Non-Additive Measures and Generalized Probabilities in Physics
University of Michigan, 7–8 July 2026

Unifying classical & quantum probability — it's mainly about symmetries

Gert de Cooman

Foundations Lab
for imprecise probabilities

Acceptability – Desirability – Accept & Reject

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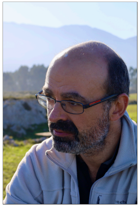

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Accept & reject statement-based uncertainty models

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ABSTRACT

We develop a framework for modelling and reasoning with uncertainty based on accept and reject statements about gambles. It generalises the frameworks found in the literature based on statements of acceptability, desirability, or favourability and clarifies their relative position. Next to the statement-based formulation, we also provide a translation in terms of preference relations, discuss—as a bridge to existing frameworks—a number of simplified variants, and show the relationship with prevision-based uncertainty models. We furthermore provide an application to modelling symmetry judgements.


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ACCEPT & REJECT STATEMENTS IN NORMED REAL VECTOR SPACES

Statement models: the basics

EXAMPLES

- gambles $f\colon \mathscr{X} \to \mathbb{R}$ on some set $\mathscr{X}$
- indifference classes of gambles on some set $\mathscr{X}$
- Hermitian operators on a complex Hilbert space

Options and statements

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

Statement models: the basics

Options and statements

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

A statement model $M = \langle A\,;R \rangle \subseteq \mathscr{U} \times \mathscr{U}$ represents Your preferences between options:

- A is the set of those options that You accept
- R is the set of those options that You reject

Statement models: the basics

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- R is the set of those options that You reject

Meaning

- $u \in A$: You want to get the option u
- $u \in R$: You don't want to get the option u
- $u \in -A$: You want to give away the option u
- $u \in -R$: You don't want to give away the option u

Statement models: the basics

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Other relevant option sets

– $A \cap R$ is Your set of conflicted options

– $\mathscr{U} \setminus (A \cup R)$ is Your set of unresolved options

Statement models: the basics

Options and statements

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Other relevant option sets

- $A \cap R$ is Your set of conflicted options
- $\mathscr{U} \setminus (A \cup R)$ is Your set of unresolved options

- $A \cap -R$ is Your set of desirable options
- $A \cap -A$ is Your set of indifferent options

Statement models: the basics

Options and statements

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

A statement model $M = \langle A; R\rangle \subseteq \mathscr{U} \times \mathscr{U}$ represents Your preferences between options:

- A is the set of those options that You accept
- R is the set of those options that You reject

Option relations

$u \trianglerighteq v \Leftrightarrow u - v \in A$	weak preference
$u \blacktriangleright v \Leftrightarrow u - v \in D$	strict preference
$u \equiv v \Leftrightarrow u - v \in I$	indifference

Statement models: rationality criteria

A statement model M satisfies:

M1. $A \cap R = \emptyset$

Rationality criteria: No Confusion

You mustn't accept and reject the same option:

$$A \cap R = \emptyset.$$

Statement models: rationality criteria

A statement model M satisfies:

M1. $A \cap R = \emptyset$

M2. A is a convex cone

Rationality criteria: Deductive Closure

You ought to accept any positive linear combination of options You accept:

$$\mathrm{posi}(A) = A, \text{ so } A \text{ is a convex cone.}$$

Statement models: rationality criteria

A statement model M satisfies:

M1. $A \cap R = \emptyset$

M2. A is a convex cone

M4. $0 \in A$

Rationality criteria: Indifference to Status Quo

You ought to accept the Status Quo:

$$0 \in A,$$

and therefore be indifferent to it, because then also

$$-0 = 0 \in A.$$

Statement models: rationality criteria

A statement model M satisfies:

M1. $A \cap R = \emptyset$

M2. A is a convex cone

M3. $\mathrm{sh}(R) - A \subseteq R$

M4. $0 \in A$

Rationality criteria: No Limbo

Consider any unresolved option u, then there are two possibilities:

- reject u: ALWAYS POSSIBLE
- accept u: IMPOSSIBLE if $\mathrm{posi}(A \cup \{u\}) \cap R \neq \emptyset$

