from D_b and D_a by finite conjunction and countable disjunction and

3.2 OUANTITIES AND ORDERING

that allows us to map statements to numbers and vice-versa.

As we want to understand quantities better, we concentrate on those experimental domains that are fully characterized by a quantity. For example, the domain for the mass of a system will be fully characterized by a rol number greater than or equal to zero. Each possibility will be identified by a number which will correspond to the mass expressed in a particular unit, say in Kg. As the values of the mass are ordered, we can also say that the possibilitie themselves are ordered. That is, "the mass of the system is 1 Ke" procedes "the mass of the sustem is 2 Ke". This ordering of the possibilities will be linked to the natural topology as he mass of the system is less than $\# Ke^*$, the distinction of all possibilities that come befor

We call a natural order for the possibility a linear order on them such that the order we can it another true in the possessing a many coder on them such that the order topology is the natural topology. An experimental domain is fully characterized by a quantity if and only if it is naturally ordered and that quantity is ordered in the same way: it is order isomorphic. In other words, we can only assign a quantity to an experimental domain if it already has a natural ordering of the same type.

CHAPTER 3. PROPERTIES AND QUANTITIES

mass of the system is more that on, K^{*} is also reduced by antronous but with the reverse ordering of the possibilities/values. These role were streament whose vertibles do end define the order topology and therefore jointly constitutes a basis for the experimental domain. Now consider the statement $y_{i} = w_{i} + w_{$ greater than I Kg we'd have $s_1 \leq s_2$. In other words, if we call B the set that includes both the less-than-or-equal and less-than statements this is also linearly ordered by narrowness But "the mass of the system is less than or equal to 1 Ka" is equivalent to - "the mass of the But the mass of the system is less than or equal for $E_i F^i$ in equivalent to $-i\hbar e$ mass of the system is less than or equal for $E_i F^i$ in the system is greater to present relationship all the autonomous like system of $E_i F^i$. In the two $E_i F^i$ is the system of $E_i F^i$ is the system of $E_i F^i$ is the system of $E_i F^i$ is the system of the system is some than $E_i F^i$ is the system of the system is some than $E_i F^i$ is the system of the system is less than or equal to $E_i F^i$ is the luminosities secrecated $E_i F^i$ is the system of the system is less than or equal to $E_i F^i$ is the luminosities secrecated $E_i F^i$ is the system of the system is less than $E_i F^i$. This is the plays efficiency and there examt be supported restaurance in $D_i F^i$ the system of the system is $E_i F^i$ is the system of the system of the system of $E_i F^i$ is the system of the system of $E_i F^i$ is the system of the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the system of $E_i F^i$ is the system of $E_i F^i$ in the

is broader than s₂ but narrower than s₁ since they differ for a single case. This will happen for any mass value. So B is composed of two exact copies of the ordering of X, where each element of one copy is immediately followed by an element of the other copy. Moreover, if a element of one copy is immediately followed by an element of the other copy. Moreover, if a statement in B has an limendate successor, there must be only one case that separates the two. If we call q_1 the value of that case, then the statement must be of the form h he mass of the system is also shad q_2 , h h with its immediate successor is of the form h^2 he mass of the system is less than or equal to q_1 , h h h h its immediate successor is broader by just the possibility associated with q_1 . Therefore statements in B that have an immediate successor must be in

The main result is that the above characterization of the basis of the domain is necessary and sufficient to order the possibilities. If an experimental domain has a basis composed of

3.2 OUANTITIES AND ORDERING

To prove (i), we have that \mathcal{B}_b and \mathcal{B}_a are linearly ordered by 3.14. We need to show that e linear ordering holds across the sets. Let $x_1, x_2 \in X$ and consider the two statements " $x < x_1$ " and " $x \le x_2$ " = -" $x > x_2$ ". As X is linearly ordered, either $\{x \in X \mid x < x_1\} \subseteq \{x \in X \mid x \le x_2\}$ or $\{x \in X \mid x \le x_2\} \subseteq \{x \in X \mid x < x_1\}$. Therefore either " $x < x_1$ " or $x \le x_2$ = D_b ∪ ¬(D_a) is also linearly ordered.

To prove (ii), let $s_b \in \mathcal{D}_b$. Take $s_a \in \mathcal{D}_a$ such that $-s_a$ is the narrowest statement in $-(\mathcal{D}_b)$ that is broader than s_b . This exists because \mathcal{D}_a is closed by infinite disjunction. As $-s_a \ge s_b$, let X_1 be the set of possibilities compatible with $-s_a$ but not compatible with s_b . The set cannot have more than one element, or we could find an element $x_1 \in X_1$ such that is empty then $s_b = -s_a$. If X_1 contains one possibility, then $-s_a$ is the immediate successor. If $s_b = -s_a$ is empty then $s_b = -s_a$. Similarly, we can start with $s_a \in \mathcal{D}_a$ and find $s_b \in \mathcal{D}_b$ such that $s_b = -s_a$. λ_1 is compt then $\lambda_2 = -\alpha_{\rm sc}$. Similarly, we must sate with $\lambda_2 = -\alpha_{\rm sc}$ and mit $\lambda_2 = 2\beta_{\rm sc}$ such mit $\lambda_2 = -\alpha_{\rm sc}$. In this compatible with $-\alpha_{\rm sc}$ but not compatible with $\alpha_{\rm sc}$. But not compatible with $\alpha_{\rm sc}$. But not compatible with $\alpha_{\rm sc}$. But $\lambda_1 = -\alpha_{\rm sc}$ is the immediate successor and if X_1 is empty then $\beta_{\rm sc} = -\alpha_{\rm sc}$. To prove (III), let $\beta_{\rm sc} > \beta_{\rm sc} > \beta_{\rm sc}$ but that $\beta_{\rm sc}$ is the immediate successor of $\beta_{\rm sc}$. This means

