The world where
$1 + 1 \neq 2!$

Assumptions of Physics Workshop:
Nonadditive Measures and Generalized Probabilities in Physics

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University of Michigan, USA (remotely)

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Summary

- Fuzzy (non-additive) measures μ

- Fuzzy integrals $\int$

- Fuzzy derivatives $\frac{d\nu}{d\mu}$

Measures

Sample space Ω with σ-algebra $\mathcal{A}$
Set function $\mu : \mathcal{A} \to \mathbb{R}$

with further conditions depending on the demand

- (groundedness) $\mu(\emptyset) = 0$
- (non-negativity) $\mu \geq 0$

$\rightarrow$ **Measure**

Measures

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Set function $\mu : \mathcal{A} \to \mathbb{R}$

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- (groundedness) $\mu(\emptyset) = 0$
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$\to$ **Measure**

plus

- (additivity) $(\forall A, B \in \mathcal{A}, A \cap B = \emptyset)\ \mu(A \cup B) = \mu(A) + \mu(B)$

$\to$ **Additive measure**

e.g.
Lebesgue measure λ, where $\lambda([a, b]) = b - a$
Probability measure P, where

$$\mathrm{P}([a, b]) = \mathcal{P}(b) - \mathcal{P}(a) \qquad \text{and} \qquad \mathrm{P}([\mathrm{a}, \mathrm{b}]) = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{p}(\mathrm{t})\,\mathrm{dt}$$

Measures

weakening additivity property

- (monotonicity) $(\forall A, B \in \mathcal{A}, A \subseteq B)\ \mu(A) \leq \mu(B)$

$\rightarrow$ **Non-additivite (fuzzy) measure**

e.g.

Minimal measure $\mu_{min}(A) = \begin{cases} 0 & A \neq \Omega \\ 1 & A = \Omega \end{cases}$

Maximal measure $\mu_{max}(A) = \begin{cases} 0 & A = \emptyset \\ 1 & A \neq \emptyset \end{cases}$

Unanimity measure $\mu_E(A) = \begin{cases} 0 & \text{otherwise} \\ 1 & E \subseteq A \end{cases}$

Power measure $\mu_{\#}(A) = \left(\frac{|A|}{|\Omega|}\right)^k$

with distortion m being non-negative, increasing function with $m(0) = 0$

Distorted Lebesgue measure λ_m, where $\lambda_m = m \circ \lambda$
Distorted probability measure P_m, where $\mathrm{P}_m = m \circ \mathrm{P}$

Measures

Non-monotone measure - can be non-monotone
from decision making and quantum mechanics - interference?

Signed measure - can have negative values
when subtracting and modelling physical quantities - net charge?

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Vector space of discrete measures - unanimity measures as basis

$$\mu(A) = \sum_{E \neq \emptyset} k(E)\, \mu_E(A)$$

Every discrete measure can be expressed as (almost) distorted probability

$$\mu = f \circ \mathrm{P} \qquad or \qquad \mu = f^+ \circ \mathrm{P} - f^- \circ \mathrm{P}$$

Continuous (general) measures?

Integrals

For additive measures:

Lebesgue integral $=$ Riemann integral

$$(\mathscr{L}) \int f \, \mathrm{d}\mu$$

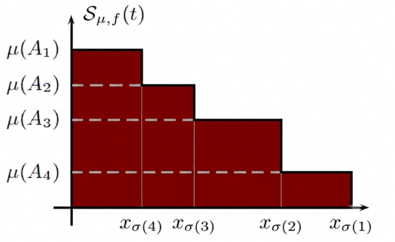
For non-additive (fuzzy) measures:

Choquet integral

$$\boxed{(C)\int f d\mu = \sum_{i=1}^{n} f(x_{\sigma(k)})\,(\mu(A_k) - \mu(A_{k+1}))}$$

with permutation σ on $\{1, 2, \dots, n\}$

$f(x_{\sigma(1)}) \leq f(x_{\sigma(2)}) \leq \cdots \leq f(x_{\sigma(n)})$ $\qquad A_k := \{x_{\sigma(k)}, x_{\sigma(k+1)}, \dots x_{\sigma(n)}\}$

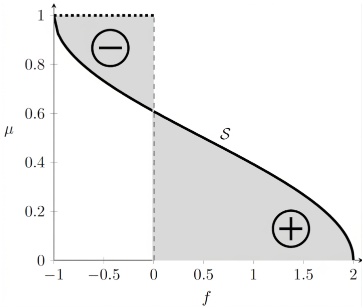


For non-additive (fuzzy) measures:

Choquet integral

$$\boxed{(C)\int f\,d\mu = -\int_{-\infty}^{0}\left[\mu(\Omega)-\mathcal{S}_{\mu,f}(t)\right]\,\mathrm{d}t + \int_{0}^{\infty}\mathcal{S}_{\mu,f}(t)\,\mathrm{d}t}$$

with $\mathcal{S}_{\mu,f}(t) = \mu(\{f \geq t\}),\ t \in \mathbb{R}$



Integrals

For non-additive (fuzzy) measures:

Choquet integral

$$(C)\int f d\mu = -\int_{-\infty}^{0} \left[\mu(\Omega) - \mathcal{S}_{\mu,f}(t)\right]\,\mathrm{d}t + \int_{0}^{\infty} \mathcal{S}_{\mu,f}(t)\,\mathrm{d}t$$

Properties:

positive homogeneity translation invariance monotonicity

negative homogeneity $(C)\int (-f)\,\mathrm{d}\mu = -(C)\int f\,\mathrm{d}\overline{\mu}$

no additivity $(C)\int (f+g)\,\mathrm{d}\mu \neq (C)\int f\,\mathrm{d}\mu + (C)\int g\,\mathrm{d}\mu$

only with comonotonicity of f, g

weaker version with sub/supermodular measure

restriction of domain $(C)\int_A f\,\mathrm{d}\mu = (C)\int f \cdot \mathbf{1}_A\,\mathrm{d}\mu$

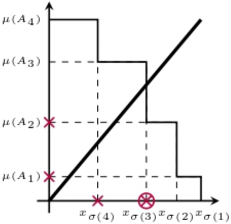
For non-additive (fuzzy) measures:

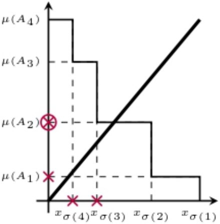
Sugeno integral

$$(S)\int f d\mu = \bigvee_{k=1}^{n} \left(f(x_{\sigma(k)}) \wedge \mu(A_k)\right)$$

with permutation σ on $\{1, 2, \ldots, n\}$

$f(x_{\sigma(1)}) \leq f(x_{\sigma(2)}) \leq \cdots \leq f(x_{\sigma(n)})$ $\qquad A_k := \{x_{\sigma(k)}, x_{\sigma(k+1)}, \ldots x_{\sigma(n)}\}$



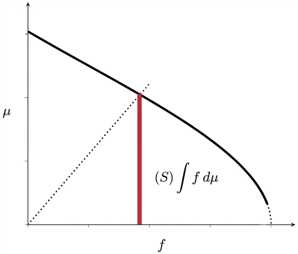


For non-additive (fuzzy) measures:

Sugeno integral

$$\boxed{(\mathsf{S})\int f\,\mathrm{d}\mu := \bigvee_{t\in[0,\infty]} (t\wedge \mathcal{S}_{\mu,f}(t))}$$

with $\mathcal{S}_{\mu,f}(t) = \mu(\{f \geq t\}), t \in \mathbb{R}$ and condition $\inf_{x\in\emptyset} f(x) = \infty$



Properties:

$\wedge$- / $\vee$-homogenity monotonicity $\vee - \wedge$ linearity . . .

For non-additive (fuzzy) measures:

Lehmann integral

$$\boxed{(\mathsf{S_L})\int f\,\mathrm{d}\mu := \bigvee_{t\in[0,\infty]} (\varphi(t)\wedge \mathcal{S}_{\mu,f}(t))}$$

Shilkret integral

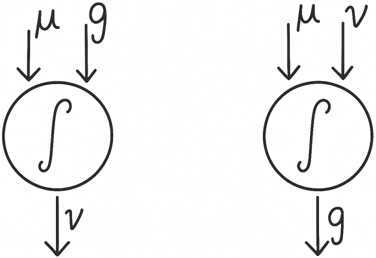
$$\boxed{(\mathsf{Sh})\int f\,\mathrm{d}\mu := \bigvee_{t\in[0,\infty]} [t\cdot \mathcal{S}_{\mu,f}(t)]}$$

Maximal chain-based integral

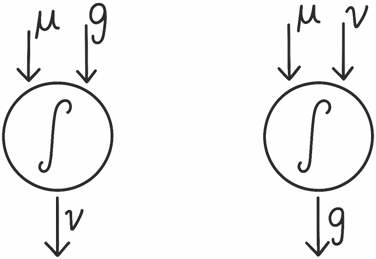
$$\boxed{(\mathrm{MC})\int f d\mu = \sum_{k=1}^{m} f(x_k)[\mu(C_k)-\mu(C_{k-1})]}$$

chain $\mathcal{C} := \{C_1, C_2, \ldots, C_m\}$ in Ω such that
$\emptyset = C_0 \subset C_1 \subset \cdots \subset C_m = \Omega$, $C_k \setminus C_{k-1} = \{x_k\}$ for all $k = 2, \ldots, m$.

