From physical assumptions to classical Hamiltonian and Lagrangian particle mechanics

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Introduction

- The purpose of this talk is to answer a (seemingly) simple type of question: what are Hamiltonian systems?
  - Or Lagrangian? Or classical (vs quantum)?
- Not all systems are Hamiltonian (or Lagrangian, or classical, or quantum)
- If I have a system in front of me, how can I tell whether it's a Hamiltonian system?
  - When is the state space T^*Q? Or a complex vector space?
- The problem is that Hamiltonian/Lagrangian classical/quantum mechanics start by setting the mathematical framework
  - Unlike Newtonian mechanics, thermodynamics and special relativity that start from physical laws or postulates
- We are left with the mathematical definition: a Hamiltonian system is one described by Hamiltonian framework
  - Many appear to be satisfied by this answer
  - To me, it begs the question

Introduction

- To answer the question, we need a way to physically motivate Hamiltonian (and Lagrangian) mechanics
- Start from physical assumptions that characterize the system under study
  – Hopefully ideas that are already familiar
- Show that those assumptions lead necessarily to the known mathematical frameworks
  – in the most direct way possible
  – not to some new theory
- Gain new insights
  – provide a math-physics dictionary
- The objective of this talk is to show that such a program is indeed possible and worthwhile

Introduction

- We limit ourselves here to classical Hamiltonian/Lagrangian particle mechanics
  - no relativity, quantum or field theory
- Go through highlights of the derivation and give physical intuition for them
  - Elements from different fields are used: vector spaces, topology, measure theory, differential/ symplectic/Riemannian geometry
- Let's start with an overview of the assumptions and what they lead to
  - The rest of the talk is more details and insight to see how it all works

Overview

Only 3 assumptions:

- Determinism and reversibility (given the initial state we can identify the final state and vice versa)
  - Dynamical system
  - State space is a topological space, evolution map is a self-homeomorphism
- Infinitesimal reducibility (describing the state of the whole is equivalent to describing the state of its infinitesimal parts)
  - Hamiltonian mechanics for infinitesimal parts
  - State of the whole is a distribution over the state space for the infinitesimal parts (i.e. phase space)
- Kinematic equivalence (studying the motion of the infinitesimal parts is equivalent to studying the state evolution)
  - Lagrangian mechanics for infinitesimal parts
  - Dynamics is the one for massive particles under scalar/vector potential forces

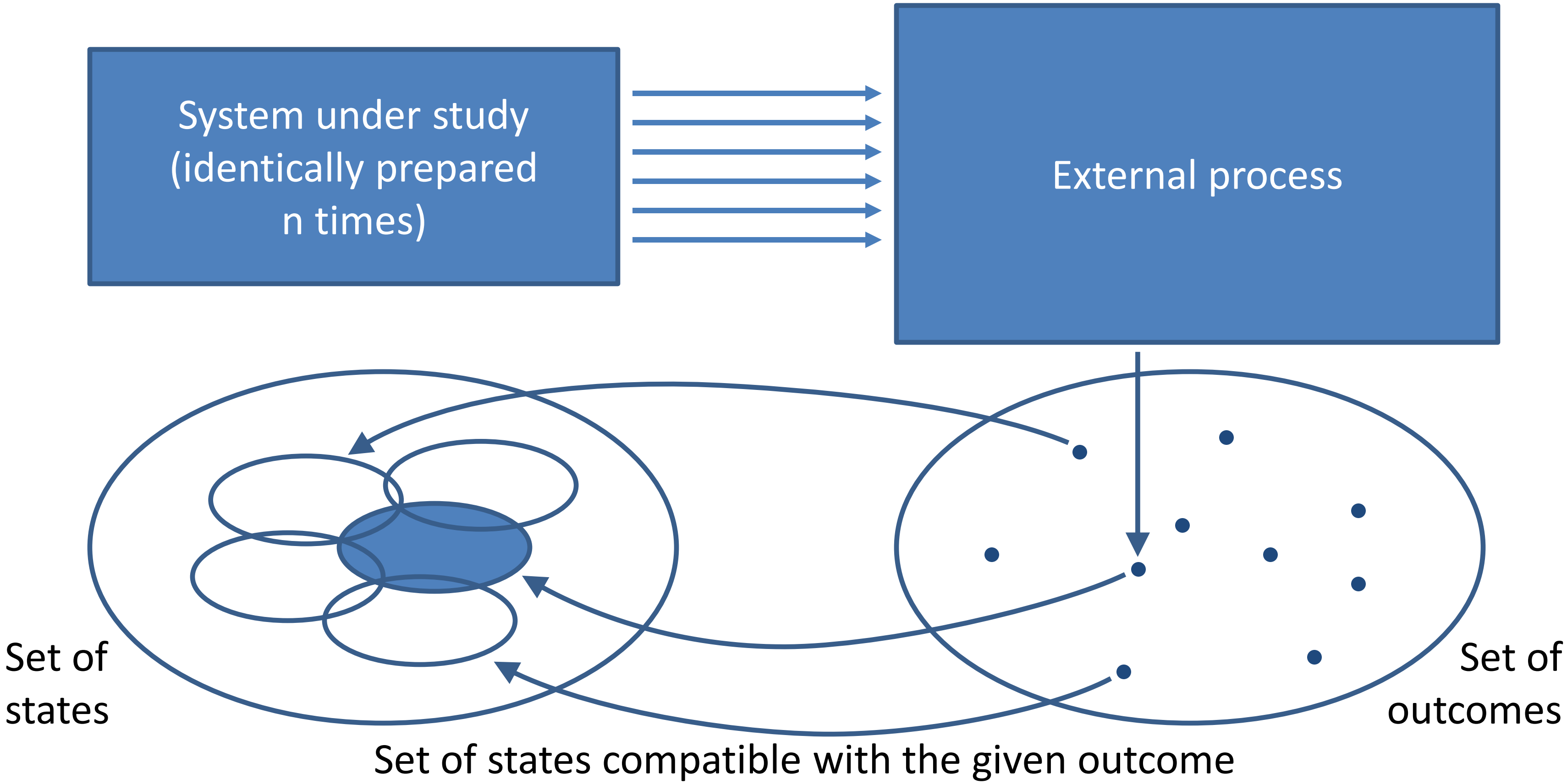
Assumption I: Determinism and reversibility