can write $s_2 \equiv s_1 \vee x_1$ for some $x_1 \in X$. This means $s_1 \equiv "x < x_1"$ while $s_2 \equiv "x \le x_1"$ and therefore $s_1 \in B_b$.

CHAPTER 3. PROPERTIES AND QUANTITIES

of the immediate successors. Let $(\cdot)^{++}: B_h \to B_n$ be the function such that $\neg(b^{++}) = \neg b^{++}$ is the immediate successor of b. Let b: $X \to B_b$ be the function such that $x_2 = b(x_2) - b(x_2)^{*+}$. On X define the ordering \le such that $x_1 \le x_2$ if and only if $b(x_1) \le b(x_2)$. Since (B_b, \le) s linearly ordered so is (X, <). To show that the ordering is natural, suppose $x_1 < x_2$ hen $b(x_1) < -b(x_1)^{++} \le b(x_2)$ and therefore $x_1 \le b(x_2)$. We also have $-b(x_1)^{++} \le b(x_2)$ then $O(1) \cap O(2) \cap O(2) \cap O(2)$ and therefore $S_1 \circ O(2) \cap O(2)$. We also have $O(2) \cap O(2) \cap O(2) \cap O(2)$. This means that given a possibility $S_1 \in X$, all and only the possibilities lower than x_1 are compatible with $O(x_1)$ and therefore $O(x_1) \equiv {}^{u}x < O(x_2) \cap O(x_1)$. x_1 , while all and only the possibilities greater than x_1 are compatible with $b(x_1)^{++}$ and herefore $b(x_1)^{++} \equiv {}^{u}x > x_1^{-v}$. The topology is the order topology and the domain has a

3.3 References and experimental ordering

In the previous section we have characterized what a quantity is and how it relates to an experimental domain. But as we saw in the first chapters, the possibilities of a domain are not objects that exist a priori: they are defined based on what can be verified experimentally. Therefore simply assigning an ordering to the possibilities of a domain does not answer the more fundamental question: how are quantities actually constructed? How do we, in practice, create a system of references that allows us to measure a quantity at a given level of precision? What are the assumptions we make in that process?

In this section we construct ordering from the idea of a reference that physically defines a boundary between a before and an ofter. In general, a reference has an extent and may overlap with others. We define ordering in terms of references that are clearly before and overlap with Unites. We desired violetting in terms no reservations can also cleanly resource and after others. We see that the possibilities have a natural ordering only if they are generated from a set of references that is refinable (we can always find finer ones that do not overlap and for which before/our/after are mutually exclusive cases. The possibilities, then, are the

We are by now so used of the ideas of real numbers, negative numbers and the number zero that it is difficult to realize that these are mental constructs that are, in the end, somewhat recent in the history of humankind. Yet geometry itself started four thousand years ago as an experimentally discovered collection of rules concerning lengths, areas and angles. That is, human beings were measuring quantities well before the real numbers were invented. So, how does one construct instruments that measure values?

To measure position, we can use a ruler, which is a series of equally spaced marks. We give a label to each mark (e.g. a number) and note which two marks are closest to the target sition (e.g. between 1.2 and 1.3 cm). To measure weight, we can use a balance and a set of ually prepared reference weights. The balance can clearly tell us whether one side is heavier than the other, so we use it to compare the target with a number of reference weights and note the two closest (e.g. between 300 and 400 grams). A clock gives us a series of events to compare to (e.g. earth's rotation on its axis, the ticks of a clock). We can pour water from a reference container into another as many times as are needed to measure its volume. In all these cases what actually happens is similar: we have a reference (e.g. a mark on a ruler, a set of equally prepared weights, a number of ticks of a clock) and it is fairly easy to tel

2.2 OUANTITIES AND ORDERING

" $x < x_1$ " \wedge " $x \ge x_1$ " $= \bigvee_{x \in X} x = \bot$. Definition 3.12, Let D_X be a naturally ordered experimental domain and X its possibilities. Define $B_b = \{ *x < x_1 * | x_1 \in X \}$, $B_a = \{ *x > x_1 * | x_1 \in X \}$ and $B = B_b \cup \neg (B_a)$.

