What do we want to do?

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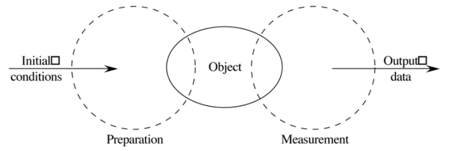
Tuesday 7th July, 2026

What is a physical theory?

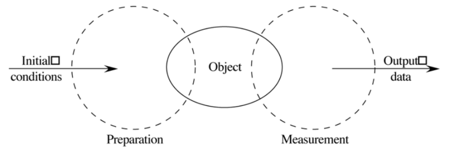


Preparation: is a list of instructions that one performs to prepare a system at the beginning of an experiment. The list of instructions may be conditioned by random events.

Measurement/Detector: is a list of instructions that we perform after preparing a state to get some numeric value.

Channels: are list of instructions that we can either append at the end of preparation or, equivalently, prepend to the beginning of a measurement.

What is a physical theory?



Preparation: is a list of instructions that one performs to prepare a system at the beginning of an experiment. The list of instructions may be conditioned by random events. **Mixture space**

Measurement/Detector: is a list of instructions that we perform after preparing a state to get some numeric value. **Random-variables**

Channels: are list of instructions that we can either append at the end of preparation or, equivalently, prepend to the beginning of a measurement.

Economics: modelling consumers

 $\succ$    preference over recipes: $\mathbf{x} \succ \mathbf{y}$

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When you buy a cake $\mathbf{x}$ you cannot be sure of its ingredients as there may be *uncertainty* in the *preparation* procedure.

We can model uncertainty by assuming that the consumer choose between *probability distributions* over X. For instance, consider two cake recipes $\mathbf{x} = [5,3]^\top$ and $\mathbf{y} = [7,7]^\top$, and consider a probability distribution with support in $\{\mathbf{x}, \mathbf{y}\}$ and such that

$p(\mathbf{x}) = 1/3$ you receive 

$p(\mathbf{y}) = 2/3$ you receive 

$q(\mathbf{x}) = 3/4$ you receive 

$q(\mathbf{y}) = 1/4$ you receive 

Economics: Von Neumann-Morgenstern (1947)

The probability distributions $p, q \in P$ are assumed to have finite support. Additionally, we can combine them

$$r = \alpha p + (1-\alpha) q \quad \text{where} \quad \alpha \in [0,1]$$

- The support of r is the union of the supports of p, q
- if $\mathbf{x}$ is some element of the support then $r(\mathbf{x}) = \alpha p(\mathbf{x}) + (1-\alpha) q(\mathbf{x})$.

$p, q, r \in P$ which is a mixture space.

In this case, we ask the consumer to give preferences about the elements of P:

$$p \succ q$$

Economics: Von Neumann-Morgenstern

Theorem (Representation theorem)

A preference relation $\succ$ on the set P of probability distributions on X with finite support, satisfies certain assumptions (Completeness, Independence, Continuity) if and only if there is a function $u : X \to \mathbb{R}$ such that

$$p \succ q \quad \textit{iff} \quad \sum_{\mathbf{x} \in supp(p)} u(\mathbf{x}) p(\mathbf{x}) \geq \sum_{\mathbf{x} \in supp(q)} u(\mathbf{x}) q(\mathbf{x})$$

Theorem (Uniqueness)

Two utility functions u, v represent the same preference relation if there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$v(\cdot) = au(\cdot) + b$$

Von Neumann never explicitly linked his work on the expected utility theorem with his mathematical foundations of QM.

Herstein and Milnor (1953) mixture space theorem

What do we want to do?

In the previous result, we focused on a mixture space whose elements are probability distributions, but we can generalise it.

Definition

A *mixture space* M is a set that satisfies the following axioms.

- **M1** for any $x, y \in M$ and any $\alpha \in [0,1]$, the α-mixture of x, y, denoted as $x\alpha y$, is a unique element of M.
- **M2** $x\alpha y = y(1-\alpha)x$
- **M3** $x\alpha(y\beta z) = \left(x \frac{\alpha}{\alpha+\beta-\alpha\beta} y\right)(\alpha+\beta-\alpha\beta)z$
- **M4** $x\alpha x = x$
- **M5** if $x\alpha z = y\alpha z$ for some $\alpha \neq 0$ then $x = y$.

Note: $x\alpha y$ is the generalisation of $\alpha x + (1-\alpha)y$.

Herstein and Milnor's mixture space theorem

Theorem

Let M be a mixture space. Then there exists a vector space V over the real numbers and a function ