The logic and topology of experimentally verifiable statements

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Motivation for this work

- This work has its root in an effort to better understand fundamental physics:
    - Why is Hamiltonian time evolution a symplectomorphism?
    - Why are classical states points of a cotangent bundle?
    - Why are they points of a manifold?
    - Why are they points of a topological space?
- Last year we presented our basic insight: a topology keeps track of what can be distinguished through experimentation
    - It seems fitting that topology maps to such a fundamental concept for experimental science
- This year we try to see if we can construct a formal framework around these ideas

Overview

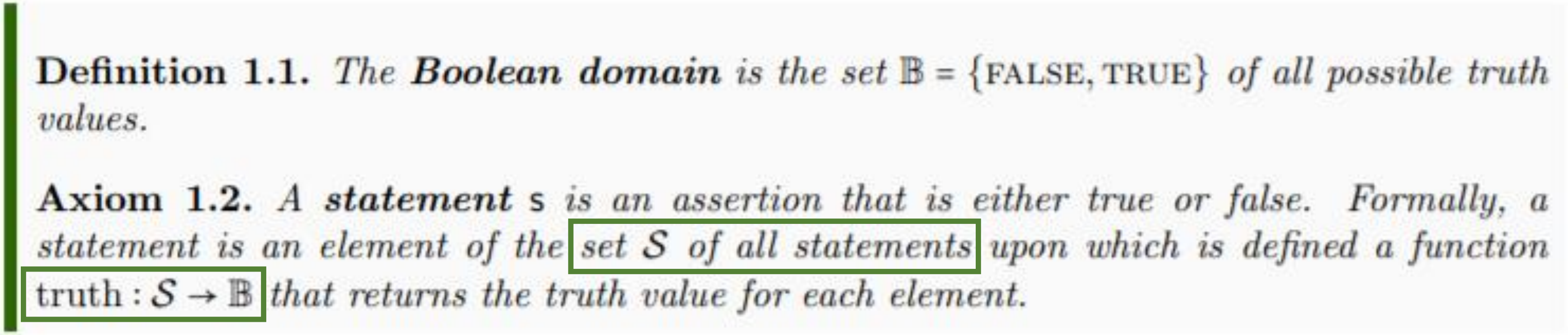
- The logic of verifiable statements
  - Develop a Boolean-like framework to capture assertions, their semantic relationships and whether they can be tested experimentally
- Experimental domains and their possibilities
  - Collect verifiable statements that can be tested together and identify the possible cases they can distinguish
- Natural topology for the possibilities
  - Show that experimental domains always provide a T_0 and second countable topology for the possible cases

The logic of verifiable statements

Requirements

- The first task is to develop a “logic framework” that recognizes these basic two requirements:
  - Our truth bearers are not sentences (e.g. sequences of symbols in a language) but the assertions they make (regardless of the language)
    - “This animal is a cat” and “Quest’animale e’ un gatto” state the same assertion
  - The truth value in science is found experimentally. The role of logic (and math) is to keep track of what is meaningful and consistent
    - “This animal is a cat” and “This animal is a dog” can’t both be true

Statements

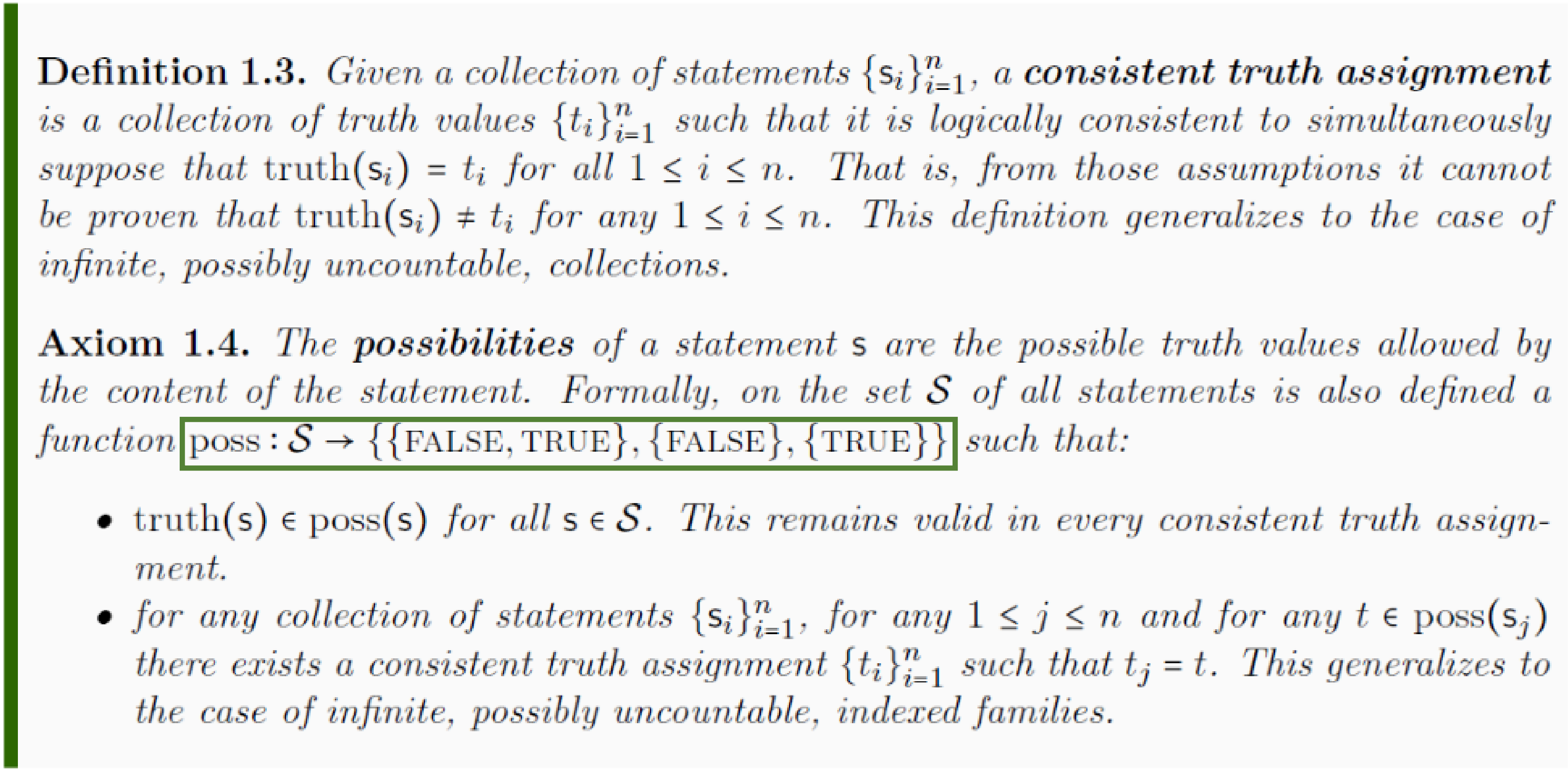


Definition 1.1. *The* ***Boolean domain*** *is the set* $\mathbb{B} = \{\text{FALSE}, \text{TRUE}\}$ *of all possible truth values.*

Axiom 1.2. *A* ***statement*** s *is an assertion that is either true or false. Formally, a statement is an element of the set* $\mathcal{S}$ *of all statements upon which is defined a function* $\text{truth} : \mathcal{S} \to \mathbb{B}$ *that returns the truth value for each element.*