The system undergoes deterministic and reversible time evolution: given the initial state, we can identify the final state; given the final state, we can reconstruct the initial state

Physical distinguishability and topological spaces

- States are physical configurations of the system at a given time
- => states must be physically distinguishable
  - There must exist an external process (i.e. that involves the environment) that is potentially able to tell two states apart
- How do we mathematically capture physical distinguishability?

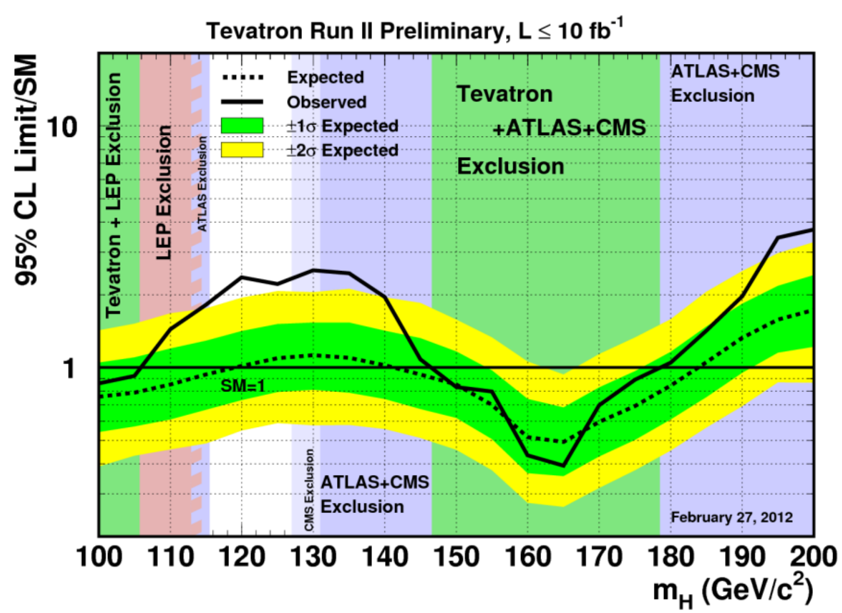
Physical distinguishability and topological spaces



