On the role of mathematics in physical theories

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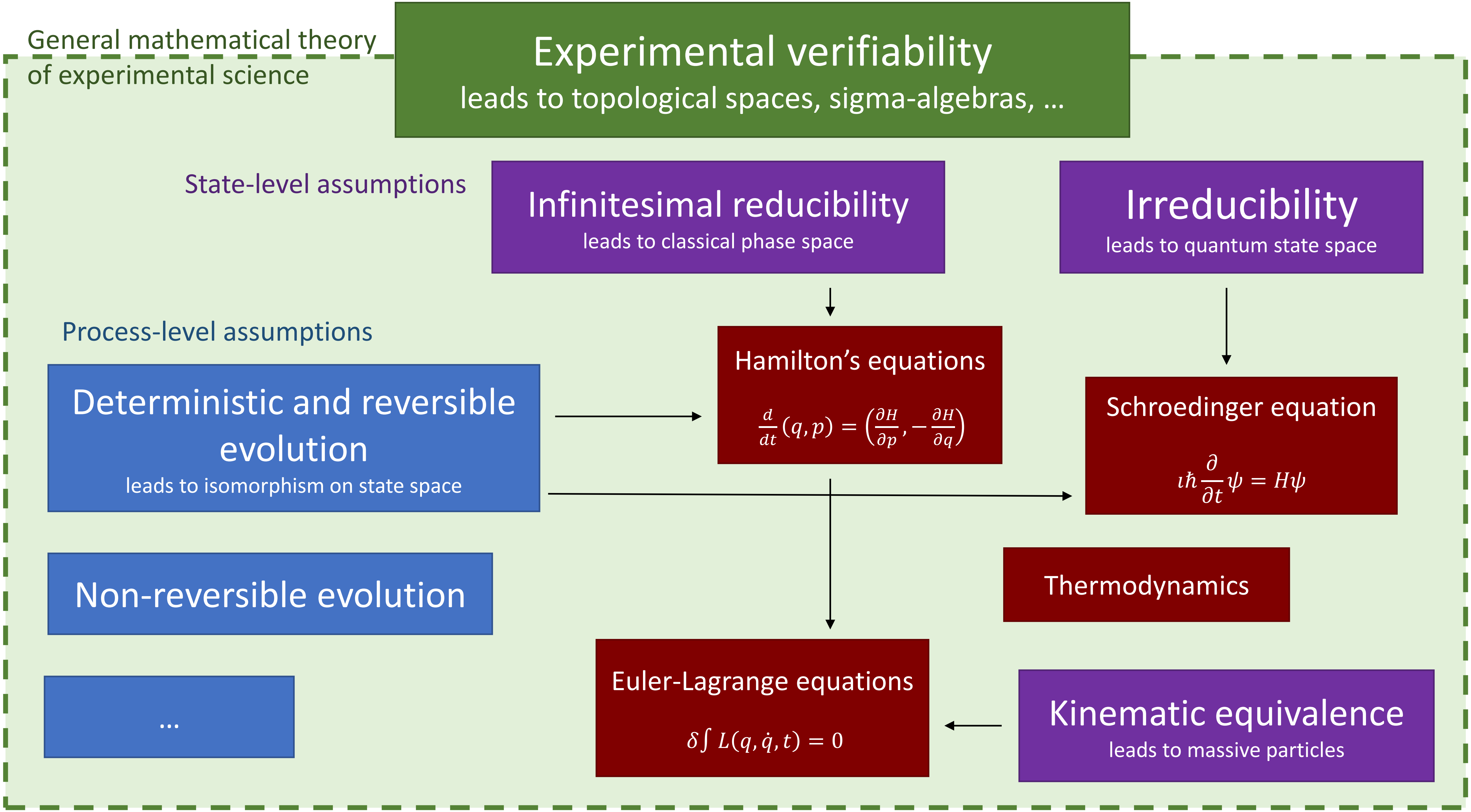
Before we start

- Disclaimer: we are not going to talk in general of the role of mathematics in science. Just its role within a physical theory.
  - By role we mean its "technical" function, what it formally captures
  - We are not going to discuss sociological roles, inspiration roles, etc…
- The viewpoint presented here is profoundly shaped by our work on our project "Assumptions of Physics"
  - It forced us to understand where the line between scientific theories and mathematical frameworks is