- Statements themselves are not formally defined
  - We are not going to try to define a grammar or try to specify what "meaning" means
- but we axiomatically give them properties from which we can construct formal propositions
  - E.g. $\text{truth}(\boldsymbol{s}_1) = \text{TRUE}$

Possibilities of statements



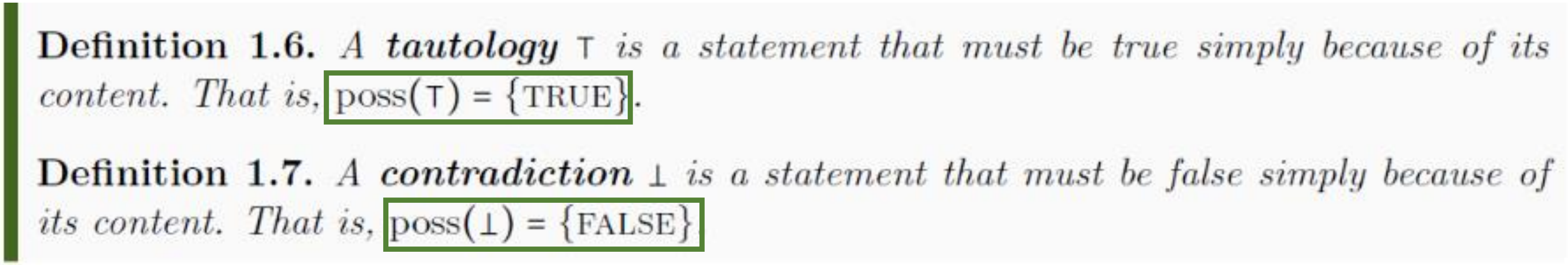
Definition 1.3. *Given a collection of statements* $\{\mathsf{s}_i\}_{i=1}^{n}$*, a* ***consistent truth assignment*** *is a collection of truth values* $\{t_i\}_{i=1}^{n}$ *such that it is logically consistent to simultaneously suppose that* $\mathrm{truth}(\mathsf{s}_i) = t_i$ *for all* $1 \le i \le n$*. That is, from those assumptions it cannot be proven that* $\mathrm{truth}(\mathsf{s}_i) \neq t_i$ *for any* $1 \le i \le n$*. This definition generalizes to the case of infinite, possibly uncountable, collections.*

Axiom 1.4. *The* ***possibilities*** *of a statement* s *are the possible truth values allowed by the content of the statement. Formally, on the set* $\mathcal{S}$ *of all statements is also defined a function* $\boxed{\mathrm{poss} : \mathcal{S} \to \{\{\mathrm{FALSE}, \mathrm{TRUE}\}, \{\mathrm{FALSE}\}, \{\mathrm{TRUE}\}\}}$ *such that:*

- $\mathrm{truth}(\mathsf{s}) \in \mathrm{poss}(\mathsf{s})$ *for all* $\mathsf{s} \in \mathcal{S}$*. This remains valid in every consistent truth assignment.*
- *for any collection of statements* $\{\mathsf{s}_i\}_{i=1}^{n}$*, for any* $1 \le j \le n$ *and for any* $t \in \mathrm{poss}(\mathsf{s}_j)$ *there exists a consistent truth assignment* $\{t_i\}_{i=1}^{n}$ *such that* $t_j = t$*. This generalizes to the case of infinite, possibly uncountable, indexed families.*

- Each statement has a set of possible truth values it can be hypothetically assigned

Possibilities of statements



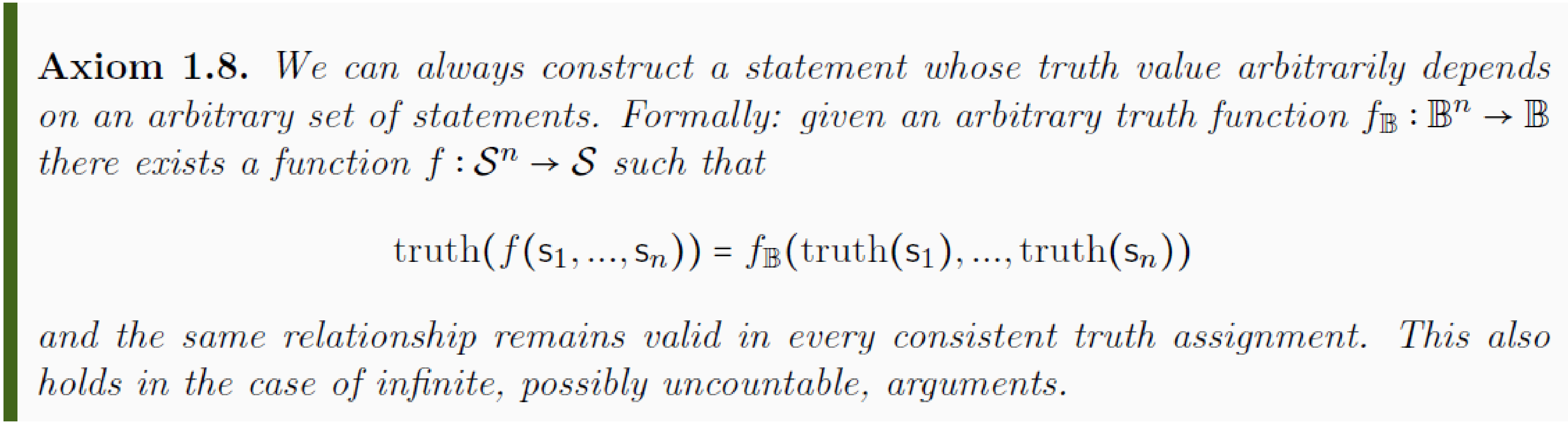
Definition 1.6. *A* ***tautology*** $\top$ *is a statement that must be true simply because of its content. That is,* $\boxed{\text{poss}(\top) = \{\text{TRUE}\}}$.

Definition 1.7. *A* ***contradiction*** $\bot$ *is a statement that must be false simply because of its content. That is,* $\boxed{\text{poss}(\bot) = \{\text{FALSE}\}}$

- For example, “This swan is a bird” is a tautology and “This cat is a dog” is a contradiction

Logic relationships

- Lastly, we need to capture logical relationships between statements



Axiom 1.8. *We can always construct a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally: given an arbitrary truth function* $f_{\mathbb{B}} : \mathbb{B}^n \to \mathbb{B}$ *there exists a function* $f : \mathcal{S}^n \to \mathcal{S}$ *such that*

$$\mathrm{truth}(f(\mathsf{s}_1, ..., \mathsf{s}_n)) = f_{\mathbb{B}}(\mathrm{truth}(\mathsf{s}_1), ..., \mathrm{truth}(\mathsf{s}_n))$$

and the same relationship remains valid in every consistent truth assignment. This also holds in the case of infinite, possibly uncountable, arguments.