Definition 3.13. Let (O, <) be an ordered set. Let $a_1, a_2 \in O$. Then a_2 is an immedi cessor of q_1 and q_2 is an immediate predecessor of q_2 if there is no tuven them in the ordering. That is, $q_1 < q_2$ and there is no $q \in Q$ such that $q_1 < q <$ we elements are consecutive if one is the immediate successor of the other.

Proposition 3.14. Let D_X be a naturally ordered experimental domain. Then $(\mathcal{B}_b, \mathcal{B}_d, z)$ and $(\mathcal{B}, \varepsilon)$ are linearly ordered sets. Moreover $(\mathcal{B}_b, \varepsilon)$, (\mathcal{B}_d, z) are order isomorphisms.

Proof. Let $f: X \to \mathcal{B}_b$ be defined such that $f(x_1) = {}^a x < x_1{}^a$. As there is one Proof. Let $f: A \to B_0$ the elements since that $\|f(x_1)\|^2 = T \in \mathbb{Z}_{p_1}^{\infty}$. As there is note in the only one asstantime $t = T_0^{\infty}$ for each $f = T_0^{\infty}$. By these $h(x_2) = \sum_{i \in \mathbb{Z}_{p_1}^{\infty}} x = \int_{\{x_1, x_2\} \in \mathbb{Z}_{p_2}^{\infty}} \left(\int_{\{x_1, x_2 = x_1\}} x = \int_{\{x_1, x_2 = x_2\} \in \mathbb{Z}_{p_2}^{\infty}} \left(\int_{\{x_1, x_2 = x_2\}} x = \int_{\mathbb{Z}_{p_2}^{\infty}} x = \int_{\mathbb{Z}_{p_2}^{\infty}} x \int_{\mathbb{Z}_{p_2}^{\infty}} \left(\int_{\{x_1, x_2 = x_2\} \in \mathbb{Z}_{p_2}^{\infty}} x \right) = \int_{\mathbb{Z}_{p_2}^{\infty}} x \int$ we $g(x_1) \equiv \bigvee_{(g \in X \mid g > g_1)} x \equiv \begin{pmatrix} \bigvee_{(g \in X \mid g > g_2)} x \end{pmatrix} \vee \begin{pmatrix} \bigvee_{(g \in X \mid g > g_2)} x \end{pmatrix} \equiv g(x_1) \vee g(x_2)$ and there

 $(\kappa X(xx)) - (\kappa X(xx)) - (\kappa X(x)x)$ $(\kappa X(x)x) - (\kappa X(x)x) - (\kappa X(x)x) - (\kappa X(x)x) - (\kappa X(x)x) + (\kappa X$

n 3.15. Let B_b and B_a be two sets of verifiable statements such that I is linearly ordered by narrowness. Let D_b and D_a be the experimental dons twby generate and $D = D_b \cup \cdot (D_a)$. Then (D_b, s) , (D_a, s) and (D, s) are linearly

rst we show that (D_k, ϵ) is linearly ordered. We have that B_k is linearly ordered ss because it is a subset of B which is linearly ordered by narrowness. Note so because it is a subsect of B which is linearly cordered by narrowness. Note junction of a finite set of statements insularly ordered by paravoness will restruct to element and the disjunction of a finite set of statements linearly ordered as will restruct the broadest defence. The contrable disjunction, instend, can we elements. But using those elements again will not introduce new ones: on of countable disjunctions will still be a countable disjunction; the finite of countable disjunction is the countable disjunction; the finite of countable disjunction of finite conjunction;

3.3. REFERENCES AND EXPERIMENTAL ORDERING

Note that determining whether the quantity is exactly equal to the reference is not as eae mark on the ruler has a width, the balance has friction, the tick of our clock will last finite amount of time. That is, the reference itself can only be compared up to a finite leve of precision. This may be a problem when constructing the references themselves: how do we to precision. In six any we a proton when to constructing the researches tensioners. In our ow-know that the marks on our rules are equally spaced, or that the weights are equally prepared, or that ticks of our clock are equally timed? It is a circular problem in the sense that, in a way, we need instruments of measurement to be able to create instruments of measurement, Yet, even if our references can't be perfectly compared and are not perfectly equal, we can

To make matters were, the object we are measuring may left have an extent. If we are measuring the position of a tipy ball, it may be clearly before or clearly after the nearest are measuring use pleation of a triy slaw, a may be examy become or examy and an enteriorise-mark, but it may also be partly before, partly on and partly after. One may try to selfestep the problem by measuring part of the object, say the position of the center of mass or of its closest part. But this assumes we have a process to interact with only part of the object, and that part can only be before, on or after the reference. It may be a reasonable assumption in many cases but we have to be mixedful that we made that assumption: our general definition will have to be able to work in the less ideal cases. In our general mathematical theory of experimental science, we can capture the above

discussion with the following definitions. A reference is represented by a set of three stateuse caseout wait use notowing trainitions. A feeting as perfect of a great content of a great or time states ments: they tell us whether the object is before, on or parter a specific reference. To make sense, these have to satisfy the following minimal requirements. The before and the after statements must be verifiable, as otherwise they would not be usable as references. As the reference must be somewhere, the on statement cannot be a contradiction. If the object is not before and not after the reference, then it must be on the reference. If the object is before and after the reference, then it must also be on the reference. These requirements recognize that, in general, a reference has an extent and so does the object being measured