Assumptions of Physics

- The aim of the project is to find a handful of physical principles and assumptions from which the basic laws of physics can be derived (see http://assumptionsofphysics.org)
- To do that we need to develop a general mathematical theory of experimental science: the theory that studies scientific theories
  - A formal framework that forces us to clarify our assumptions
  - From those assumptions the mathematical objects are derived
  - Each mathematical object has a clear physical meaning and no object is unphysical
  - Gives us concepts and tools that span across different disciplines
  - Gives us a better understanding of what the laws of physics are and what they represent


General mathematical theory of experimental science
Experimental verifiability
leads to topological spaces, sigma-algebras, …
State-level assumptions
Infinitesimal reducibility
leads to classical phase space
Irreducibility
leads to quantum state space
Process-level assumptions
Hamilton's equations
$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$
Deterministic and reversible evolution
leads to isomorphism on state space
Schroedinger equation
$\imath\hbar\frac{\partial}{\partial t}\psi = H\psi$
Thermodynamics
Non-reversible evolution
Euler-Lagrange equations
$\delta\int L(q,\dot{q},t) = 0$
Kinematic equivalence
leads to massive particles
…

Outline

- First we will get familiar with the basic ideas
  - We do not have time to see all details, but we have to see at least some parts to get a sense of what it means to properly formalize scientific concepts
  - We will present a formalism to properly capture semantic properties and relationships of the type we have in science and see how this formalism leads necessarily to topologies and sigma-algebras (the foundations of differential geometry, measure theory, probability theory, …)
- Once we have seen exactly how physical concepts are encoded into mathematical structures, we can draw conclusions

Principle of scientific objectivity. Science is universal, non-contradictory and evidence based.

- The principle of scientific objectivity tells us that science deals with assertions that are:
  - either true or false (non-contradictory)
  - for everybody (universal)
  - and experimentally verifiable (evidence-based)
- We call such assertions **verifiable statements**
  - The first two requirements are the same as in classical logic
  - The third means we have an experimental test that we can run and, if the statement is true, it completes successfully in **finite time**

Examples of verifiable statements

- Examples:
  - The mass of the photon is less than 10^{-13} eV
  - If the height of the mercury column is between 24 and 25 millimeters then its temperature is between 24 and 25 Celsius
  - If I take 2 ± 0.01 Kg of Sodium-24 and wait 15 ± 0.01 hours there will be only 1 ± 0.01 Kg left
- Counterexamples:
  - Chocolate tastes good (not universal)
  - It is immoral to kill one person to save ten (not universal and/or evidence-based)
  - The number 4 is prime (not evidence-based)
  - This statement is false (not non-contradictory)
  - The mass of the photon is exactly 0 eV (not verifiable due to infinite precision)
- We need a mathematical framework to capture these concepts

Statements

- In mathematical logic, statements are sentences constructed from a well defined set of symbols
  - I.e. $\{\neg, \wedge, \vee, \exists, \forall, a, b, c, \ldots\}$
  - They are so defined to avoid paradoxes, the “web of meaning” and therefore give a rigorous foundation to mathematics, study proof theory, etc…
- This does not work for us
  - Science started well before we had those symbols
  - Science is about the web of meaning
- In science, statements are the assertions represented by the sentences
  - “This animal is a cat” is the same statement as “Quest’animale e’ un gatto”
  - “The desk is 1 meter wide” is the same statement as “The desk is 3.28083989501 feet wide”

Statements

> **Definition 1.1.** *The* ***Boolean domain*** *is the set* $\mathbb{B} = \{\text{FALSE}, \text{TRUE}\}$ *of all possible truth values.*
>
> **Axiom 1.2.** *A* ***statement*** s *is an assertion that is either true or false. A* ***logical context*** $\mathcal{S}$ *is a collection of statements with well defined logical relationships. Formally, a logical context* $\mathcal{S}$ *is a collection of elements called statements upon which is defined a function* $\text{truth} : \mathcal{S} \to \mathbb{B}$.