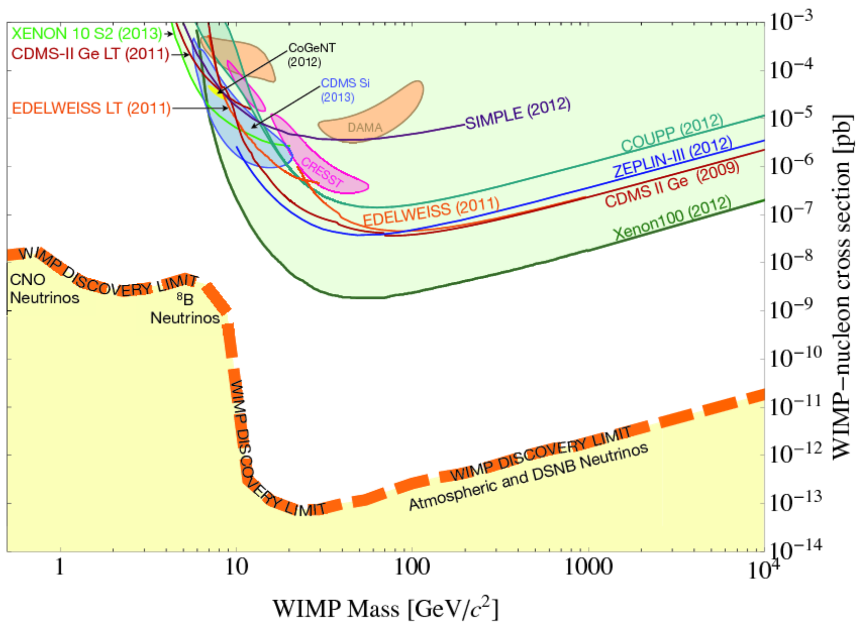
The collection of all sets of states compatible with all potential outcomes of all potential processes form a Hausdorff and second countable topology

Physical distinguishability and topological spaces

- Use of sets and set operations is apparent in exclusion plots



Range of Higgs masses excluded by different experiments



Range of WIMP (dark matter) mass/cross section excluded by different experiments

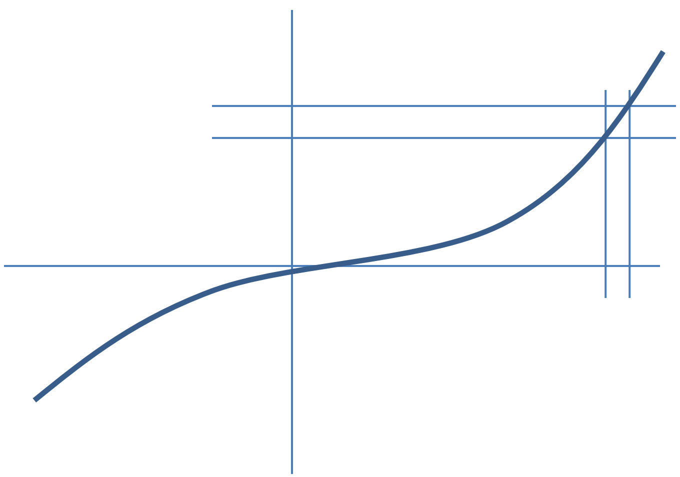
Mapping physical objects and continuous functions

- Physical distinguishability -> topological space
- We now want to associate initial and final states through the evolution map
- The map has to be compatible with physical distinguishability: distinguishing final (or initial) states is also distinguishing initial (or final states)
- How do we mathematically capture maps between physically distinguishable objects?
  - If U is a distinguishable set of final states, then $f^{-1}(U)$ is a distinguishable set of initial states: $f^{-1}(U)$ is in the topology
- A map between distinguishable objects is a continuous map by definition

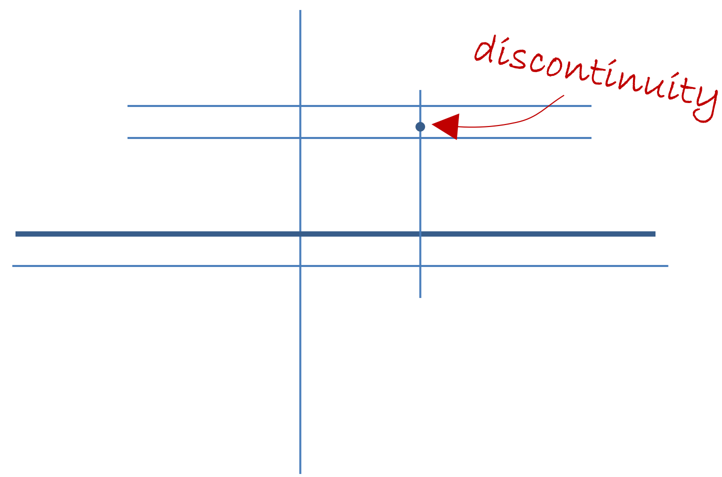
topological sense

Mapping physical objects and continuous functions

Continuous functions are fundamental in physics as they preserve what constitutes a set of states that can be associated to the outcome of a physical process. Standard topology for $\mathbb{R}^n$ excludes infinite precision knowledge of one continuous quantity (sets with one point) and implies topological continuity <=> analytical continuity