We can compare the extent of two references and say that one is finer than the other if the on statement is narrower than the other, and the before and after statements are wider. This corresponds to a finer take of a rule or a finer pulse in our timing system. We say that a reference is strict if the before, on and after statements are incompatible. That is, the three ses are distinct and can't be true at the same time

Definition 3.17. A reference defines a before, an on and an after relationship between iself and another object. Formally a reference r = (b, o, a) is a tuple of three statements

- 1. we can verify whether the object is before or after the reference: b and a are verifiable
- 2. the object can be on the reference: o ± 1
- 3. if it's not before or after, it's on the reference: ¬b ∧ ¬a ≤ o
 4. if it's before and after, it's also on the reference: b ∧ a ≤ o

A beginning reference has nothing before it. That is, $b \equiv \pm$. An ending reference has nothing after it. That is, $a \equiv \pm$. A terminal reference is either beginning or ending.

CHAPTER 3. PROPERTIES AND QUANTITIES

Proof. By definition, we have $\neg b \land \neg a \le o$ and by 1.23 $\neg (\neg b \land \neg a) \lor o \equiv \top \equiv b \lor a \lor o$.

Definition 3.19. A reference $r_1 = (b_1, o_1, a_1)$ is finer than another reference $r_2 = (b_2, o_2, a_2)$

Corollary 3.20. The finer relationship between references is a partial order

Proof. As the finer relationship is directly based on narrowness, it inherits its reflexivity, stisymmetry and transitivity properties and is therefore a partial order.

Definition 3.21. A reference is strict if its before on end ofter statements are incomatible. Formally, r = (b, o, a) is such that $b \neq a$ and $o = \neg b \land \neg a$. A reference is loose if it s not strict

Remark. In general, we can't turn a loose reference into a strict one. The on statement an be made strict by replacing it with $\neg b \land \neg a$. This is possible because o is not required to e verifiable. The before (and after) statements would need to be replaced with statements like $b \land \neg a$, which are not in general verifiable because of the negation.

To measure a quantity we will have many references one after the other; a ruler will have many marks, a scale will have many reference weights, a clock will keep ticking. What does

it mean that a reference comes after another in terms of the before/on/after statements? If reference \mathbf{r}_1 is before reference \mathbf{r}_2 we expect that if the value measured is before the first it will also be before the second, and if it is after the second it will also be after the first. Note that this is not enough, though, because as references have an extent they may overlap. And if they overlap one can't be after the other. To have an ordering properly defined we must have that the first reference is entirely before the second. That is, if the value measured is on the first it will be before the se

Mathematically, this type of order before and strictly after. It does no One may be tempted to define the requires refining the references and refined references, not the original or

Definition 3.22. A reference is be he first it cannot be on or after the roposition 3.23. Reference ord

• irreflexivity: not r < r

• transitivity: if $r_1 < r_2$ and r_2

and is therefore a strict partial (

Proof. For irreflexivity, since tl d therefore by osova. Therefore

therefore the before statement is narrower than the not-after statement. This means that given a set of aligned strict references, the set of all before and not-after statements is linearly

Proof. We have $b_1 \vee o_1 \equiv (b_1 \vee o_1) \wedge T \equiv (b_1 \vee o_1) \wedge (b_2 \vee o_2 \vee a_2) \equiv ((b_1 \vee o_1) \wedge b_2) \vee (b_2 \vee o_2 \vee a_2) = ((b_1 \vee o_1) \wedge b_2) \vee (b_2 \vee o_2 \vee a_2) = ((b_1 \vee o_1) \wedge b_2) \vee ((b_2 \vee o_2 \vee a_2) \otimes ((b_1 \vee o_1) \wedge b_2) \vee ((b_2 \vee o_2 \vee a_2) \otimes ((b_2 \vee o_$ $(b_1 \vee o_1) \wedge (o_2 \vee a_2) \equiv ((b_1 \vee o_1) \wedge b_2) \vee \bot \equiv (b_1 \vee o_1) \wedge b_2$. Therefore $b_1 \vee o_1 \leq b_2$. And ince $b_1 \leq b_1 \vee o_1$, we have $b_1 \leq b_2$.