- Statements themselves are not formally defined
  - We are not going to try to define a grammar or try to specify what "meaning" means, we just have symbols to represent them mathematically
- but we axiomatically give them properties from which we can construct formal propositions
  - E.g. $\text{truth}(\boldsymbol{s}_1) = \text{TRUE}$

Assignments

- In mathematical logic, logical relationships are defined by their truth value
  - For example $p \to q$ is false if p is true and q is false
  - Therefore “4 is a prime number” implies “π is transcendental”
  - Moreover, the semantic is said to define what is true or not true
- This does not work for us
  - “This animal is a dog” implies “this animal is a mammal” is about what we could possibly find, not on whether it happens to be true or not
  - The meaning (i.e. the semantic) of “Secretariat will win the race tomorrow” is clear even if we don’t know the truth value
- In science, the truth is found experimentally and the semantic has to be clear **before** we run the tests
- Logical consistency is about what hypothetical truths we can find within a certain model
  - For example, “this animal is a cat” and “this animal is a dog” can’t be both assigned true

Assignments

Definition 1.3. *An* ***assignment*** *for a logical context $\mathcal{S}$ is a map $a : \mathcal{S} \to \mathbb{B}$ that assigns a truth value to each statement. Formally, an assignment is simply a map $a \in \mathbb{B}^{\mathcal{S}}$. An assignment for a set of statements $S \subseteq \mathcal{S}$ is a map $a : S \to \mathbb{B}$ while an assignment for a statement $\mathsf{s} \in \mathcal{S}$ is a truth value $t \in \mathbb{B}$.*

Axiom 1.4. *A* ***possible assignment*** *for a logical context $\mathcal{S}$ is a map $a : \mathcal{S} \to \mathbb{B}$ that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$ such that $\mathrm{truth} \in \mathcal{A}_{\mathcal{S}}$. A map $a : \mathcal{S} \to \mathbb{B}$ is a possible assignment for $\mathcal{S}$ if $a \in \mathcal{A}_{\mathcal{S}}$.*

- As with statements, we don't try to formalize how the possible assignments are derived
  - It would require "meaning"
- but we axiomatically give them properties from which we can construct formal propositions

Statements as functions of other statements

Axiom 1.10. *We can always construct a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let* $S \subseteq \mathcal{S}$ *be a set of statements and* $f_{\mathbb{B}} : \mathbb{B}^S \to \mathbb{B}$ *an arbitrary function from an assignment of* S *to a truth value. Then we can always find a statement* $\bar{\mathsf{s}} \in \mathcal{S}$ *such that*

$$a(\bar{\mathsf{s}}) = f_{\mathbb{B}}(\{a(\mathsf{s})\}_{\mathsf{s} \in S})$$

for every possible assignment $a \in \mathcal{A}_S$*.*

- This allows us to create functions of statements
  - In particular, we have negation (logical NOT), conjunction (logical AND) and disjunction (logical OR)
- Note that logical operators are not symbolic connectors
  - They are algebraic operations

Formal system for informal statements

- Those three axioms are enough to capture all the semantic structure we need to guarantee logical consistency even on informal statements
  - Any set of consistent statements that have a universal truth will fit the axioms
  - A set of inconsistent statements violates the second axiom as there would be no possible assignment

Statement equivalence

- They allow to distinguish between different notions of equivalence

Definition 1.16. *Two statements* s_1 *and* s_2 *are* ***equivalent*** $s_1 \equiv s_2$ *if they must be equally true or false simply because of their content. Formally,* $s_1 \equiv s_2$ *if and only if* $a(s_1) = a(s_2)$ *for all possible assignments* $a \in \mathcal{A}_S$.

Are the same statement

"This animal is a bird" = "Questo animale e' un uccello"

"This animal is a bird" ≡ "This animal has feathers" ← Must have the same truth

truth("This animal is a bird") = truth("That animal is a mammal")

Happen to have the same truth

Tautologies and contradictions

- We can distinguish between statements based on the truth values they are allowed

Definition 1.6. *A* ***tautology*** $\top$ *is a statement that must be true simply because of its content. That is,* $a(\top) = \text{TRUE}$ *for all possible assignments* $a \in \mathcal{A}_{\mathcal{S}}$.