Finite precision knowledge of one continuous quantity corresponds to finite precision knowledge of the other



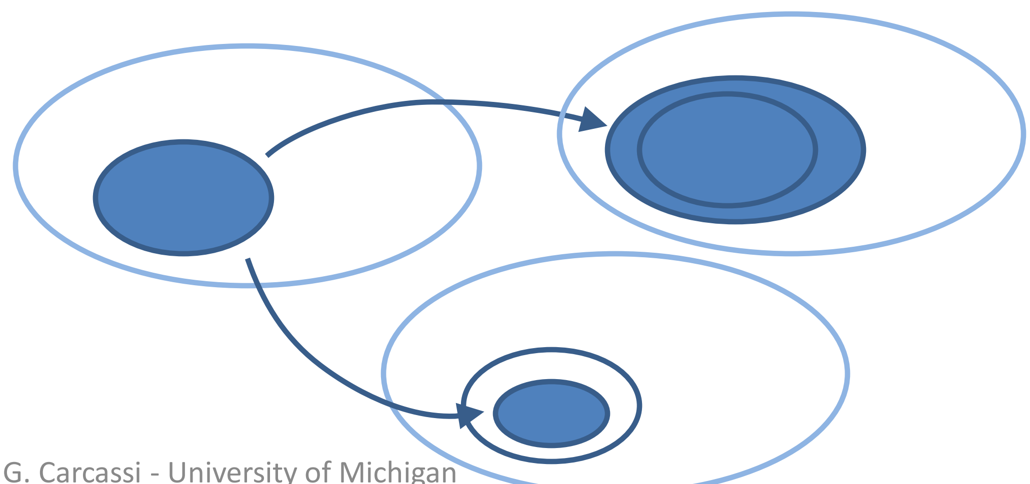
Finite precision knowledge of one continuous quantity does not always correspond to finite precision knowledge of the other

Deterministic and reversible maps and homeomorphisms

- Physical distinguishability -> topological space
- Map between physically distinguishable objects -> continuous map
- Deterministic and reversible evolution is a bijective map
- A bijective continuous map is a homeomorphism: deterministic and reversible evolution is a self-homeomorphism
  - A map onto the same space that preserves what is physically distinguishable (i.e. the topology)

Not all self-homeomorphisms are deterministic and reversible maps

Discrete topology	Standard (continuous) topology
Map is a self-homeomorphism $f: \mathcal{S} \leftrightarrow \mathcal{S}$	*Map is a self-homeomorphism* $f: \mathcal{S} \leftrightarrow \mathcal{S}$
Cardinality of states is conserved $\#(U) = \#\big(f(U)\big)$	?
Independent variables remain independent $\mathscr{s}: \mathcal{Q}_1 \times \mathcal{Q}_2 \leftrightarrow \mathcal{S}$ $\#\big(\mathscr{s}(U,V)\big) = \#(U)\#(V) = \#\big(f(U)\big)\#\big(f(V)\big)$	?



A self-homeomorphism that maps a set to a proper superset or subset is clearly not a deterministic or reversible process!

Need to keep track of the cardinality of states and possibilities. We need a measure.

measure theory sense

From assumption I

- Physical distinguishability -> topological space
- Map between physically distinguishable objects -> continuous map
- Deterministic and reversible evolution -> self-homeomorphism
  - Not all self-homeomorphisms -> need a measure to count states