You ought to reject all unresolved options that couldn't be accepted without creating conflict:

$$\mathrm{sh}(R) - A \subseteq R.$$

Statement models: rationality criteria

A statement model M satisfies:

M1. $A \cap R = \emptyset$

M2. A is a convex cone

M3. $\mathrm{sh}(R) - A \subseteq R$

M4. $0 \in A$

M5. $D = -R$

Rationality criteria: Rejection Restriction

You should only reject options that You're willing to give away:

$$R \subseteq -A$$

or equivalently

$$D = A \cap -R = -R.$$

Hence, the model

$$M = \langle A\,; R \rangle = \langle A\,; -D \rangle$$

is completely determined by the accepted and desirable options.

Statement models: rationality criteria

A statement model M satisfies:

M1. $A \cap R = \emptyset$

M2. A is a convex cone

M3. $\mathrm{sh}(R) - A \subseteq R$

M4. $0 \in A$

M5. $D = -R$

Rationality criteria: Summary

A couple $M = \langle A\,;\, -D \rangle$ is an Accept-Desirability model, or AD-model, so $M \in \mathbf{M}_{\text{AD}}$, if:

ADM1. $0 \in A$

ADM2. $0 \notin D$

ADM3. $D \subseteq A$

ADM4. A, D are convex cones

ADM5. $D + A \subseteq D$

These conditions are closed under arbitrary non-empty (component-wise) intersections:

There is a conservative inference mechanism that takes any assessment to the smallest AD-model that includes it (componentwise), if it exists.

Statement models: backgrounds

$M = \langle A\,;-D \rangle$ is AD-model if:

ADM1. $0 \in A$

ADM2. $0 \notin D$

ADM3. $D \subseteq A$

ADM4. A, D are convex cones

ADM5. $D + A \subseteq D$

A background AD-model $V = \langle V_{\trianglerighteq}\,; V_{\blacktriangleright} \rangle \in \mathbf{M}_{\mathrm{AD}}$ exhibits the weak and strict preferences and indifferences that are always there, even before You've started thinking about the decision problem.

The background AD-model $V \in \mathbf{M}_{\mathrm{AD}}$ represents complete ignorance.

We also call it the vacuous AD-model.

Statement models: backgrounds

$M = \langle A; -D \rangle$ is AD-model if:

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We also call it the vacuous AD-model.

A couple $M = \langle A; -D \rangle$ is an AD-model that respects the AD-background $V \in \mathbf{M}_{\mathrm{AD}}$, so $M \in \mathbf{M}_{\mathrm{AD}}(V)$, if:

ADM1. $V \subseteq M$, so $V_{\trianglerighteq} \subseteq A$ and $V_{\blacktriangleright} \subseteq D$

ADM2. $0 \notin D$

ADM3. $D \subseteq A$

ADM4. A, D are convex cones

ADM5. $D + A \subseteq D$

“It’s probability theory, Jim, but not as we know it.”

Archimedean models


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Archimedean models: the basics

Properties of a norm $\|\bullet\|_{\mathscr{U}}$:

(i) $\|u\|_{\mathscr{U}} \geq 0$;

(ii) $\|u\|_{\mathscr{U}} = 0 \Leftrightarrow u = 0$;

(iii) $\|u+v\|_{\mathscr{U}} \leq \|u\|_{\mathscr{U}} + \|v\|_{\mathscr{U}}$;

(iv) $\|\lambda u\|_{\mathscr{U}} = |\lambda| \|u\|_{\mathscr{U}}$.

Structural assumptions

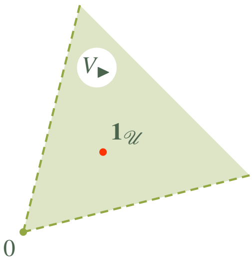
The option space $\mathscr{U}$, provided with a norm $\|\bullet\|_{\mathscr{U}}$, is a normed linear space.

The norm $\|\bullet\|_{\mathscr{U}}$ induces a metric topology on $\mathscr{U}$, with interior operator Int and closure operator Cl.

A real functional $\Gamma\colon \mathscr{U} \to \mathbb{R}$ is bounded if its operator norm $\|\Gamma\|_{\mathscr{U}^\circ}$ is:

$$\|\Gamma\|_{\mathscr{U}^\circ} := \sup_{u \in \mathscr{U} \setminus \{0\}} \frac{|\Gamma(u)|}{\|u\|_{\mathscr{U}}} < +\infty.$$

Archimedean models: the basics



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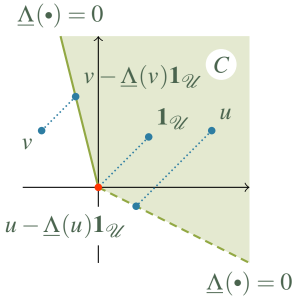
Take as unit element $\mathbf{1}_{\mathscr{U}}$ any element in the interior of $V_{\blacktriangleright}$:

$$\mathbf{1}_{\mathscr{U}} \in \mathrm{Int}(V_{\blacktriangleright})$$

Archimedean models: buying and selling price functionals



Other ways to characterise Your preferences?

Let C be any convex cone in $\mathscr{U}$ that includes $\mathrm{Int}(V_{\blacktriangleright})$, then:

Buying price functional:

$$\underline{\Lambda}_C(u) := \sup\{\alpha \in \mathbb{R} \colon u - \alpha\mathbf{1}_{\mathscr{U}} \in C\} \text{ for all } u \in \mathscr{U}$$

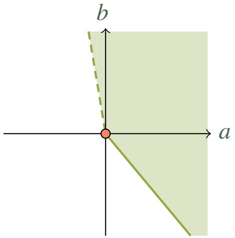
Selling price functional:

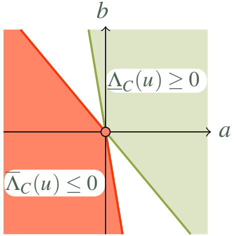
$$\overline{\Lambda}_C(u) := \inf\{\beta \in \mathbb{R} \colon \beta\mathbf{1}_{\mathscr{U}} - u \in C\} \text{ for all } u \in \mathscr{U}$$

Conjugacy:

$$\overline{\Lambda}_C(u) = -\underline{\Lambda}_C(-u) \text{ for all } u \in \mathscr{U}$$

Archimedean models: buying and selling price functionals





Other ways to characterise Your preferences?

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Conjugacy:

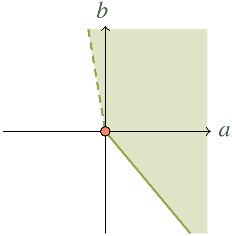
$$\overline{\Lambda}_C(u) = -\underline{\Lambda}_C(-u) \text{ for all } u \in \mathscr{U}$$

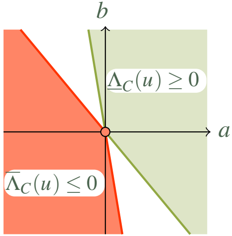
Converse relation to the convex cone C

$$u \in \mathrm{Int}(C) \Leftrightarrow \underline{\Lambda}_C(u) > 0 \text{ and } u \in \mathrm{Cl}(C) \Leftrightarrow \underline{\Lambda}_C(u) \geq 0$$

The real functional $\underline{\Lambda}_C$ characterises C up to its topological boundary.

Archimedean models: buying and selling price functionals





Price functionals for an AD-model $M \in \mathbf{M}_{\mathsf{AD}}(V)$

Buying price functional:

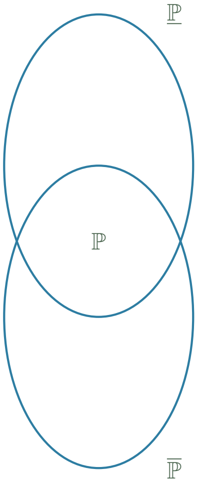
$$\begin{aligned}\underline{\Lambda}_M(u) &:= \sup\{\alpha \in \mathbb{R} \colon u - \alpha \mathbf{1}_{\mathscr{U}} \in A\}\\ &= \sup\{\alpha \in \mathbb{R} \colon u - \alpha \mathbf{1}_{\mathscr{U}} \in D\}\end{aligned}$$