Definition 1.7. *A* ***contradiction*** $\perp$ *is a statement that must be false simply because of its content. That is,* $a(\perp) = \text{FALSE}$ *for all possible assignments* $a \in \mathcal{A}_{\mathcal{S}}$.

Definition 1.8. *A statement is* ***contingent*** *if it is neither a contradiction nor a tautology.*

"This swan is a bird" is a tautology, "This cat is a dog" is a contradiction and "This animal is a cat" is contingent

Other semantic relationships

Definition 1.21. *Given two statements* s_1 *and* s_2*, we say that:*

- s_1 ***is narrower than*** s_2 *(noted* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$*) if* s_2 *is true whenever* s_1 *is true simply because of their content. That is, for all* $a \in \mathcal{A}_{\mathcal{S}}$ *if* $a(\mathsf{s}_1) = \text{TRUE}$ *then* $a(\mathsf{s}_2) = \text{TRUE}$.
- s_1 ***is broader than*** s_2 *(noted* $\mathsf{s}_1 \succcurlyeq \mathsf{s}_2$*) if* $\mathsf{s}_2 \preccurlyeq \mathsf{s}_1$.
- s_1 ***is compatible to*** s_2 *(noted* $\mathsf{s}_1 \mathrel{\hat{=}} \mathsf{s}_2$*) if their content allows them to be true at the same time. That is, there exists* $a \in \mathcal{A}_{\mathcal{S}}$ *such that* $a(\mathsf{s}_1) = a(\mathsf{s}_2) = \text{TRUE}$.

The negation of these properties will be noted by $\npreccurlyeq$, $\nsucccurlyeq$, $\not\mathrel{\hat{=}}$ *respectively.*

Definition 1.22. *The elements of a set of statements* $S \subseteq \mathcal{S}$ *are said to be* ***independent*** *(noted* $\mathsf{s}_1 \perp\!\!\!\perp \mathsf{s}_2$ *for a set of two) if the assignment of any subset of statements does not depend on the assignment of the others. That is, a set of statements* $S \subseteq \mathcal{S}$ *is independent if given a family* $\{t_{\mathsf{s}}\}_{\mathsf{s} \in S}$ *such that each* $t_{\mathsf{s}} \in \mathbb{B}$ *is a possible assignment for the respective* s *we can always find* $a \in \mathcal{A}_{\mathcal{S}}$ *such that* $a(\mathsf{s}) = t_{\mathsf{s}}$ *for all* $\mathsf{s} \in S$.

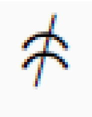
Other semantic relationships

For example:

narrower than

“This animal is a cat” $\preccurlyeq$ “This animal is a mammal”

incompatible

“This animal is a cat”  “This animal is a dog”

independent

“This animal is a cat” $\perp\!\!\!\perp$ “This animal is black”

Logical context as an algebraic structure

- We can prove that a logical context is a complete Boolean algebra, that narrowness imposes a partial order, …
- The algebraic structure we defined on the logical context captures the minimum logical and semantic relationships between our statements to guarantee universality and non-contradiction
  - Any further structure we impose on a logical context to capture other semantic relationships will need to interact with the previous structure in a well defined manner
  - That is, the notions of equivalence, independence, ordering, etc… defined on this structure will be inherited in some fashion by all other structures

Verifiable statements

- Now that we have a framework rich enough to capture all the statement relationships we need, we turn our attention to experimental verification
- A statement is verifiable if we have a repeatable procedure that terminates successfully in finite time if and only if the statement is true
  - This is hard to define formally so we won't
- Note that not all statements are experimentally verifiable
  - We can verify that "there exists extra-terrestrial life" or that "the mass of the photon is less than 10^{-18} eV"
  - We cannot verify that "there exists no extra-terrestrial life" or that "the mass of the photon is exactly 0 eV"

Verifiable statements

Axiom 1.28. *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_v \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

- *every tautology $\top \in \mathcal{S}$ is verifiable*
- *every contradiction $\perp \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

- What makes a statement verifiable is not formally defined
  - There is a sense that trying to fully specify what can be measured is equivalent to already knowing the laws of physics
- But we have to ask: under what operations is the set of verifiable statements closed?
  - Is it a Boolean algebra?