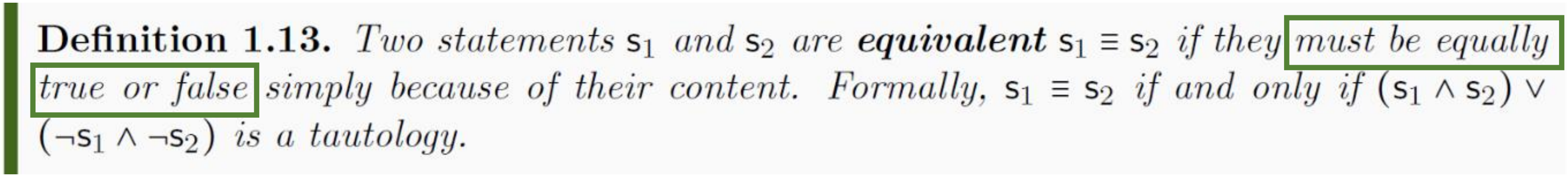
- We'll use the standard symbols for negation (¬), conjunction (∧) and disjunction (∨)

Logic relationships

- Given:
  - $\boldsymbol{s}_1$=“This animal is a cat”
  - $\boldsymbol{s}_2$=“This animal is a dog”
  - $\mathrm{poss}(\boldsymbol{s}_1) = \mathrm{poss}(\boldsymbol{s}_2) = \{\mathrm{TRUE}, \mathrm{FALSE}\}$
  - $\mathrm{poss}(\boldsymbol{s}_1 \wedge \boldsymbol{s}_2) = \{\mathrm{FALSE}\}$
- We can deduce that:
  - $\mathrm{truth}(\boldsymbol{s}_1) = \mathrm{truth}(\boldsymbol{s}_2) = \mathrm{TRUE}$ is not a consistent truth assignment
- The possibilities of statements allow us to rule out cases that are not meaningful

Statement equivalence

- From these starting points, we can:
  - Define statement equivalence



Definition 1.13. *Two statements* s_1 *and* s_2 *are* ***equivalent*** $s_1 \equiv s_2$ *if they must be equally true or false simply because of their content. Formally,* $s_1 \equiv s_2$ *if and only if* $(s_1 \wedge s_2) \vee (\neg s_1 \wedge \neg s_2)$ *is a tautology.*

Are the same statement

“This animal is a bird” = “Questo animale e’ un uccello”

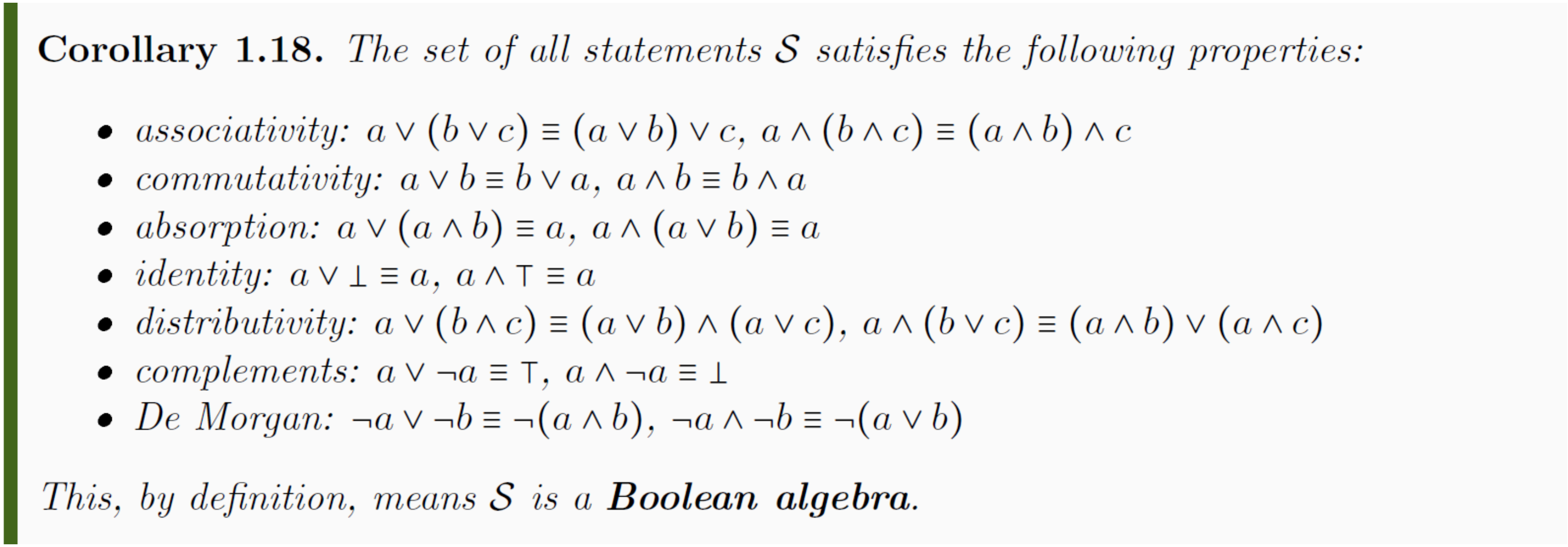
“This animal is a bird” ≡ “This animal has feathers” ← Must have the same truth

truth(“This animal is a bird”) = truth(“That animal is a mammal”)

Happen to have the same truth

Boolean algebra

- From these starting points, we can:
  - Show the set of all statements is a (complete) Boolean algebra (in terms of the equivalence classes)



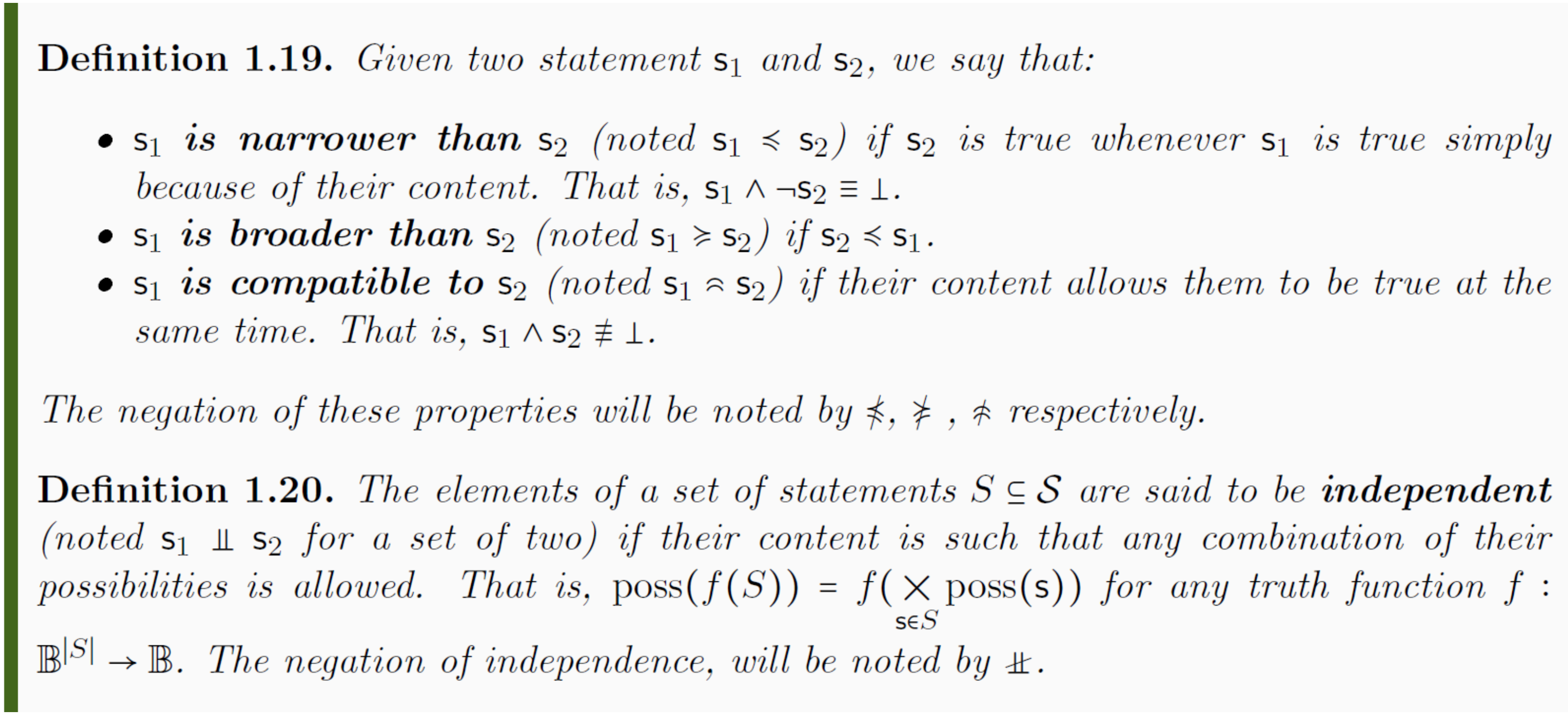
Corollary 1.18. *The set of all statements $\mathcal{S}$ satisfies the following properties:*

- *associativity:* $a \vee (b \vee c) \equiv (a \vee b) \vee c$, $a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$
- *commutativity:* $a \vee b \equiv b \vee a$, $a \wedge b \equiv b \wedge a$
- *absorption:* $a \vee (a \wedge b) \equiv a$, $a \wedge (a \vee b) \equiv a$
- *identity:* $a \vee \bot \equiv a$, $a \wedge \top \equiv a$
- *distributivity:* $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$
- *complements:* $a \vee \neg a \equiv \top$, $a \wedge \neg a \equiv \bot$
- *De Morgan:* $\neg a \vee \neg b \equiv \neg(a \wedge b)$, $\neg a \wedge \neg b \equiv \neg(a \vee b)$

This, by definition, means $\mathcal{S}$ is a ***Boolean algebra****.*

Other operators

- From these starting points, we can:
    - Define other statement relationships



Definition 1.19. *Given two statement* s_1 *and* s_2*, we say that:*

- s_1 ***is narrower than*** s_2 *(noted* $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$*) if* s_2 *is true whenever* s_1 *is true simply because of their content. That is,* $\mathsf{s}_1 \wedge \neg \mathsf{s}_2 \equiv \perp$.
- s_1 ***is broader than*** s_2 *(noted* $\mathsf{s}_1 \succcurlyeq \mathsf{s}_2$*) if* $\mathsf{s}_2 \preccurlyeq \mathsf{s}_1$.
- s_1 ***is compatible to*** s_2 *(noted* $\mathsf{s}_1 \frown \mathsf{s}_2$*) if their content allows them to be true at the same time. That is,* $\mathsf{s}_1 \wedge \mathsf{s}_2 \not\equiv \perp$.

The negation of these properties will be noted by $\not\preccurlyeq$, $\not\succcurlyeq$, $\not\frown$ *respectively.*

Definition 1.20. *The elements of a set of statements* $S \subseteq \mathcal{S}$ *are said to be* ***independent*** *(noted* $\mathsf{s}_1 \perp\!\!\!\perp \mathsf{s}_2$ *for a set of two) if their content is such that any combination of their possibilities is allowed. That is,* $\mathrm{poss}(f(S)) = f(\bigtimes_{\mathsf{s} \in S} \mathrm{poss}(\mathsf{s}))$ *for any truth function* $f : \mathbb{B}^{|S|} \to \mathbb{B}$*. The negation of independence, will be noted by* $\not\perp\!\!\!\perp$.

Other operators

For example:

narrower than

“This animal is a cat” $\preccurlyeq$ “This animal is a mammal”

incompatible

“This animal is a cat” $\not\overset{\frown}{\frown}$ “This animal is a dog”

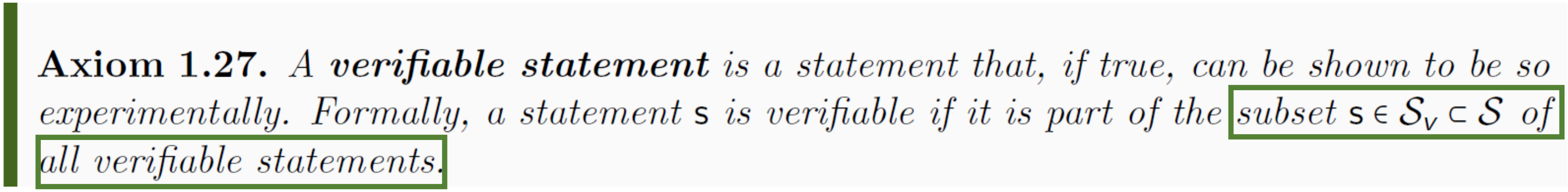
independent

“This animal is a cat” $\perp\!\!\!\perp$ “This animal is black”

Verifiable statements

- Now that we have a framework rich enough to capture all the statement relationships we need, we turn our attention to experimental verification
- A statement is verifiable if we have a repeatable procedure that terminates successfully in finite time if and only if the statement is true
    - This is hard to define formally so we won't
- Note that not all statements are experimentally verifiable
    - We can verify that "there exists extra-terrestrial life" or that "the mass of the photon is less than 10^{-18} eV"
    - We cannot verify that "there exists no extra-terrestrial life" or that "the mass of the photon is exactly 0 eV"

Verifiable statements



Axiom 1.27. *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, a statement* s *is verifiable if it is part of the subset* $\mathsf{s} \in \mathcal{S}_{\mathsf{v}} \subset \mathcal{S}$ *of all verifiable statements.*

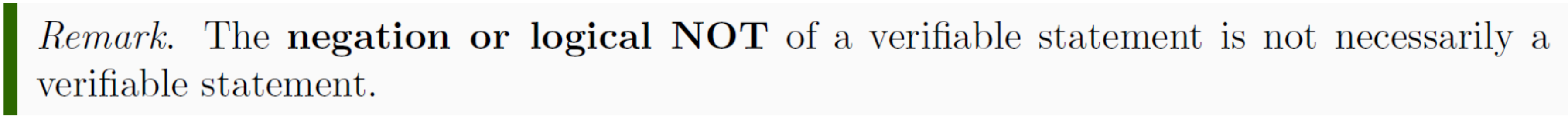
- What makes a statement verifiable is not formally defined
    - There is a sense that trying to fully specify what can be measured is equivalent to already knowing the laws of physics
- But we have to ask: under what operations is the set of verifiable statements closed?
    - Is it a Boolean algebra?

Logic of verifiable statements

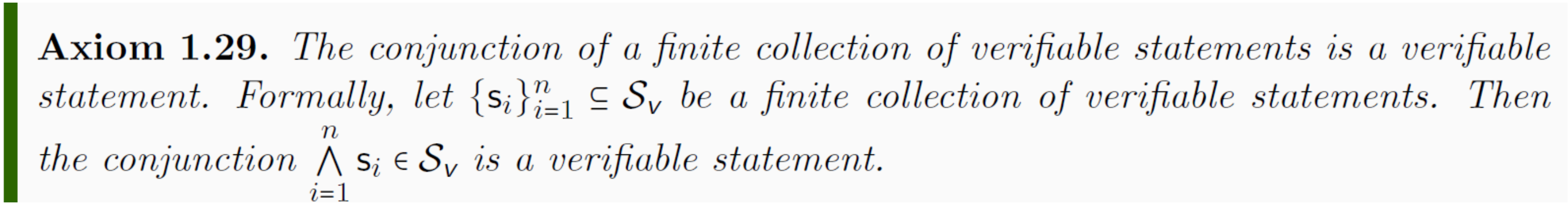
- We can't always test negation
    - The test is not guaranteed to terminate if the test is unsuccessful
        - If we can, we say the statement is *decidable*
- We can always test the finite conjunction
    - Just test one statement at a time: if they are all true all tests will terminate in finite time
    - We cannot have infinitely many tests though: we wouldn't terminate in finite time
- We can always test the countable disjunction
    - Once just one test terminates successfully, we are done
    - We cannot extend to uncountably many: we wouldn't be able to find the test that terminates in finite time

Logic of verifiable statements

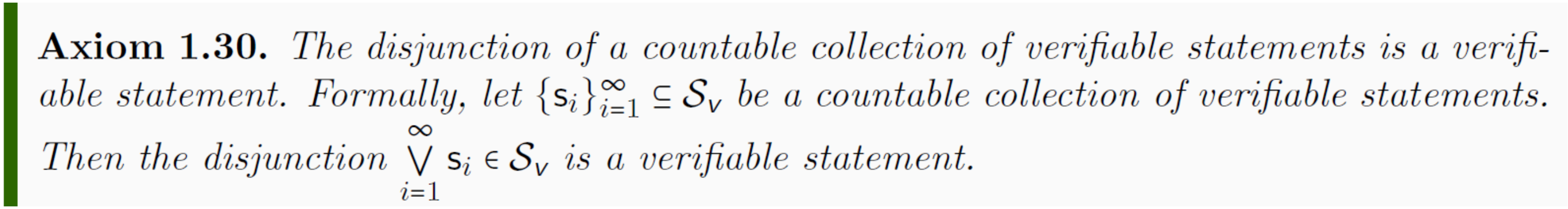
- We capture what operations are allowed on verifiable statements with the following axioms



Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.

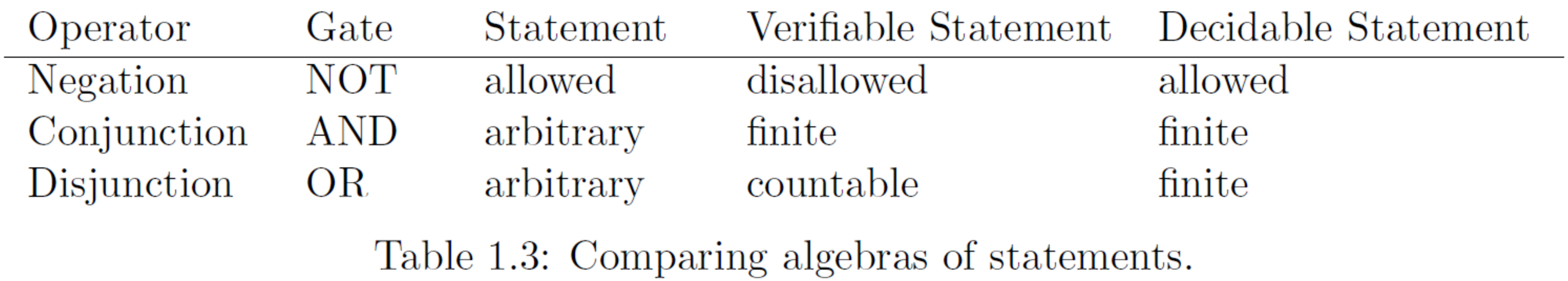


Axiom 1.29. *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{n} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a finite collection of verifiable statements. Then the conjunction* $\bigwedge\limits_{i=1}^{n} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*



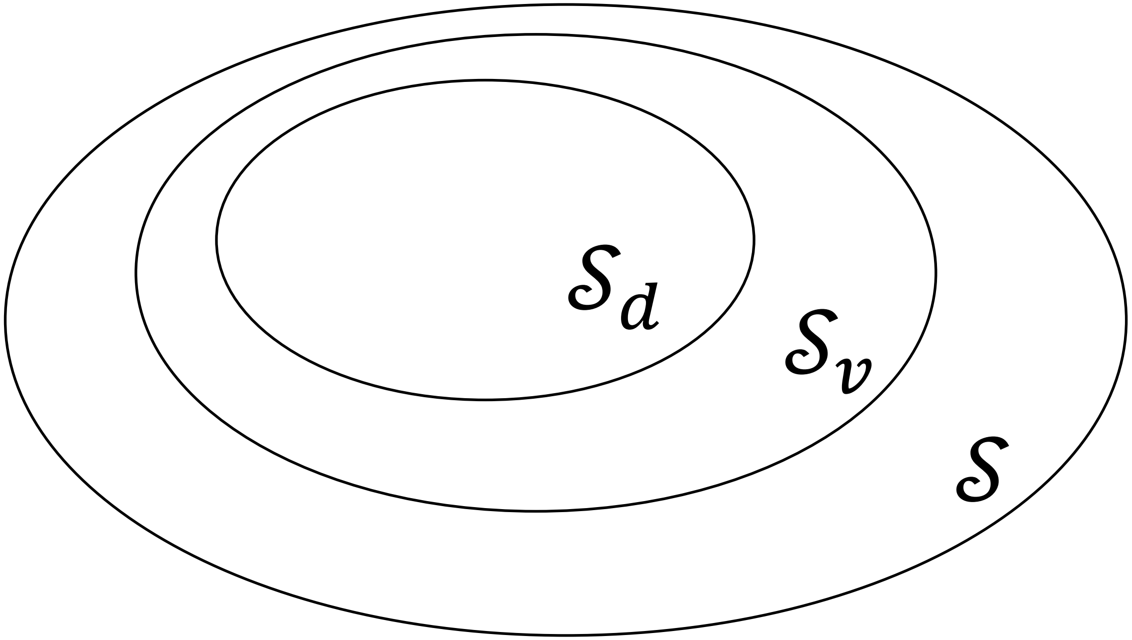
Axiom 1.30. *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let* $\{\mathsf{s}_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_{\mathsf{v}}$ *be a countable collection of verifiable statements. Then the disjunction* $\bigvee\limits_{i=1}^{\infty} \mathsf{s}_i \in \mathcal{S}_{\mathsf{v}}$ *is a verifiable statement.*

Comparing algebras



Operator	Gate	Statement	Verifiable Statement	Decidable Statement
Negation	NOT	allowed	disallowed	allowed
Conjunction	AND	arbitrary	finite	finite
Disjunction	OR	arbitrary	countable	finite

Table 1.3: Comparing algebras of statements.

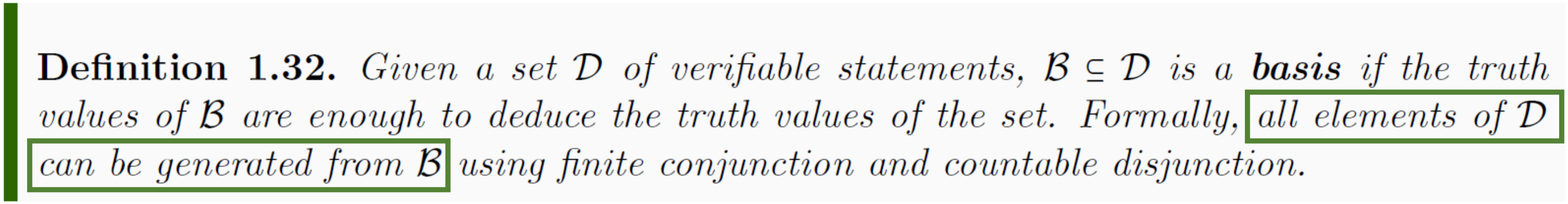


Experimental domains and their possibilities

Sets of verifiable statements

- Now that we have captured how we verify single statements, what can we say about verifying a collection of statements?
- Specifically, what's the biggest set of verifiable statements we can verify?
- Clearly, we do not need to run the test for all the elements
  - Once we verify that $\boldsymbol{s}_1$ is true we already know that $\boldsymbol{s}_1 \vee \boldsymbol{s}_2$ is also true

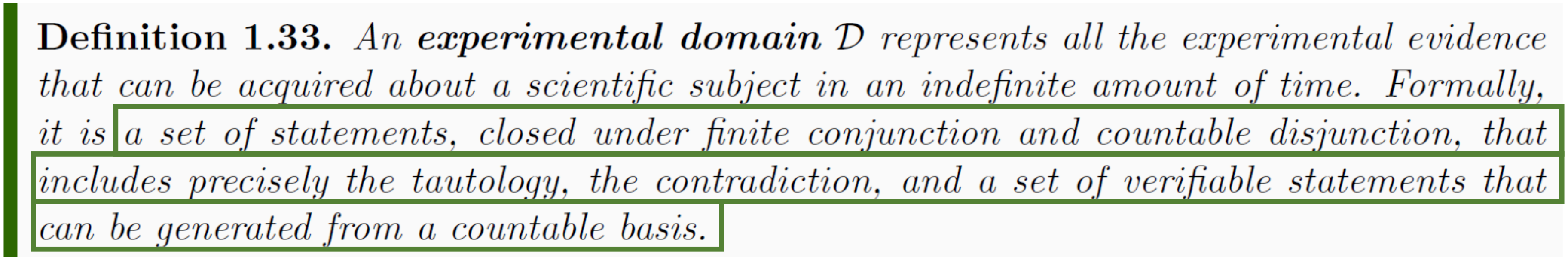
Basis



Definition 1.32. *Given a set* $\mathcal{D}$ *of verifiable statements,* $\mathcal{B} \subseteq \mathcal{D}$ *is a* **basis** *if the truth values of* $\mathcal{B}$ *are enough to deduce the truth values of the set. Formally, all elements of* $\mathcal{D}$ *can be generated from* $\mathcal{B}$ *using finite conjunction and countable disjunction.*

- What is the biggest basis we can experimentally test?
- A countable set
  - Even with unlimited time, we can only test countably many statements

Experimental domain



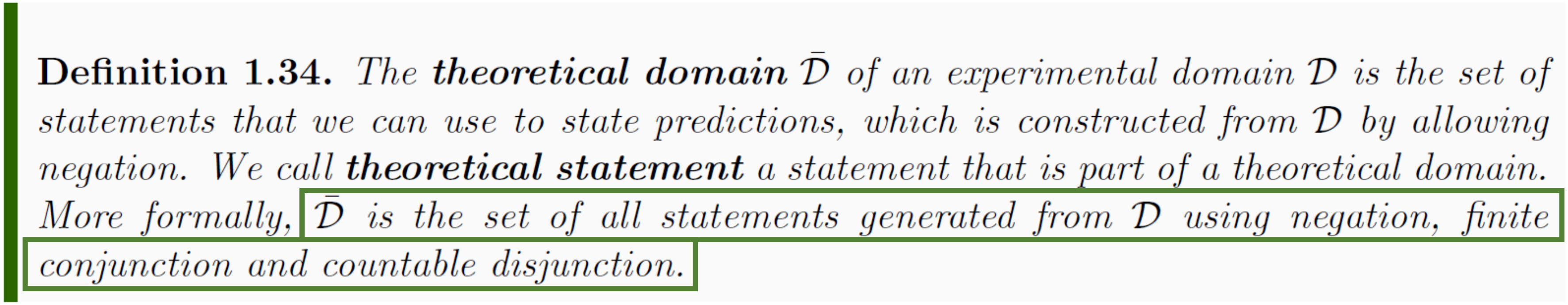
Definition 1.33. *An* ***experimental domain*** $\mathcal{D}$ *represents all the experimental evidence that can be acquired about a scientific subject in an indefinite amount of time. Formally, it is a set of statements, closed under finite conjunction and countable disjunction, that includes precisely the tautology, the contradiction, and a set of verifiable statements that can be generated from a countable basis.*

- This represents be biggest set of verifiable statements we can test
    - Any scientific theory, in the end, is equivalent to a set of verifiable statements, which forms at most an experimental domain

Predictions

- Science is also about making predictions, but not all predictions are directly verifiable
- For example, “there exists no extra-terrestrial life” predicts that the test for “there exists extra-terrestrial life” is never going to terminate
- While we cannot always experimentally confirm negation, it still makes sense logically as a possible way things could be

Theoretical domain



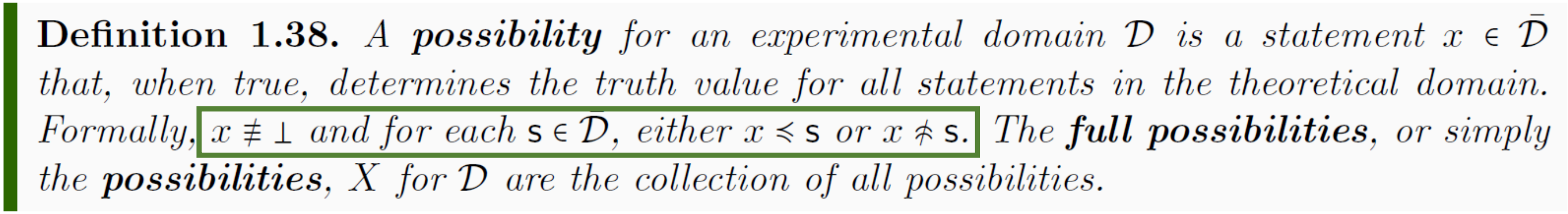
Definition 1.34. *The* ***theoretical domain*** $\bar{\mathcal{D}}$ *of an experimental domain* $\mathcal{D}$ *is the set of statements that we can use to state predictions, which is constructed from* $\mathcal{D}$ *by allowing negation. We call* ***theoretical statement*** *a statement that is part of a theoretical domain. More formally,* $\bar{\mathcal{D}}$ *is the set of all statements generated from* $\mathcal{D}$ *using negation, finite conjunction and countable disjunction.*

- This represents all statements that give meaningful predictions to (and only to) the verifiable statements in the domain

Possibilities for the domain

- Among all the predictions we look for the ones that give the full picture
  - For example, if we knew “This animal is a cat” to be true, we would also know that “This animal has whiskers” and “This animal is a mammal” are true while “This animal has feathers” is false