Selling price functional:

$$\begin{aligned}\overline{\Lambda}_M(u) &:= \inf\{\beta \in \mathbb{R} \colon \beta \mathbf{1}_{\mathscr{U}} - u \in A\}\\ &= \inf\{\beta \in \mathbb{R} \colon \beta \mathbf{1}_{\mathscr{U}} - u \in D\}\end{aligned}$$

$\underline{\Lambda}_M = \underline{\Lambda}_A = \underline{\Lambda}_D$ and $\overline{\Lambda}_M = \overline{\Lambda}_A = \overline{\Lambda}_D$.

Archimedean models: coherent (lower and upper) previsions



Coherent lower prevision

A real functional $\underline{P} \colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if there is some AD-model $M \in \mathbf{M}_{\mathrm{AD}}(V)$ such that $\underline{P} = \underline{\Lambda}_M$.

Coherent upper prevision

A real functional $\overline{P} \colon \mathscr{U} \to \mathbb{R}$ is a coherent upper prevision if and only if there is some AD-model $M \in \mathbf{M}_{\mathrm{AD}}(V)$ such that $\overline{P} = \overline{\Lambda}_M$.

Coherent prevision

A real functional $P \colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if there is some AD-model $M \in \mathbf{M}_{\mathrm{AD}}(V)$ such that $P = \underline{\Lambda}_M = \overline{\Lambda}_M$.

Archimedean models: coherent (lower and upper) previsions

Characterisation

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if

L1. $\underline{P}(u+v) \geq \underline{P}(u)+\underline{P}(v)$ for all $u,v \in \mathscr{U}$

L2. $\underline{P}(\lambda u) = \lambda \underline{P}(u)$ for all $u \in \mathscr{U}$ and all real $\lambda > 0$

L3. $\|\underline{P}\|_{\mathscr{U}^\circ} < +\infty$

L4. $\underline{P}(u+\alpha \mathbf{1}_{\mathscr{U}}) = \underline{P}(u)+\alpha$ for all $u \in \mathscr{U}$ and all real α

L5. if $u \in V_\blacktriangleright$ then $\underline{P}(u) \geq 0$ for all $u \in \mathscr{U}$

A real functional $P\colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if

P1. $P(u+v) = P(u)+P(v)$ for all $u,v \in \mathscr{U}$

P2. $\|P\|_{\mathscr{U}^\circ} < +\infty$

P3. $P(\mathbf{1}_{\mathscr{U}}) = 1$

P4. if $u \in V_\blacktriangleright$ then $P(u) \geq 0$ for all $u \in \mathscr{U}$

Archimedean models: coherent (lower and upper) previsions

Vacuous lower and upper previsions

Price functionals associated with the background model V:

$$\mathrm{Inf}_V := \underline{\Lambda}_V \text{ and } \mathrm{Sup}_V := \overline{\Lambda}_V.$$

Archimedean models: coherent (lower and upper) previsions

Vacuous lower and upper previsions

Price functionals associated with the background model V:

$$\mathrm{Inf}_V := \underline{\Lambda}_V \text{ and } \mathrm{Sup}_V := \overline{\Lambda}_V.$$

Simpler characterisation

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if

L0. $\underline{P}(u) \geq \mathrm{Inf}_V u$

L1. $\underline{P}(u+v) \geq \underline{P}(u) + \underline{P}(v)$ for all $u, v \in \mathscr{U}$

L2. $\underline{P}(\lambda u) = \lambda \underline{P}(u)$ for all $u \in \mathscr{U}$ and all real $\lambda > 0$

A real functional $P\colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if

P0. $P(u) \geq \mathrm{Inf}_V u$

P1. $P(u+v) = P(u) + P(v)$ for all $u, v \in \mathscr{U}$

Archimedean models: coherent (lower and upper) previsions

Lower envelope theorem

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if it is the lower envelope of some set $\mathscr{M}$ of coherent previsions:

$$\underline{P}(u) = \inf\{P(u)\colon P \in \mathscr{M}\} \text{ for all } u \in \mathscr{U}.$$

In that case, the largest such set is the convex and (weak*)-closed

$$\mathscr{M}(\underline{P}) := \{P\colon (\forall u \in \mathscr{U})P(u) \geq \underline{P}(u)\}.$$

This is an instance of the Hahn–Banach Theorem.

The vacuous lower prevision Inf_V is the lower envelope of the set of all coherent previsions $\mathbb{P}$.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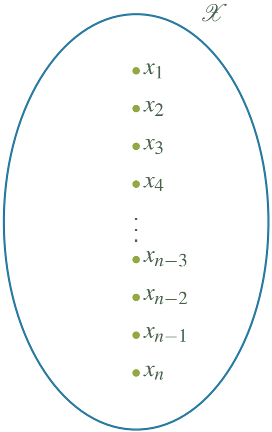
CLASSICAL PROBABILITY

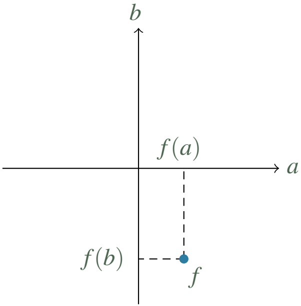
Classical probability: gambles as options



Consider a variable X that assumes values in some non-empty set $\mathscr{X}$, but whose value You don't know.

Classical probability: gambles as options

$\mathscr{X} = \{a, b\}$

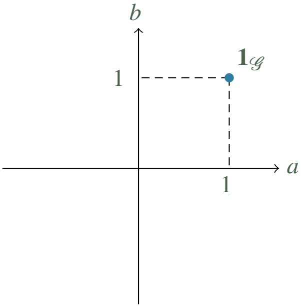


Consider a variable X that assumes values in some non-empty set $\mathscr{X}$, but whose value You don't know.

A bounded map $f\colon \mathscr{X} \to \mathbb{R}$ represents an uncertain reward $f(X)$; we call it a gamble.

Classical probability: gambles as options

$\mathscr{X} = \{a, b\}$



Consider a variable X that assumes values in some non-empty set $\mathscr{X}$, but whose value You don't know.

A bounded map $f\colon \mathscr{X} \to \mathbb{R}$ represents an uncertain reward $f(X)$; we call it a gamble.

The gambles constitute a normed linear space $\mathscr{G}$ with norm

$$\|f\|_\infty := \sup_{x \in \mathscr{X}} |f(x)| = \sup |f|$$

and as unit gamble $\mathbf{1}_{\mathscr{G}}$ the constant map 1:

$$\mathbf{1}_{\mathscr{G}}(x) = 1 \text{ for all } x \in \mathscr{X}.$$

Classical probability: background model

The background model V

is the statement model on the vector space $\mathscr{G}$ that is always there, regardless of what You may believe or prefer:

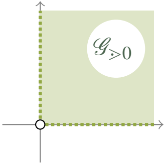
it represents COMPLETE IGNORANCE.

So, what is Your background model $V = \langle A\,; R\rangle$ under complete ignorance?

Classical probability: background model

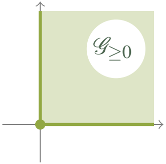
strong dominance ordering

$$f \gtrdot 0 \Leftrightarrow \inf f > 0$$



weak dominance ordering

$$f \geq 0 \Leftrightarrow \inf f \geq 0$$



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Between strong and weak dominance ordering:

$$\langle \mathscr{G}_{\geq 0}\,; -\mathscr{G}_{\gtrdot 0}\rangle \subseteq V \subseteq \langle \mathscr{G}_{\geq 0}\,; -\mathscr{G}_{>0}\rangle.$$

This implies that

$$\operatorname{Inf}_V f = \inf f = \inf\{f(x) : x \in \mathscr{X}\}, \text{ for all gambles } f.$$

Classical probability: background model

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So, what is Your background model $V = \langle A\,; R \rangle$ under complete ignorance?