Logic of verifiable statements

- We can't always test negation
  - The test is not guaranteed to terminate if the test is unsuccessful
    - If we can, we say the statement is *decidable*
- We can always test the finite conjunction
  - Just test one statement at a time: if they are all true all tests will terminate in finite time
  - We cannot have infinitely many tests though: we wouldn't terminate in finite time
- We can always test the countable disjunction
  - Once just one test terminates successfully, we are done
  - We cannot extend to uncountably many: we wouldn't be able to find the test that terminates in finite time

Logic of verifiable statements

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.

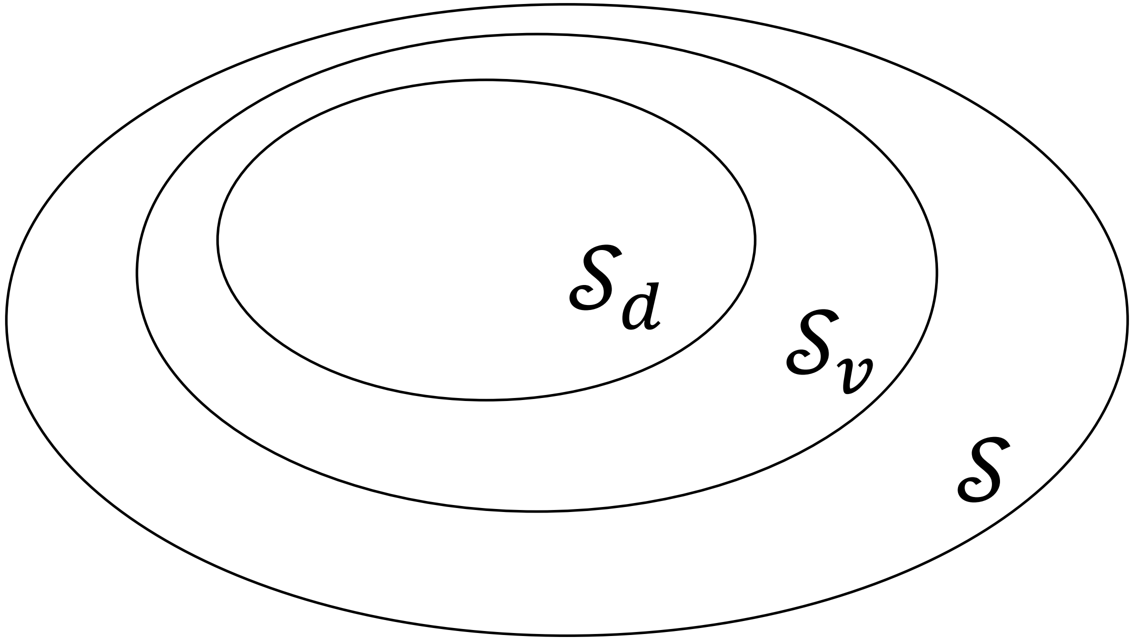
Axiom 1.31. *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a finite collection of verifiable statements. Then the conjunction* $\bigwedge_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Axiom 1.32. *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a countable collection of verifiable statements. Then the disjunction* $\bigvee_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Comparing algebras

Operator	Gate	Statement	Verifiable Statement	Decidable Statement
Negation	NOT	allowed	disallowed	allowed
Conjunction	AND	arbitrary	finite	finite
Disjunction	OR	arbitrary	countable	finite

Table 1.3: Comparing algebras of statements.



Sets of verifiable statements

- Now that we have captured how we verify single statements, what can we say about verifying a collection of statements?
- Specifically, what's the biggest set of verifiable statements we can verify?
- Clearly, we do not need to run the test for all the elements
  - Once we verify that $\boldsymbol{s}_1$ is true we already know that $\boldsymbol{s}_1 \vee \boldsymbol{s}_2$ is also true

Basis

Definition 1.32. *Given a set $\mathcal{D}$ of verifiable statements, $\mathcal{B} \subseteq \mathcal{D}$ is a* **basis** *if the truth values of $\mathcal{B}$ are enough to deduce the truth values of the set. Formally, all elements of $\mathcal{D}$ can be generated from $\mathcal{B}$ using finite conjunction and countable disjunction.*