Possibilities for the domain



Definition 1.38. *A **possibility** for an experimental domain $\mathcal{D}$ is a statement $x \in \bar{\mathcal{D}}$ that, when true, determines the truth value for all statements in the theoretical domain. Formally, $x \not\equiv \bot$ and for each $\mathsf{s} \in \bar{\mathcal{D}}$, either $x \preccurlyeq \mathsf{s}$ or $x \not\curlyeqprec \mathsf{s}$. The **full possibilities**, or simply the **possibilities**, X for $\mathcal{D}$ are the collection of all possibilities.*

- A possibility, if true, gives a prediction for all theoretical and verifiable statements
- The set of possibilities corresponds to all the cases we can experimentally distinguish given the experimental domain

Start with a countable set of verifiable statements (the most we can verify experimentally)

Basis $\mathcal{B}$			
$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	…

Construct all verifiable statements that can be verified from the basis
(close under finite conjunction and countable disjunction)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$		
$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	…	$\boldsymbol{s}_1 = \boldsymbol{e}_1 \vee \boldsymbol{e}_2$	$\boldsymbol{s}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_3$	…

Construct all statements that give a prediction for those verifiable statements (close under negation as well)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	…	$\boldsymbol{s}_1 = \boldsymbol{e}_1 \vee \boldsymbol{e}_2$	$\boldsymbol{s}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_3$	…	$\overline{\boldsymbol{s}_1} = \boldsymbol{e}_1 \vee \neg \boldsymbol{e}_2$	$\overline{\boldsymbol{s}_2} = \neg \boldsymbol{e}_1$	…

Consider all truth assignments: it is sufficient to assign the basis

Basis $\mathcal{B}$				**Verifiable statements $\mathcal{D}_X$**			**Theoretical statements $\overline{\mathcal{D}_X}$**		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
T	T	F	…	T	F	…	T	F	…
…	…	…	…	…	…	…	…	…	…

Remove truth assignments that are not consistent with the possibilities of the statements

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Each consistent truth assignment is associated with a possibility of the domain

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

Each consistent truth assignment is associated with a possibility of the domain

Basis $\mathcal{B}$				**Verifiable statements $\mathcal{D}_X$**			**Theoretical statements $\overline{\mathcal{D}_X}$**			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	
F	F	F	…	F	F	…	T	T	…	**Possibilities X**
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	
…	…	…	…	…	…	…	…	…	…	

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

The role of logic (and math) in science is to capture what is consistent (i.e. the possibilities) and what is verifiable (i.e. the verifiable statements)

Natural topology for the possibilities

Each consistent truth assignment is associated with a possibility of the domain

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	
F	F	F	…	F	F	…	T	T	…	Possibilities X
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	
…	…	…	…	…	…	…	…	…	…	

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

Where is the topology?!?

Each statement can be expressed as the disjunction of a set of possibilities

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	
F	F	F	…	F	F	…	T	T	…	Possibilities X
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	
…	…	…	…	…	…	…	…	…	…	

$$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$$

For each consistent truth assignment we have a minterm that is true only in that case. Each minterm is a possibility of the domain.

$$U: \mathcal{D}_X \to 2^X$$

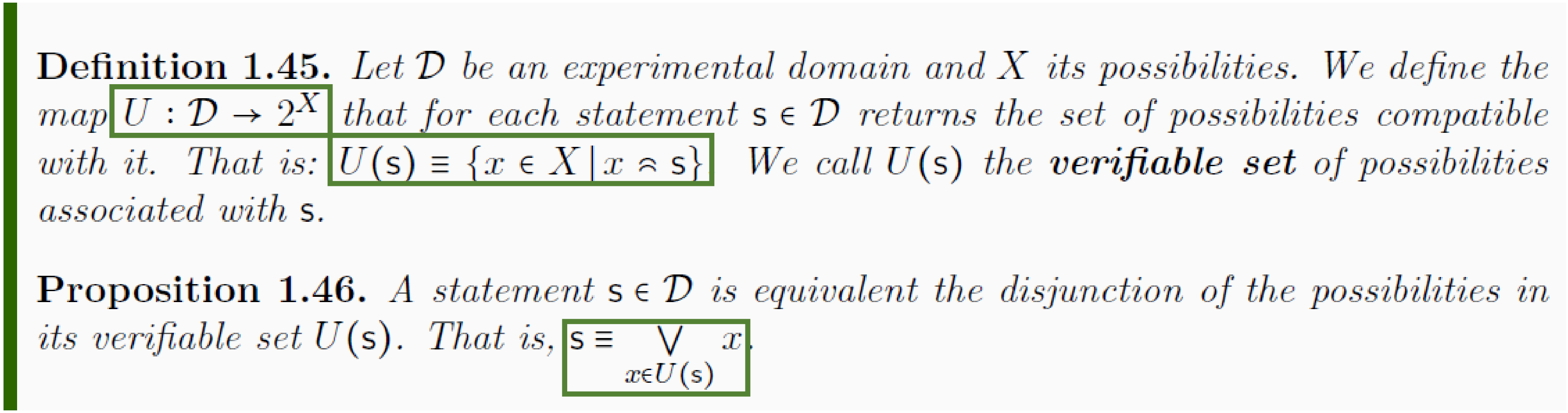
$$s = \bigvee_{x \in U(s)} x$$

$$A: \overline{\mathcal{D}_X} \to 2^X$$

$$\bar{s} = \bigvee_{x \in A(\bar{s})} x$$

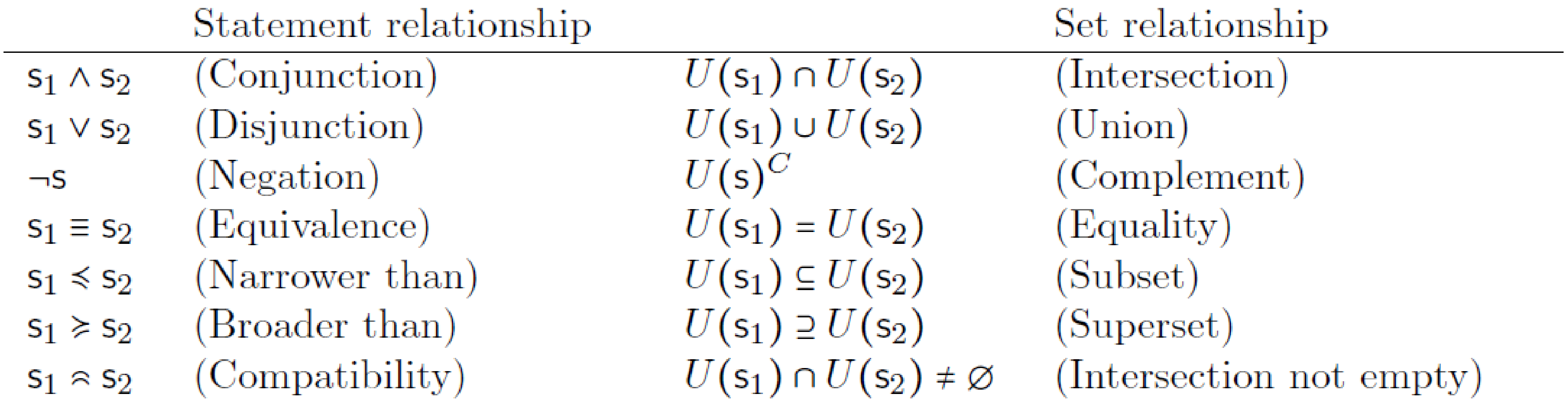
Each statement can be expressed in the disjunctive normal form (disjunction of minterms, OR of ANDs): a disjunction of possibilities.