$A :=$ Your set of accepted gambles

$R :=$ Your set of rejected gambles

$D :=$ Your set of desirable gambles $= A \cap -R$

$I :=$ Your set of indifferent gambles $= A \cap -A$

NO CONFUSION: $A \cap R = \emptyset$.

ACCEPT-DESIRABILITY: $0 \in A$ and $R \subseteq -A$.

Classical probability: background ordering

The one with constants:

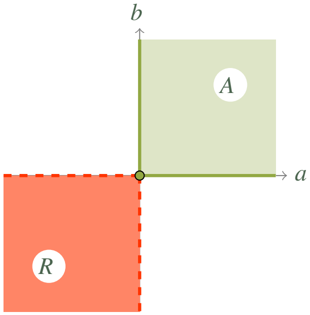
$$\frac{\mu \geq 0 \Rightarrow \mu\mathbf{1}_{\mathscr{G}} \in A \qquad \left(\inf f \geq \mu \text{ and } \mu\mathbf{1}_{\mathscr{G}} \in A\right) \Rightarrow f \in A}{\inf f \geq 0 \Rightarrow f \in A}$$

$$\frac{\mu < 0 \Rightarrow \mu\mathbf{1}_{\mathscr{G}} \in R \qquad \left(\sup f \leq \mu \text{ and } \mu\mathbf{1}_{\mathscr{G}} \in R\right) \Rightarrow f \in R}{\sup f < 0 \Rightarrow f \in R.}$$

Classical probability: background ordering

The one with constants:

$\mathscr{X} = \{a,b\}$



$$\mu \geq 0 \Rightarrow \mu \mathbf{1}_{\mathscr{G}} \in A$$

$$\frac{\left(\inf f \geq \mu \text{ and } \mu \mathbf{1}_{\mathscr{G}} \in A\right) \Rightarrow f \in A}{\inf f \geq 0 \Rightarrow f \in A}$$

$$\mu < 0 \Rightarrow \mu \mathbf{1}_{\mathscr{G}} \in R$$

$$\frac{\left(\sup f \leq \mu \text{ and } \mu \mathbf{1}_{\mathscr{G}} \in R\right) \Rightarrow f \in R}{\sup f < 0 \Rightarrow f \in R.}$$

So:

$$\{f \in \mathscr{G} :\ \inf f \geq 0\} \subseteq A \text{ and } \{f \in \mathscr{G} :\ \sup f < 0\} \subseteq R \tag{1}$$

Classical probability: background ordering

The one with complete ignorance:

A is invariant under permuting and lumping elements of $\mathscr{X}$.

$$\big(f \in A \text{ and } g(\mathscr{X}) \subseteq f(\mathscr{X})\big) \Rightarrow g \in A$$

As a consequence:

$$\begin{aligned} f \in A &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \in A \text{ for all } x \in \mathscr{X} && \text{[Complete Ignorance]} \\ &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \notin R \text{ for all } x \in \mathscr{X} && \text{[No Confusion]} \\ &\Rightarrow f(x) \geq 0 \text{ for all } x \in \mathscr{X} && \text{[Eq. (1)]} \\ &\Rightarrow \inf f \geq 0 \end{aligned}$$

Classical probability: background ordering

The one with complete ignorance:

A is invariant under permuting and lumping elements of $\mathscr{X}$.

$$\big(f \in A \text{ and } g(\mathscr{X}) \subseteq f(\mathscr{X})\big) \Rightarrow g \in A$$

As a consequence:

$$\begin{aligned} f \in A &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \in A \text{ for all } x \in \mathscr{X} && \text{[Complete Ignorance]} \\ &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \notin R \text{ for all } x \in \mathscr{X} && \text{[No Confusion]} \\ &\Rightarrow f(x) \geq 0 \text{ for all } x \in \mathscr{X} && \text{[Eq. (1)]} \\ &\Rightarrow \inf f \geq 0 \end{aligned}$$

So:

$$A \subseteq \{f \in \mathscr{G} : \inf f \geq 0\} \tag{2}$$

$$\{f \in \mathscr{G} : \inf f \geq 0\} \subseteq A$$

Classical probability: background ordering

The one with the conclusion:

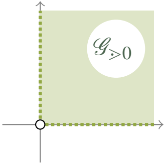
Combining Eqs. (1) and (2) with No Confusion yields:

$$\frac{\dfrac{\begin{gathered} A = \{f \in \mathscr{G} \colon \inf f \geq 0\} \\ \{f \in \mathscr{G} \colon \sup f < 0\} \subseteq R \subseteq \{f \in \mathscr{G} \colon \inf f < 0\} \end{gathered}}{\begin{gathered} -A = \{f \in \mathscr{G} \colon \sup f \leq 0\} \\ \{f \in \mathscr{G} \colon \inf f > 0\} \subseteq -R \subseteq \{f \in \mathscr{G} \colon \sup f > 0\} \end{gathered}}}{\begin{gathered} \{f \in \mathscr{G} \colon \inf f > 0\} \subseteq D \subseteq \{f \in \mathscr{G} \colon \inf f \geq 0 \text{ and } f \neq 0\} \\ I = \{0\}. \end{gathered}}$$

Classical probability: background ordering

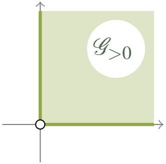
strong dominance ordering

$$f \gtrdot 0 \Leftrightarrow \inf f > 0$$



strict weak dominance ordering

$$f > 0 \Leftrightarrow \inf f \geq 0 \text{ and } f \neq 0$$



The one with the conclusion:

Combining Eqs. (1) and (2) with No Confusion yields:

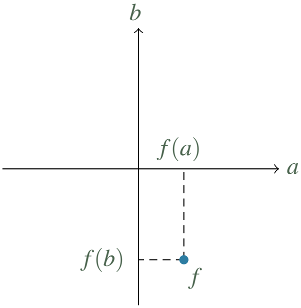
$$\begin{array}{c}
A = \{f \in \mathscr{G} \colon \inf f \geq 0\} \\
\{f \in \mathscr{G} \colon \sup f < 0\} \subseteq R \subseteq \{f \in \mathscr{G} \colon \inf f < 0\} \\
\hline
-A = \{f \in \mathscr{G} \colon \sup f \leq 0\} \\
\{f \in \mathscr{G} \colon \inf f > 0\} \subseteq -R \subseteq \{f \in \mathscr{G} \colon \sup f > 0\} \\
\hline
\{f \in \mathscr{G} \colon \inf f > 0\} \subseteq D \subseteq \{f \in \mathscr{G} \colon \inf f \geq 0 \text{ and } f \neq 0\} \\
I = \{0\}.
\end{array}$$

Hence,

$$A = \mathscr{G}_{\geq 0} \text{ and } \mathscr{G}_{\gtrdot 0} \subseteq D \subseteq \mathscr{G}_{>0} \text{ and } I = \{0\}.$$

Classical probability: coherent previsions

$$\mathscr{X} = \{a, b\}$$



The coherent previsions on $\mathscr{G}$

Expansion in the standard basis:

$$f = \sum_{x \in \mathscr{X}} f(x) \mathbb{I}_{\{x\}}.$$

By linearity:

$$P(f) = \sum_{x \in \mathscr{X}} f(x) \underbrace{P(\mathbb{I}_{\{x\}})}_{p(x)} = (p|f).$$

By the background condition $P \geq \mathrm{Inf}_V = \inf$:

$$\sum_{x \in \mathscr{X}} p(x) = 1 \text{ and } p(x) \geq 0.$$

P is a coherent prevision if and only if it's the expectation operator associated with a (finitely additive) probability measure.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QUANTUM PROBABILITY

Quantum IP

ALESSIO BENAVOLI

ALESSANDRO FACCHINI

MARCO ZAFFALON

KEANO DE VOS

JASPER DE BOCK

ALEXANDER ERREYGERS

GERT DE COOMAN

Quantum probability: measurements as options

Consider a quantum system whose unknown state $|\Psi\rangle$ lives in an n-dimensional complex Hilbert state space $\mathscr{X}$, so

$$|\Psi\rangle \in \mathscr{X} \text{ and } \langle\Psi|\Psi\rangle = 1.$$

Any measurement corresponds to some Hermitian operator $\hat{A}$, and its outcome is uncertain. Its reward is the measurement outcome.