- What is the biggest basis we can experimentally test?
- A countable set
  - Even with unlimited time, we can only test countably many statements

Experimental domain

Definition 1.33. *An **experimental domain** $\mathcal{D}$ represents all the experimental evidence that can be acquired about a scientific subject in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the tautology, the contradiction, and a set of verifiable statements that can be generated from a countable basis.*

- This represents the biggest set of verifiable statements we can test
  - Any scientific theory, in the end, is equivalent to a set of verifiable statements, which forms at most an experimental domain

Predictions

- Science is also about making predictions, but not all predictions are directly verifiable
- For example, “there exists no extra-terrestrial life” predicts that the test for “there exists extra-terrestrial life” is never going to terminate
- While we cannot always experimentally confirm negation, it still makes sense logically as a possible way things could be

Theoretical domain

Definition 1.34. *The* ***theoretical domain*** $\bar{\mathcal{D}}$ *of an experimental domain* $\mathcal{D}$ *is the set of statements that we can use to state predictions, which is constructed from* $\mathcal{D}$ *by allowing negation. We call* ***theoretical statement*** *a statement that is part of a theoretical domain. More formally,* $\bar{\mathcal{D}}$ *is the set of all statements generated from* $\mathcal{D}$ *using negation, finite conjunction and countable disjunction.*

- This represents all statements that give meaningful predictions to (and only to) the verifiable statements in the domain

Possibilities for the domain

- Among all the predictions we look for the ones that give the full picture
  - For example, if we knew “This animal is a cat” to be true, we would also know that “This animal has whiskers” and “This animal is a mammal” are true while “This animal has feathers” is false

Possibilities for the domain

Definition 1.38. *A* ***possibility*** *for an experimental domain $\mathcal{D}$ is a statement $x \in \bar{\mathcal{D}}$ that, when true, determines the truth value for all statements in the theoretical domain. Formally, $x \not\equiv \bot$ and for each $\mathsf{s} \in \mathcal{D}$, either $x \preccurlyeq \mathsf{s}$ or $x \not\preccurlyeq \neg\mathsf{s}$. The* ***full possibilities****, or simply the* ***possibilities****, X for $\mathcal{D}$ are the collection of all possibilities.*

- A possibility, if true, gives a prediction for all theoretical and verifiable statements
- The set of possibilities corresponds to all the cases we can experimentally distinguish given the experimental domain

Start with a countable set of verifiable statements (the most we can verify experimentally)

Basis $\mathcal{B}$			
e_1	e_2	e_3	…

Construct all verifiable statements that can be verified from the basis
(close under finite conjunction and countable disjunction)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$		
$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	…	$\boldsymbol{s}_1 = \boldsymbol{e}_1 \vee \boldsymbol{e}_2$	$\boldsymbol{s}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_3$	…

Construct all statements that give a prediction for those verifiable statements (close under negation as well)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	…	$\boldsymbol{s_1 = e_1 \vee e_2}$	$\boldsymbol{s_2 = e_1 \wedge e_3}$	…	$\boldsymbol{\overline{s_1} = e_1 \vee \neg e_2}$	$\boldsymbol{\overline{s_2} = \neg e_1}$	…

Consider all truth assignments: it is sufficient to assign the basis

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
T	T	F	…	T	F	…	T	F	…
…	…	…	…	…	…	…	…	…	…

Remove truth assignments that are not possible

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Each consistent truth assignment is associated with a possibility of the domain

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

Each consistent truth assignment is associated with a possibility of the domain

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

The role of logic (and math) in science is to capture what is consistent (i.e. the possibilities) and what is verifiable (i.e. the verifiable statements) and the corresponding logical relationships

Each consistent truth assignment is associated with a possibility of the domain

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

For example, verifiable statements corresponds to open sets in a topology while theoretical statements correspond to Borel sets in a σ-algebra

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
T	T	F	…	T	F	…	T	F	…
…	…	…	…	…	…	…	…	…	…

Possibilities X

The lines of the truth table (i.e. the possible assignments) are the points

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	Possibilities X
F	F	F	…	F	F	…	T	T	…	
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
T	T	F	…	T	F	…	T	F	…	
…	…	…	…	…	…	…	…	…	…	