We can express each verifiable statement in terms of a set of possibilities. Relationships between statements become relationships between sets.



Definition 1.45. *Let $\mathcal{D}$ be an experimental domain and X its possibilities. We define the map* $\boxed{U : \mathcal{D} \to 2^X}$ *that for each statement $\mathsf{s} \in \mathcal{D}$ returns the set of possibilities compatible with it. That is:* $\boxed{U(\mathsf{s}) \equiv \{x \in X \,|\, x \mathrel{\overset{\frown}{-}} \mathsf{s}\}}$ *We call $U(\mathsf{s})$ the* ***verifiable set*** *of possibilities associated with s.*

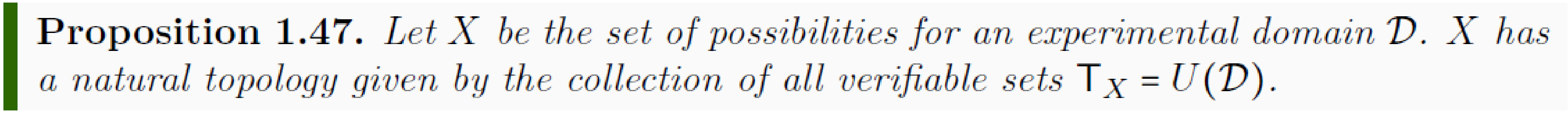
Proposition 1.46. *A statement $\mathsf{s} \in \mathcal{D}$ is equivalent the disjunction of the possibilities in its verifiable set $U(\mathsf{s})$. That is,* $\boxed{\mathsf{s} \equiv \bigvee_{x \in U(\mathsf{s})} x}$.



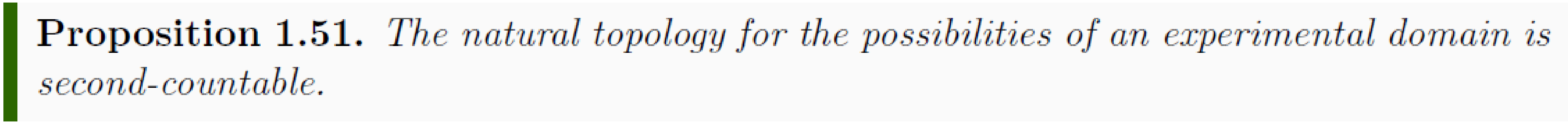
	Statement relationship		Set relationship
$\mathsf{s}_1 \wedge \mathsf{s}_2$	(Conjunction)	$U(\mathsf{s}_1) \cap U(\mathsf{s}_2)$	(Intersection)
$\mathsf{s}_1 \vee \mathsf{s}_2$	(Disjunction)	$U(\mathsf{s}_1) \cup U(\mathsf{s}_2)$	(Union)
$\neg \mathsf{s}$	(Negation)	$U(\mathsf{s})^C$	(Complement)
$\mathsf{s}_1 \equiv \mathsf{s}_2$	(Equivalence)	$U(\mathsf{s}_1) = U(\mathsf{s}_2)$	(Equality)
$\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$	(Narrower than)	$U(\mathsf{s}_1) \subseteq U(\mathsf{s}_2)$	(Subset)
$\mathsf{s}_1 \succcurlyeq \mathsf{s}_2$	(Broader than)	$U(\mathsf{s}_1) \supseteq U(\mathsf{s}_2)$	(Superset)
$\mathsf{s}_1 \mathrel{\overset{\frown}{-}} \mathsf{s}_2$	(Compatibility)	$U(\mathsf{s}_1) \cap U(\mathsf{s}_2) \neq \varnothing$	(Intersection not empty)

Table 1.4: Correspondence between statement operators and set operators.

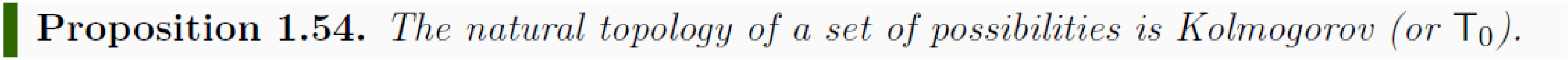
Closure of the experimental domain under finite conjunction and countable disjunction means the set of verifiable sets is closed under finite intersection and countable union.



Proposition 1.47. *Let X be the set of possibilities for an experimental domain $\mathcal{D}$. X has a natural topology given by the collection of all verifiable sets* $\mathsf{T}_X = U(\mathcal{D})$.



Proposition 1.51. *The natural topology for the possibilities of an experimental domain is second-countable.*



Proposition 1.54. *The natural topology of a set of possibilities is Kolmogorov (or* T_0*).*

Similarly, the theoretical domain provides a natural ***σ***-algebra for the possibilities (i.e. the Borel algebra)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	
F	F	F	…	F	F	…	T	T	…	Possibilities X
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	
…	…	…	…	…	…	…	…	…	…	

The lines of the truth table (i.e. the consistent truth assignments) are the points

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

Each column is a set (i.e. the set of possibilities that are true in that column)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

The experimental domain is the topology (i.e. each verifiable statement is an open set)

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

The basis of the experimental domain is a sub-basis of the topology

Basis $\mathcal{B}$				Verifiable statements $\mathcal{D}_X$			Theoretical statements $\overline{\mathcal{D}_X}$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities X

The theoretical domain is the σ-algebra (i.e. each theoretical statement is a Borel set)

Examples

- Discrete topology means every statement is decidable
    - All tests terminate and we can experimentally tell whether any statement is true or false
- Standard topology on the real numbers means we can experimentally measure with arbitrarily small, but always finite, precision
- A Hausdorff topology means that each possibility can be seen as a limit of a sequence of verifiable statements
    - The statements become narrower and narrower

General results

- Every set of experimentally distinguishable objects will be a Kolmogorov second-countable topological space
  - Each point is the finest description the experimental technique allows
  - Each open set is a partial description that can be verified experimentally
- This is true no matter which branch of science, what experimental technique or how clever we are
  - All we are doing is keeping track of the consistent truth assignments in that gigantic truth table
- Consequence: we cannot experimentally distinguish elements of sets with cardinality greater than the continuum

Peculiarities of this framework

- In topology the empty set is the same in all cases but the full set is not. In these topologies, the full set corresponds to the tautology, so it is the "same" in all cases as well.
- In topology, points and open sets are different objects. In these topologies they are all statements.
- In topology, first you must define the points and then you define the open sets. In these topologies you only define the open sets (i.e. the verifiable statements) and the points (i.e. the possibilities) are generated.

Conclusion

- We believe this framework successfully formalizes the fundamental structure for experimental science, shedding light as to what role mathematical structures, such as topologies, play
- We are working to extend this framework to other areas of math and science
  - For example, states are possibilities of some experimental domain, deterministic and reversible evolution is equivalence between past and future domains which will correspond to an isomorphism in the category

Acknowledgements and further information

- We'd like to thank all the individuals in this community for the warm welcome, interest and insights, all of which helped push this project further
  - Mathew Timm and John Mayer in particular for the help throughout the year

- For further information and details, see the draft of our open access book:
  - http://assumptionsofphysics.org/book/