All measurements constitute a real n^2-dimensional Hilbert space $\mathscr{H}$, with (Frobenius) inner product

$$(\hat{A},\hat{B}) := \mathrm{Tr}(\hat{A}^\dagger\hat{B}) = \mathrm{Tr}(\hat{A}\hat{B}), \text{ for all } \hat{A},\hat{B} \in \mathscr{H},$$

corresponding (Frobenius) norm

$$\|\hat{A}\|_{\mathscr{H}} := \sqrt{(\hat{A},\hat{A})} = \sqrt{\mathrm{Tr}(\hat{A}^\dagger\hat{A})} \text{ for all } \hat{A} \in \mathscr{H},$$

and unit measurement $\mathbf{1}_{\mathscr{H}} := \hat{I}$.

Quantum probability: background model

The background model

is the statement model on the vector space $\mathscr{H}$ that is always there, regardless of what You may believe or prefer:

it represents COMPLETE IGNORANCE.

So, what is Your background model $V = \langle A\,; R\rangle$ under complete ignorance?

Quantum probability: background model

strong dominance ordering

$$\hat{A} \gtrdot \hat{0} \Leftrightarrow \min \operatorname{spec} \hat{A} > 0$$

so $\hat{A}$ is positive definite.

weak dominance ordering

$$\hat{A} \geq \hat{0} \Leftrightarrow \min \operatorname{spec} \hat{A} \geq 0$$

so $\hat{A}$ is positive semi-definite.

strict weak dominance ordering

$\hat{A} > \hat{0} \Leftrightarrow \min \operatorname{spec} \hat{A} \geq 0$ and $\hat{A} \neq \hat{0}$

so $\hat{A}$ is positive semi-definite and non-null.

The background model

is the statement model on the vector space $\mathscr{H}$ that is always there, regardless of what You may believe or prefer:

it represents COMPLETE IGNORANCE.

So, what is Your background model $V = \langle A ; R \rangle$ under complete ignorance?

Between strong and weak dominance ordering:

$$A = \mathscr{H}_{\geq 0} \text{ and } \mathscr{H}_{\gtrdot 0} \subseteq D = -R \subseteq \mathscr{H}_{>0}$$

This implies that

$$\operatorname{Inf}_V \hat{A} = \min \operatorname{spec} \hat{A}, \text{ for all measurements } \hat{A} \in \mathscr{H}.$$

Quantum probability: coherent previsions

Recall that a density operator $\hat{\rho}$ is a positive semidefinite Hermitian operator such that $\operatorname{Tr}\hat{\rho} = 1$.

The coherent previsions on $\mathscr{H}$

By linearity and the Riesz Representation Theorem, there is some unique $\hat{B}_P \in \mathscr{H}$ such that

$$P(\hat{A}) = (\hat{B}_P, \hat{A}) = \operatorname{Tr}(\hat{B}_P \hat{A}) \text{ for all } \hat{A} \in \mathscr{H}.$$

By the background condition $P \geq \min\operatorname{spec}(\bullet)$:

$$\hat{B}_P \text{ is positive semidefinite and } \operatorname{Tr}(\hat{B}_P) = 1,$$

so there's a density operator $\hat{\rho} \in \mathscr{H}$ such that

$$P(\hat{A}) = \operatorname{Tr}(\hat{\rho}\hat{A}) \text{ for all } \hat{A} \in \mathscr{H} \quad \text{[Born's rule]}$$

CONCLUSIONS

In summary:

- Probability theory — in the form of AD-models, acceptability, desirability — is a deductive inference system, if we allow for partial specification.
- precise probability = linearity + background ordering.
- the difference between classical and quantum probability lies only in the background ordering.
- the connection between classical and quantum probability is deep-rooted, and stares you in the face as soon as you have the right language to express probabilities in.
- the background ordering is determined by what it means to be completely ignorant, and therefore by the 'physics' of the problem.
- working with sets of desirable measurements = Heisenberg picture
- working with sets of density operators = Schrödinger picture