Each column is a set (i.e. the set of possibilities that are true in that column)

conjunction and disjunction becomes intersection and union

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
T	T	F	…	T	F	…	T	F	…
…	…	…	…	…	…	…	…	…	…

Possibilities X

The experimental domain is the topology (i.e. each verifiable statement is an open set)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

The basis of the experimental domain is a sub-basis of the topology

negation and countable disjunction becomes complement and countable union

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	Possibilities X
F	F	F	…	F	F	…	T	T	…	
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
T	T	F	…	T	F	…	T	F	…	
…	…	…	…	…	…	…	…	…	…	

The theoretical domain is the σ-algebra (i.e. each theoretical statement is a Borel set)

Logical consistency and mathematics

- Mathematical structures deal with logical consistency
  - Most mathematicians, in fact, focus on whether the objects are well defined and not whether they can be found in practice (e.g. axiom of choice)
- As we want science to be logically consistent, physical theories will be mathematical structures but not all mathematical structures can be physical theories
  - For example, as an experimental domain must have a countable basis, the cardinality of the possibilities (the cases we can distinguish experimentally) is at most that of the continuum

Mathematics captures the semantic structure

- Experimentally verifiable statements are represented by topologies
- Theoretically sound statements are represented by σ-algebras
- Composition of parts into wholes is represented by vector spaces
- Objects whose description can be given by a set of numbers are represented by manifolds
- Descriptions with commensurable quantities give rise to geometrical structures
- If our objects allow distributions and densities then they are represented by differentiable manifolds
- If those densities are coordinate invariant then we have symplectic manifolds
- Deterministic and reversible evolution is logical equivalence between statements about future and past states which leads to isomorphism in whatever category is used to describe states
- …

Mathematics captures the semantic structure

- Once the semantic structure of the theory is made clear, the mathematical structure follows and there is a perfect mapping between mathematical concepts and physical concepts
  - Every mathematical theorem can be read line by line as a physical argument
  - If there is no perfect mapping, then either the theory is incomplete (i.e. some physical concept is not captured by the mathematical framework) or not completely physical (i.e. some mathematical objects do not correspond to physical ones)
- The converse is not true, even if the mathematical structure of a theory is made clear, its semantic structure cannot be reconstructed
  - Two different scientific theories can have the same mathematical structure

Science → Math is a forgetful functor (Mathematics is meaningless)