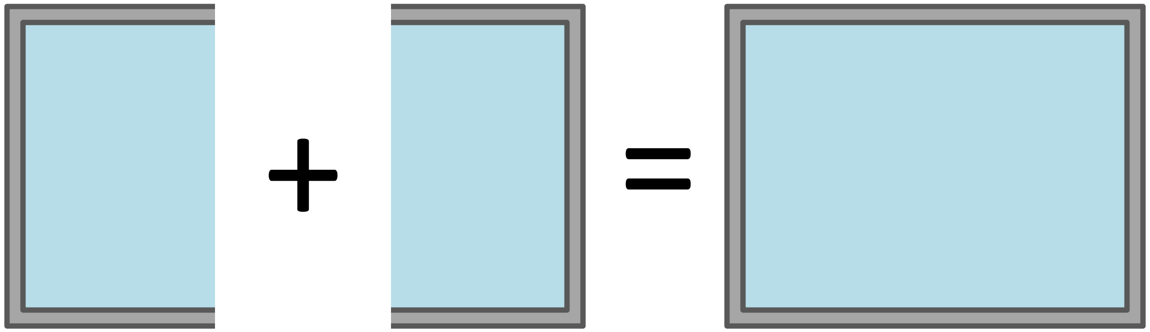
Assumption II: infinitesimal reducibility (classical material)

The system is composed of a homogeneous infinitesimally reducible material. Each part undergoes deterministic and reversible evolution.

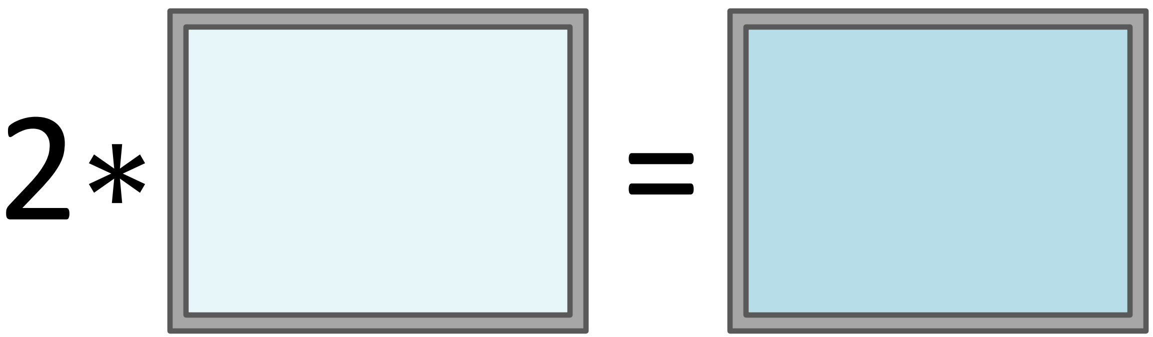
Classical homogeneous material and real vector spaces

- A "classical material" is:
  - Decomposable: can be divided into smaller amounts
  - Infinitesimally so: can keep dividing arbitrarily ("particles" are the limits of such division)
  - Reducible: giving state of whole equivalent to giving state of parts
  - Homogeneous: state space of each part is the same
- A classical fluid is an example
- How do we mathematically capture the structure of such material?

Classical homogeneous material and real vector spaces



Homogeneous and decomposable => law of composition $+ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$



Infinitesimally decomposable => increase/decrease the amount of material by a real number

These two operations give the structure of a vector space

Classical homogeneous material and function spaces

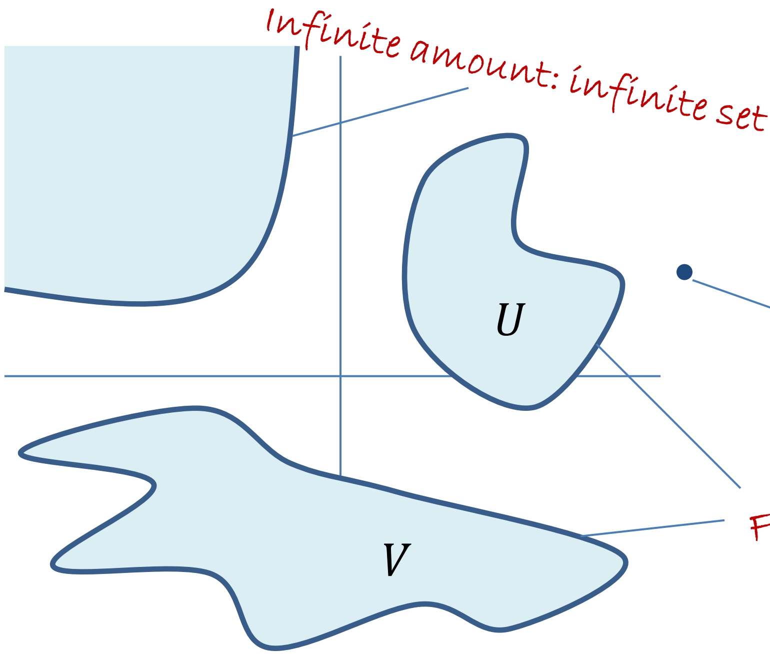
- Infinitesimally reducible homogeneous material -> real vector space
- We want to characterize the material in terms of the state of each particle, the infinitesimal part that is the limit of the process of recursive subdivision
- Let $\mathcal{S}$ be the state space for particles. What can we tell about $\mathcal{C}$ the state space for the composite system?
  - For each composite state $c \in \mathcal{C}$ we have a distribution $\rho_c : \mathcal{S} \to \mathbb{R}$ that tells us the amount of material (or density) for that particle state
  - The function is continuous as particle states and amounts of material are physically distinguishable
- The state space $\mathcal{C}$ is isomorphic as a real vector space to a subspace of the real valued continuous functions over $\mathcal{S}$, the state space for particles: $\mathcal{C} \cong G \subseteq C(\mathcal{S})$

Integration and measure

- Infinitesimally reducible homogeneous material -> real vector space
- Composite state -> continuous distributions over particle state space
- In practice, we want amounts of material for a set $U \subseteq \mathcal{S}$ of the particle states
  - We don't measure the density of a fluid at a specific position and momentum
  - We measure how much material is in this volume with this momentum range
- How do we mathematically capture the relationship between the distribution ρ_c and the amount of material found in $U \subseteq \mathcal{S}$?
  - Given a region $U \subseteq \mathcal{S}$ we have a functional that given a distribution returns the amount of material: $\Lambda_U : C(\mathcal{S}) \to \mathbb{R}$
  - By Riesz representation theorem, exists **unique** Borel (i.e. compatible with the topology) measure μ such that $\Lambda_U(\rho_c) = \int_U \rho_c d\mu$
    A way to count states!
  - Only measure finite amounts of material: the distributions are integrable
- Particle state space $\mathcal{S}$ is a measure space. Composite state space $\mathcal{C}$ is isomorphic to subspace of Lebesgue integrable functions: $\mathcal{C} \cong G \subseteq L^1(\mathcal{S})$

Integration and measure

Because we have densities and integration we can assign a measure (a “size”) to each set of states



Take a set of particle states. Fill it with a uniform density. Look at the total amount of material.

Zero amount: zero measure set

Finite amount: finite set

V double amount of U:

V is double the size

Analogous to using water to measure the capacity of a container

Invariant densities and differentiability

- …
- Finite amounts of material -> Measure and integrability
- For discrete states, $\Lambda_U(\rho_c) = \sum \rho_c(q)$ with q a discrete state variable
- For continuous states, expect $\Lambda_U(\rho_c) = \int \rho_c(q)dq$ with q a continuous state variable
  - This does not work! Changing variables $\rho_c\left(q^j\right) = \rho_c\left(q^i\right)\left|\frac{\partial q^j}{\partial q^i}\right|$ the density changes: densities in general are a function of **coordinates**
  - Our distributions are a function of the **state** $\rho_c(\mathscr{s})$: they are invariant densities
- We need to make sure we can have invariant densities
- Jacobian must be well defined => differentiability
  - $\mathcal{S}$ is a differentiable manifold
  - $\mathcal{C}$ isomorphic to the space of integrable differentiable functions: $\mathcal{C} \cong C^1(\mathcal{S}) \cap L^1(\mathcal{S})$
  - Note: $C^1(\mathcal{S}) \cap L^1(\mathcal{S})$ is not a complete metric space

Invariant densities and symplectic manifolds

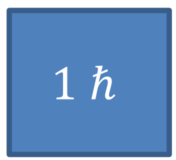
- …
- Finite amounts of material -> Measure and integrability
- Invariant distributions -> Differentiability
- Jacobian must be unitary
  - Changing physical units should not change our distribution
  - Let $\mathcal{Q}$ be the manifold that defines the system of units, invariant densities are defined on $\mathrm{T}^*\mathcal{Q}$
  - Canonical two-form $\omega = dq^i \wedge \hbar dk_i = dq^i \wedge dp_i$ allows "counting" the number of possibilities defined on each independent d.o.f.
  - $\mathcal{S}$ is a symplectic manifold formed by $\mathrm{T}^*\mathcal{Q}$ and the symplectic form ω

Invariant densities and symplectic manifolds

$$\omega_{ab} = \begin{vmatrix} 0 & I_n \\ -I_n & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \omega = dq^i \wedge \hbar dk_i = dq^i \wedge dp_i$$

Wedge product (area) within a d.o.f.