Derivatives



$$\nu(A) = \int_A f \, \mathrm{d}\mu \qquad \Leftrightarrow \qquad f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$$



$$\boxed{\nu(A) = \int_A f \, \mathrm{d}\mu \qquad \Leftrightarrow \qquad f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}}$$

for Lebesgue integral - **Radon-Nikodym derivative**

Let μ and ν being two σ-finite additive measures. If $\nu \ll \mu$, then there exists a nonnegative measurable function f, such that for any measurable set A

$$\nu(A) = (\mathscr{L}) \int_A f \, \mathrm{d}\mu.$$

Derivatives

$$\nu(A) = (C)\int_A f \,\mathrm{d}\mu$$

$f = \dfrac{\mathfrak{d}\nu}{\mathfrak{d}\mu}$ is the **Choquet-Radon-Nikodym derivative**

- $\mu,\ \nu$ are submodular
- $\mu,\ \nu$ are subadditive and couple (ν, μ) possesses the SDP
- with Resulting measure approach

Derivatives

$$\nu(A) = (C)\int_A f\,\mathrm{d}\mu$$

$f = \dfrac{\mathfrak{d}\nu}{\mathfrak{d}\mu}$ is the **Choquet-Radon-Nikodym derivative**

$$\frac{\mathfrak{d}(k\nu)}{\mathfrak{d}\mu} = k\frac{\mathfrak{d}\nu}{\mathfrak{d}\mu} \quad \mu\text{-a.e.} \quad k>0$$

$$\frac{\mathfrak{d}\nu}{\mathfrak{d}\tau},\frac{\mathfrak{d}\mu}{\mathfrak{d}\tau} \text{ comonotone} \quad \rightarrow \quad \frac{\mathfrak{d}(\mu+\nu)}{\mathfrak{d}\tau} = \frac{\mathfrak{d}\mu}{\mathfrak{d}\tau} + \frac{\mathfrak{d}\nu}{\mathfrak{d}\tau} \quad \tau-\text{a.e.}$$

$$\tau \text{ submodular} \quad \rightarrow \quad \frac{\mathfrak{d}\mu}{\mathfrak{d}\tau} + \frac{\mathfrak{d}\nu}{\mathfrak{d}\tau} \leq \frac{\mathfrak{d}(\mu+\nu)}{\mathfrak{d}\tau} \quad \tau-\text{a.e.}$$

$$g \text{ non}-\text{negative} \quad \rightarrow \quad (C)\int g\,\mathrm{d}\nu = (C)\int g\frac{\mathfrak{d}\nu}{\mathfrak{d}\mu}\,\mathrm{d}\mu$$

$$\frac{\mathfrak{d}\nu}{\mathfrak{d}\tau}\frac{\mathfrak{d}\tau}{\mathfrak{d}\mu} = \frac{\mathfrak{d}\nu}{\mathfrak{d}\mu} \quad \mu-\text{a.e.}$$

$$\frac{\mathfrak{d}\nu}{\mathfrak{d}\mu} = \left(\frac{\mathfrak{d}\mu}{\mathfrak{d}\nu}\right)^{-1} \quad \mu-\text{a.e. and } \nu-\text{a.e.}$$

Summary

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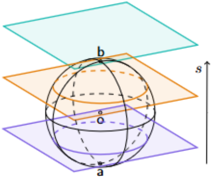
Summary

For an ensamble space $\mathcal{E}$, given an element $e \in \mathcal{E}$,
the fraction capacity of $A \subseteq \mathcal{E}$ with respect to e is defined as

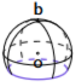
$$fcap_e(A) = \sup\{p \in [0,1] : \exists a \in A, b \in \mathcal{E} \text{ s.t. } \mathrm{e} = \mathrm{pa} + \overline{\mathrm{p}}\mathrm{b}\}$$



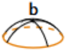
$$(C)\int F \,\mathrm{d} fcap_O = \int_0^{\infty} fcap_O(\{x : F(x) \geq s\})\,\mathrm{d}s$$

$$A(s) = \{x : F(x) \geq s\}$$

$=$



$s < \frac{1}{2}F(a) + \frac{1}{2}F(b)$



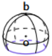
$s \in \left[\frac{1}{2}F(a) + \frac{1}{2}F(b), F(b)\right]$

$\emptyset$ $\quad s > F(b)$

Summary

$$fcap_e(A) = \sup\{p \in [0,1] : \exists a \in A, b \in \mathcal{E} \text{ s.t. } \mathrm{e} = \mathrm{pa} + \overline{\mathrm{p}}\mathrm{b}\}$$

$A(s) =$



$s < \frac{1}{2}F(a) + \frac{1}{2}F(b)$

$s \in \left[\frac{1}{2}F(a) + \frac{1}{2}F(b), F(b)\right]$

$\emptyset$ $\quad s > F(b)$

$$\mu(A(s)) = \begin{cases} ?? & s < \frac{1}{2}F(a) + \frac{1}{2}F(b) \\ ?? & s \in \left[\frac{1}{2}F(a) + \frac{1}{2}F(b), F(b)\right] \\ 0 & s > F(b) \end{cases}$$