Similarly, we have $o_2 \vee a_2 \equiv (o_2 \vee a_2) \wedge \top \equiv (o_2 \vee a_2) \wedge (b_1 \vee o_1 \vee a_1) \equiv ((o_2 \vee a_2) \wedge (b_1 \vee a_2) \wedge (b_2 \vee a_2) \wedge$ $(o_2 \lor a_2) \land a_1) \equiv \bot \lor ((o_2 \lor a_2) \land a_1) \equiv (o_2 \lor a_2) \land a_1$. Therefore $a_2 \lor o_2 \preccurlyeq a_1$. And since

CHAPTER 3. PROPERTIES AND QUANTITIES

 $a_2 \leq a_2 \vee a_2$, we have $a_2 \leq a_1$. Since $b_1\vee o_1\neq o_2\vee a_2,$ we have $b_1\neq a_2$ which means $b_1\preccurlyeq \neg a_2$ Since $b_1 \vee o_1 \vee a_1 \equiv \top$, we have $\neg a_1 \leq b_1 \vee o_1$. Similarly $\neg b_2 \leq o_2 \vee a_2$. Since $b_1 \vee o_1 \neq o_2 \vee a_2$,

a₁ ≠ ¬b₂ and therefore ¬a₁ ≤ b₂. Since $b_1 \le b_2$, $a_2 \le a_1$, $b_1 \le \neg a_2$ and $\neg a_1 \le b_2$, the two references are aligned.

Proposition 3.28. Let $r_1 = (b_1, o_1, a_1)$ and $r_2 = (b_2, o_2, a_2)$ be two strict references. Then $1 < r_2$ if and only if $-a_1 \le b_2$

 $\textit{Proof.} \ \, \mathrm{Let} \ \, \mathbf{r}_1 < \mathbf{r}_2. \ \, \mathrm{By} \ \, 3.27, \ \, \mathrm{we \ have} \ \, \neg \mathsf{a}_1 \not < \mathsf{b}_2. \ \, \mathrm{Conversely, \ \, let} \ \, \neg \mathsf{a}_1 \not < \mathsf{b}_2. \ \, \mathrm{Then} \ \, \neg \mathsf{a}_1 \not < \neg \mathsf{b}_2.$ ecause the references are strict, $\neg a_1 \equiv b_1 \lor o_1$ and $\neg b_2 \equiv o_2 \lor a_2$. Therefore $b_1 \lor o_1 \not = o_2 \lor a_2$ and $\mathbf{r}_1 < \mathbf{r}_2$ by definition.

Definition 3.29. A reference is the immediate predecessor of another if nothing can be bound before the second and after the first. Formally, $r_1 < r_2$ and $a_1 \neq b_2$. Two references ecutive if one is the immediate suc

Proposition 3.30. Let $r_1 = (b_1, o_1, a_1)$ and $r_2 = (b_2, o_2, a_2)$ be two references. If r_1 is mmediately before \mathbf{r}_2 then $\mathbf{b}_2 \equiv \neg \mathbf{a}_1$. *Proof.* Let \mathbf{r}_1 be immediately before \mathbf{r}_2 . Then $a_1 \neq b_2$ which means $b_2 \leqslant \neg a_1$. By 3.27

e also have $\neg a_1 \leq b_2$. Therefore $b_2 \equiv \neg a_1$.

Proposition 3.31. Let $r_1 = (b_1, o_1, a_1)$ and $r_2 = (b_2, o_2, a_2)$ be two strict references. Then is immediately before r2 if and only if b2 = -a1.

 $\textit{Proof.} \ \, \text{Let} \ \mathbf{r}_1 \ \text{be immediately before} \ \mathbf{r}_2. \ \, \text{Then} \ \mathsf{b}_2 \equiv -\mathsf{a}_1 \ \text{by 3.30.} \ \, \text{Conversely, let} \ \mathsf{b}_2 \equiv -\mathsf{a}_1.$ Then $\mathbf{r}_1 < \mathbf{r}_2$ by 3.28. We also have $\mathbf{a}_1 \neq \neg \mathbf{a}_1$, therefore $\mathbf{a}_1 \neq \mathbf{b}_2$ and \mathbf{r}_1 is immediately before rs by definition.

CHAPTER 3. PROPERTIES AND QUANTITIES

neans we can find $\mathbf{r}_3 = (b, \neg b \land \neg a_2, a_2)$ for some $b \in \mathcal{D}_k$ such that $\mathbf{r}_3 < \mathbf{r}_2$ and therefore ¬a1 ≼ b < ¬a2.

For the third, suppose $a_1 \in D_a$ and $b_2 \in D_b$ such that $\neg a_1 \prec b_2$. Then $r_1 = (\bot, \neg a_1, a_1)$ and $\mathbf{r}_2 = (b_2, \neg b_2, \bot)$ are strict references aligned with the domain such that $\mathbf{r}_1 < \mathbf{r}_2$ but \mathbf{r}_2 flate successor of \mathbf{r}_1 . This means we can find $\mathbf{r}_3 = (\mathbf{b}, \neg \mathbf{b} \land \neg \mathbf{a}, \mathbf{a})$ such that $r_1 < r_3 < r_2$ and therefore $\neg a_1 \le b < \neg a \le \neg b_2$.

Proposition 3.37. Let D be an experimental domain generated by a set of refinable aligned references. Then all elements of D are part of a pair $(s_b, \neg s_a)$ such that $s_b \in D_b$, $s_a \in D_a$ and $\neg s_a$ is the immediate successor of s_b in D or $s_b \equiv -s_a$. Moreover if $s \in D$ has liate successor, then $s \in D_b$.

Proof. Let $\mathcal D$ be an experimental domain generated by a set of refinable aligned strict eferences. Let $s_b \in \mathcal{D}_b$. Let $A = \{a \in \mathcal{D}_a \mid a \lor s_b \not\equiv \top\}$. Let $s_a = \bigvee_{a \in A} a$. First we show that $s_b \le \neg s_a$. We have $s_b \land \neg s_a \equiv s_b \land \neg \bigvee_{a \equiv s_b} \land \bigwedge_{a = s_b} \neg a \equiv \bigwedge_{a \in A} s_b \land \neg a$. For all $a \in A$ we have $a \leftarrow A$ $a \leftarrow A$ which means $a \leftarrow A$ because of the total order of $a \leftarrow A$. This means that $\land \neg a \equiv s_b$ for all $a \in A$, therefore $s_b \land \neg s_a \equiv s_b$ and $s_b \preccurlyeq \neg s_a$.

Next we show that no statement $s \in D$ is such that $s_h < s < -s_a$. Let $a \in D_a$ such that $s_i < -a$. By construction $a \in A$ and therefore $-s_a \le -a$. Therefore we can't have $s_b < a < -s_a$. We also can't have $b \in D_b$ such that $s_b < b < -s_a$: by 3.36 we'd find $a \in D_a$ such that $s_b < a \le b < -s_a$ which was ruled out. So there are two cases. Either $s_b \not\equiv -s_a$ then $s_b < -s_a$: $-s_a$ is the immediate successor of b. Or $s_b \equiv -s_a$.

The same reasoning can be applied starting from $s_a \in \mathcal{D}_a$ to find a $s_b \in \mathcal{D}_b$ such that s_b is he immediate predecessor of $\neg s_a$ or an equivalent statement. This shows that all elements of D are paired

To show that if a statement in D has a successor then it must be a before statement, let $s_1, s_2 \in D$ such that s_2 is the immediate successor of s_1 . By 3.36, in all cases where $s_1 \notin D_b$ and $s_2 \notin D_a$ we can always find another statement between the two. Then we must have that $s_1 \in \mathcal{D}_k$ and $s_2 \in \mathcal{D}_n$.

Theorem 3.38 (Reference ordering theorem). An experimental domain is naturally orlered if and only if it can be generated by a set of refinable aligned strict references

Proof. Suppose Dx is an experimental domain generated by a set of refinable aligned ict references. Then by 3.34 and 3.37 the domain satisfies the requirement of theorem .16 and therefore is naturally ordered.

Now suppose D_X is naturally ordered. Define the set B_b , B_a and D as in 3.12. Let $R = \{(b, \neg b \land \neg a, a) \mid b \in B_b, a \in B_a, b < \neg a\}$ be the set of all references constructed from the basis. First let us verify they are references. The before and after statements are verifiable since they are part of the basis. The on statement $\neg b \land \neg a$ is not a contradiction since $b < \neg a$ means $b \not= a$ and $b \not= \neg a$. The on statement is broader than $\neg b \land \neg a$ as they are equivalent and it is broader than $b \land a$ as that is a contradiction since $b < \neg a$. Therefore Rs a set of references. Since the before and after statements of R coincide with the basis of the domain, D_X is generated by R.

3.3. REFERENCES AND EXPERIMENTAL ORDERING

3.3. REFERENCES AND EXPERIMENTAL ORDERING

defined. But references can overlap in different ways.

which means $b_1 \lor o_1 \neq o_3 \lor a_3$

For transitivity, if $r_1 < r_2$, we have $b_1 \lor o_1 * o_2 \lor a_2$ and therefore $\neg(b_1 \lor o_1) \succeq o_2 \lor a_2$ by 1.23. Since $b_1 \lor o_1 \lor a_1 \equiv \tau$, we have $a_1 \succeq \neg(b_1 \lor o_1)$. Similarly if $\mathbf{r}_2 \lessdot \mathbf{r}_3$ we'll have $a_2 \succeq \neg(b_2 \lor o_2) \succeq o_3 \lor a_3$. Putting it all together $\neg(b_1 \lor o_1) \succeq o_2 \lor a_2 \succeq a_2 \succeq \neg(b_2 \lor o_2) \succeq o_3 \lor a_3$.

Corollary 3.24. The relationship $r_1 \le r_2$, defined to be true if $r_1 < r_2$ or $r_1 = r_2$, is a

As we saw, two references may overlap and therefore an ordering between them cannot be

Suppose we have a vertical line one millimeter thick and call the left side the part before

the line and the right side the part after. We can have another vertical line of the same

overlap. The case of the two vertical lines is something that, through finding finer references

can be given a linear order. The case of the vertical and horizontal line, instead, cannot

Conceptually, the overlapping vertical lines are aligned because we can imagine narrower

lines around the borders, and those lines will be ordered references in the above sense; each

not-after statements of one reference are either narrower or broader than the before and not

after statements of the other. That is, alignment can also be defined in terms of narrowner

Note that if a reference is strict, before and after statements are not compatible and

line would be completely before or after, without intersection. This means that the before and

Intuitively, the vertical lines are aligned while the horizontal and vertical are not.

thickness overlapping but we can also have a horizontal line which will also, at some point,

Definition 3.33. Let D be a domain generated by a set of references R. A reference - (b o a) is said to be aliened with D if b c D, and a c D.

Proposition 3.34. Let D be an experimental domain generated by a set of aligned strict ferences R and let $D = D_b \cup \neg(D_a)$. Then (D, \preccurlyeq) is linearly ordered.

Proof. By 3.26 we have that $B = \mathcal{B}_b \cup \neg(\mathcal{B}_a)$ is aligned by narrowness. By 3.15 the rdering extends to D.

Having a set of aligned references is not necessarily enough to cover the whole space at all levels of precision. To do that we need to make sure that, for example, between two references that are not consecutive we can at least put a reference in between. Or that if we have two references that overlap, we can break them apart into finer ones that do not overlap and on is after the other.

We call a set of references refinable if the domain they generate has the above mentioned properties. This allows us to break up the whole domain into a sequence of references that not overlap, are linearly ordered and that cover the whole space. As we get to the fines references, their before statements will be immediately followed by the negation of their after statements, since there can't be any reference in between. Conceptually, this will give us the second and the third condition of the domain ordering theorem 3.16.

Definition 3.35. Let D be an experimental domain generated by a set of aligned reference . The set of references is refinable if, given two strict references $r_1 = (b_1, o_1, a_1)$ and $\mathbf{p}_2 = (\mathbf{b}_2, \mathbf{o}_2, \mathbf{a}_2)$ aligned with \mathcal{D} , we can always:

- \bullet find an intermediate one if they are not consecutive; that is, if $\mathbf{r}_1 < \mathbf{r}_2$ but \mathbf{r}_2 is no the immediate successor of r., then we can find a strict reference r. aligned with D
- refine overlapping references if one is finer than the other; that is, if o₂ < o₁, we can find a strict reference r_3 aligned with \mathcal{D} such that $o_3 \leq o_1$ and either $b_3 \equiv b_1$ and $r_3 < r_2$ or $a_3 \equiv a_1$ and $r_2 < r_3$.

Proposition 3.36. Let D be an experimental domain generated by a set of refinable aligned

3.4. DISCRETE QUANTITIES

Now we show that R consists of aligned strict references. We already saw that b * aand we also have ¬b∧¬a is incompatible with both b and a. The references are strict. To show they are aligned, take two references. The before and not after statements are inearly ordered by 3.14 which means the references are aligned.

To show R is refinable, note that each reference can be expressed as (" $x < x_1$ ", " $x_1 \le x \le x_2$ ", " $x > x_2$ ") where $x_1, x_2 \in X$ and " $x_1 \le x \le x_2$ " = " $x \ge x_1$ " \wedge " $x \le x_2$ ". That is, very reference is identified by two possibilities x_1, x_2 such that $x_1 \le x_2$. Therefore take two references $\mathbf{r}_1, \mathbf{r}_2 \in R$ and let (x_1, x_2) and (x_3, x_4) be the respective pair of possibilities we can use to express the references as we have shown. Suppose $\mathbf{r}_1 < \mathbf{r}_2$ but they are not consecutive. Then " $x \le x_2$ " < " $x < x_3$ ". That is, we can find $x_5 \in X$ such that $x_2 < x_5 < x_3$ which means " $x \le x_2$ " \le " $x < x_5$ " and " $x \le x_5$ " \le " $x < x_3$ ". Therefore the reference $\mathbf{r}_3 \in R$ dentified by (x_5, x_5) is between the two references. On the other hand, assume the second reference is finer than the first. Then $x_1 \le x_2$ and $x_4 \le x_2$ with either $x_1 \ne x_3$ or $x_4 \ne x_2$. Consider the references $\mathbf{r}_3, \mathbf{r}_4 \in R$ identified by (x_1, x_1) and (x_2, x_2) . Either $\mathbf{r}_3 < \mathbf{r}_2$ or $\mathbf{r}_2 < \mathbf{r}_4$. Also note that the before statements of \mathbf{r}_1 and \mathbf{r}_3 are the same and the after statements of \mathbf{r}_1 and \mathbf{r}_4 are the same. Therefore we satisfy all the requirements and the set R is refinable by definition.

To recap, experimentally we construct ordering by placing references and being able to tell whether the object measured is before or after. We can define a linear order on the possibilities, and therefore a quantity, only when the set of references meets special conditions.

The references must be strict, meaning that before, on and after are mutually exclusive. They must be aligned, meaning that the before and not-after statement must be ordered by narrowness. They must be refinable, meaning when they overlap we can always find finer references with well defined before/after relationships. If all these conditions apply, we have a linear order. If any of these conditions fail, a linear order cannot be defined.