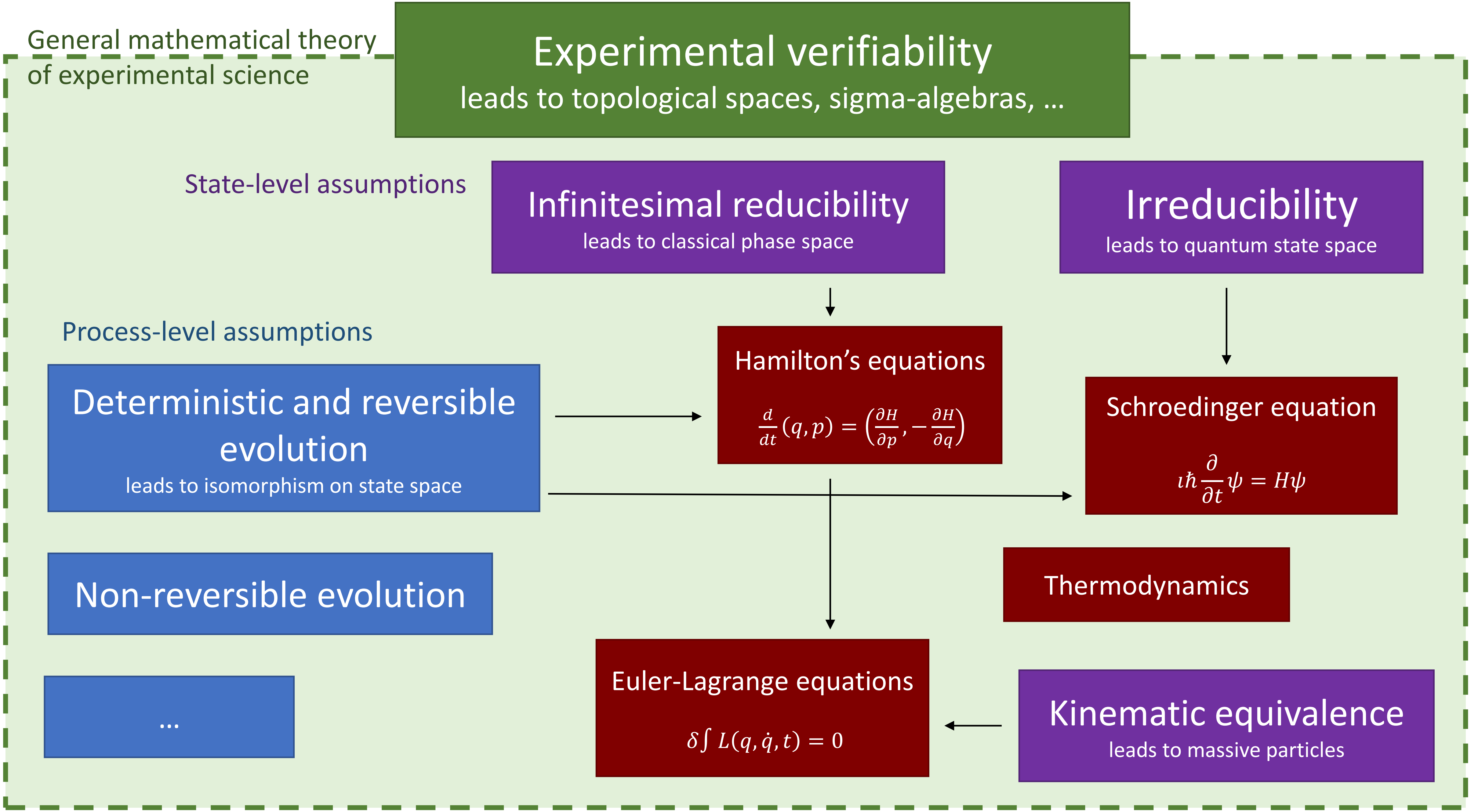
- Note that the way we made our discourse “precise” was not by making everything “precise” but by dropping what couldn’t be made precise
  - What is the meaning of a statement, what does it mean for it to be true, how does the meaning rule out assignments, when can we say that a statements is verifiable, … these cannot be formalized
- But what is left out is the actual physics
  - the fact that those symbols represent statements in the real world, that the labels we use to identify statements represent quantities that correspond to specific measurements, that those statements have relationships

Reformulation

- A model change, then, can then happen in two ways: within the same context or by changing context altogether
- If the context/experimental domain is the same, we have a purely mathematical reformulation
  - the meaning of the statements must remain the same, what is verifiable is the same
  - we can only change how the relationships are captured mathematically
- If the context/experimental domain is different, we have a different physical theory
  - the meaning of the statement may change, how tests are mapped to statements may change
  - unavoidable with new experimental techniques, different assumptions,
  - whether or not the mathematical structure is the same is inconsequential
- Naturally, model changes are a continuum, they are never formalized, so whether a particular case falls in one category or the other can be a matter of opinio

Conclusion

- Mathematics studies logically consistent structures
- Science studies theories and models that are not only logically consistent, but are universal and allow experimental verification
  - The mathematical structures we use in physics (e.g. topology, σ-algebras, measures, metrics, …) are there precisely to keep track of the logical structures of statements: how are they logically related, which ones are verifiable, which ones are more precise and by how much, and so on
- As we are developing a general mathematical theory for experimental science, we had to clearly demarcate the line between scientific ideas and mathematical ideas
- The result is that the only role mathematics has within a physical theory is to keep track of logical/semantic relationships between scientific statements within the theory. For example:
  - If "that animal is a dog" is true then "that animal is a cat" is not true
  - "The mass of the photon is less than 10^-10 eV" can be verified experimentally while "the mass of the photon is exactly 0" cannot be verified experimentally
  - If "within the volume V there is 1 Kg of gaseous helium" is true then "within the volume v there is 1/2 Kg of gaseous helium" is true for some sub volume v



For more information, see http://assumptionsofphysics.org