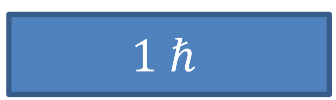
$\Delta k = 1\ m^{-1}$



$\Delta q = 1\ m$

$\bar{q} = 100\ cm/m\ q$

$\Delta\bar{k} = 0.01\ cm^{-1}$

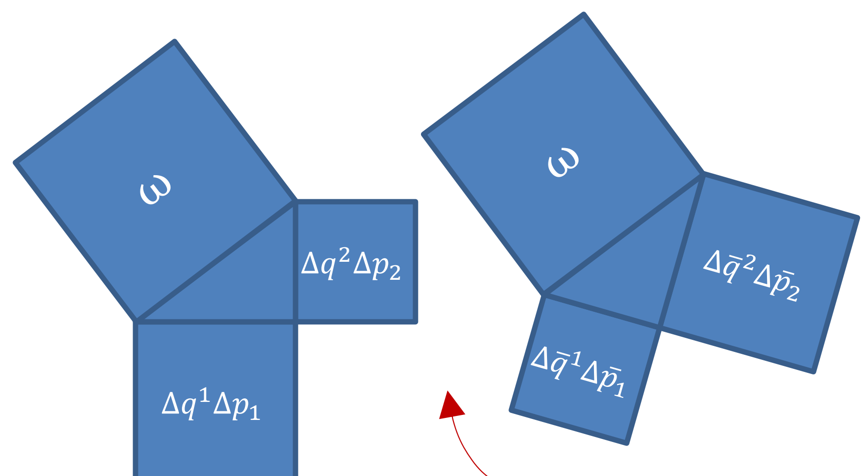


$\Delta\bar{q} = 100\ cm$

Number of possibilities $= \hbar\Delta q\Delta k$
is invariant

Scalar product across independent d.o.f.

Number of states = $\prod \hbar\Delta q^i \Delta k_i$
=> independent d.o.f are orthogonal
=> ω sum of projections



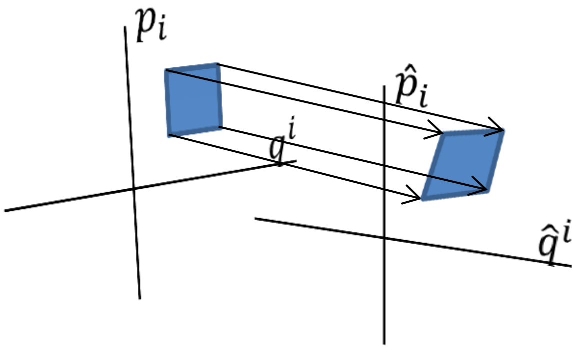
2D rendition of 4D space

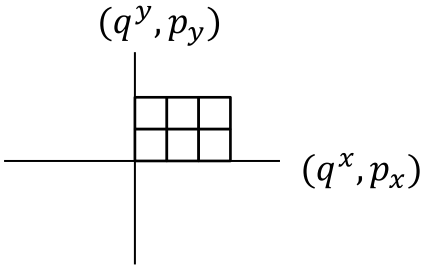
Deterministic and reversible maps and symplectomorphisms

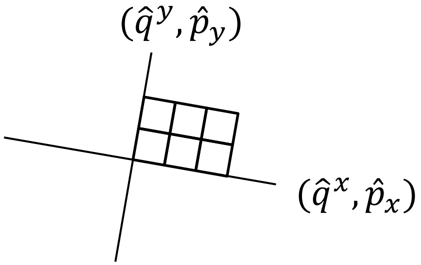
- …
- Invariant distributions -> Differentiability; symplectic (metric) space (T^*Q, ω)
- Now characterize deterministic and reversible evolution
  - Each particle state mapped to one and only one particle state, density values transported to new states, cardinality of possibilities and states preserved
  - Deterministic and reversible time evolution is a symplectomorphism (an isometry): metric ω is preserved
  - Infinitesimal symplectomorphism admits potential H such that $d_t q^i = \partial_{p_i} H \qquad d_t p_i = \partial_{q^i} H$
- Deterministic and reversible time evolution for a particle of classical material follows Hamilton's equations