The possibilities, then, correspond to the finest references we can construct within the domain. That is, given a value qo, we have the possibility "the value of the property is qo" and we have the reference ("the value of the property is less than q_0 ", "the value of the property is q_0 ", "the value of the property is more than q_0 ").

3.4 Discrete quantities

Now that we have seen the general conditions to have a naturally ordered experimental domain, we study common types of quantities and under what conditions they arise. We start with discrete ones: the number of chromosomes for a species, the number of inhabitants of a country or the atomic number for an element are all discrete quantities. These are quantities

that are fully characterized by integers (positive or negative) We will see that discrete quantities have a simple characterization: between two references there can only be a finite number of other references.

The first thing we want to do is characterize the ordering of the integers. That is, we want to find necessary and sufficient conditions for an ordered set of elements to be isomorphic to a subset of integers. First we note that between any two integers there are always finitely many elements. Let's call sparse an ordered set that has that property: that between two elements there are only finitely many. This is enough to say that the order is isomorphic to

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Reference ordering theorem

 An experimental domain is fully characterized by a quantity if and only if it can be generated by a set of refinable aligned strict references

Property of references	Meaning
Strict	The quantity is always only before/on/after the reference. This can be assumed if the extent of what we measure is smaller than the extent of the reference.
Aligned	The before/after statement have an ordering in term of narrowness (specificity). Necessary to have a coherent before and after over the whole range.
Refinable	If we have overlaps, we can always construct finer references. Necessary to create smallest mutually exclusive cases that correspond to the values.

Integers and reals

- If we assume that between two non-overlapping references we can only put finitely many references, then the ordering is the one of the integers
 - Equality can be tested as well
- If we assume that between two non-overlapping references we can always put another, then the ordering is the one of the reals
 - Equality cannot be tested in this case
- These are the only two orderings that are homogeneous, where all references have the same properties
 - And that is why they are the most fundamental in physics

Are these requirements tenable at Planck scale?

Property of references	Meaning	Problems
Strict	The quantity is always only before/on/after the reference. This can be assumed if the extent of what we measure is smaller than the reference.	Objects measured and references are ultimately of the same kind; their extent should be comparable
Aligned	The before/after statements have an ordering in term of narrowness (specificity). Necessary to have a coherent before and after over the whole range.	If indistinguishable particles are the smallest references and are placed very close to each other, it is not clear how can be sure they haven't switched
Refinable	If we have overlaps, we can always construct finer references. Necessary to create smallest mutually exclusive cases that correspond to the values.	The whole point of reaching Planck length is that we cannot further refine our references

Are these requirements tenable at Planck scale?

- If we take the quantum nature of the references into consideration, all the requirements seem untenable
 - Note that all three are necessary: if even only one fails we have a problem
- What fails is ordering itself
 - Is not that the real numbers need to be changed to rationals or integers: we don't have numbers to begin with

Failure of ordering

- Riemannian manifold
- Differentiable munifold + innercoroduct Topological manifold + differentiable structure Ordered topological space + locally \mathbb{R}^n Topological space + order topology

 If ordering fails, all the structures that are based on ordering fail as well. No manifold, no differentiability, no calculus, no inner product, no geometry. We need to develop a new chain of mathematical tools.

Conclusion

- Topology, the simplest mathematical structure needed for geometry, has a clear well-defined meaning in terms of experimental verifiability
 - This is appropriate as experimental verifiability is the foundation of science
- Order topology, the next required structure, formally captures the ability to experimentally compare quantities
 - The ordering is generated by logical relationships: if "x<8" then also "x<10"
- For real numbers, the requirements can only be satisfied ideally, most likely leading to a breakdown at Planck scale
 - The idea that our "measurement device" is "classical" is baked into the very nature of the order topology, which can't then be undone up the stack

Conclusion

- The standard mathematical toolchain (i.e. manifolds, differentiability/integration, differential geometry, Riemannian geometry, ...) needs to be rethought
 - The idea that we can take something and divide it into infinitesimal contributions is intrinsically classical
- In the same way that the geometry of space-time (i.e. the metric tensor) depends on the energy/mass distribution, the topology may depend on it as well
- The foundations of physics lie in understanding the most basic mathematical structures, their physical significance and how they can be generalized

General mathematical theory of experimental science

Experimental verifiability

leads to topological spaces, sigma-algebras, ...

State-level assumptions

Infinitesimal reducibility

leads to classical phase space

Process-level assumptions

Deterministic and reversible evolution

leads to isomorphism on state space

Non-reversible evolution

•••

Hamilton's equations

$$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$$

Irreducibility

leads to quantum state space

Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi$$

Thermodynamics

Euler-Lagrange equations

$$\delta \int L(q,\dot{q},t) = 0$$

Kinematic equivalence

leads to massive particles

For more information

- Assumptions of Physics project website: http://assumptionsofphysics.org/
- Topology and Experimental Distinguishability
 Christine A. Aidala, Gabriele Carcassi, and Mark J. Greenfield, *Top. Proc.* 54 (2019) pp. 271-282