Deterministic and reversible maps and symplectomorphisms

Possibilities (area) within a d.o.f. conserved







Independent d.o.f. remain independent (orthogonal)

Total number of states (volume) is conserved

All that Hamiltonian mechanics does is to conserve the number of states and possibilities

It's the continuous equivalent of "3 possibilities for x times 2 possibilities for y gives 6 total states"

From assumption II

- Infinitesimally reducible homogeneous material -> real vector space
- Composite state -> continuous distributions over particle state space
- Finite amounts of material -> Measure and integrability
- Invariant distributions -> Differentiability; symplectic (metric) space (T^*Q, ω)
- Deterministic and reversible map -> symplectomorphism (isometry) and Hamilton's equations

Assumption III:
kinematic equivalence

Studying the motion (kinematics) of the system is equivalent to studying its state evolution (dynamics)

Kinematic equivalence and Lagrangian systems

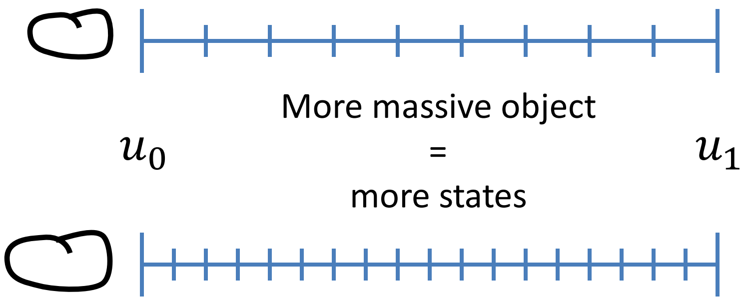
- Each particle has no relevant internal dynamics: motion tells us everything
  - Falling rock has no relevant internal dynamics
  - A helicopter has relevant internal dynamics
- How do we capture mathematically a system where kinematic equivalence applies?
  - One-to-one relationship (homeomorphism) between state variables (q, p) and initial conditions $(x, \dot{x})$. State will be identified by only position and velocity
  - Set $x^i = q^i$. We have $\dot{x}^i = u^i(q^i, p_i) = d_t q^i = \partial_{p_i} H$. u^i is invertible, monotonic in p_i. H is convex, admits a Lagrangian
- A Hamiltonian system where kinematic equivalence applies is a Lagrangian system

Kinematic equivalence, inertial mass and conservative forces

- Can convert between state variables and initial conditions -> Convex Hamiltonian; existence of Lagrangian
- Distribution must be expressible in terms of initial conditions
  - Use initial conditions to count states. ω must be proportional to an invariant bilinear function: $\omega = dq^i \wedge dp_i \propto g(dx^i, d\dot{x}^i)$
  - $dq^i \wedge dp_i = m dx^i g_{ij} du^j = m dx^i du_i$. $\quad dp_i = m du_i$.
    $p_i = m u_i + A_i(x^j)$
  - $\partial_{p_i} H = d_t q^i = u^i$
    $H = \frac{1}{2m}(p_i - A_i) g^{ij} (p_j - A_j) + V(q^k)$
- The Hamiltonian limited to particle under scalar and vector potential forces

Kinematic equivalence, inertial mass and conservative forces

What is inertial mass? It's the constant that tells us how many possibilities (i.e. possible states) there are for a unit range of velocity.



Why is a more massive body more difficult to accelerate? Because for the same change in velocity it has to go through more states.

What are conservative forces? The ones that conserve the number of states (deterministic and reversible forces).

From assumption III

- Can convert between state variables and initial conditions -> Convex Hamiltonian; existence of Lagrangian
- Invariant densities on initial conditions -> linear relationship between conjugate momentum and velocity; inertial mass; scalar/vector potential forces

Conclusion

- Physically motivating Hamiltonian and Lagrangian mechanics is possible and offers insights
- Classical Hamiltonian and Lagrangian mechanics founded upon three physical assumptions, which give the definition of states and their laws of evolution
  – Only distributions give meaningful account for use of cotangent bundle/Hamilton's equations
  – Kinematic equivalence gives potential forces, tells why second order equations of motion, what is inertial mass
- Extending this approach to other fundamental theories may give insights useful to address open problems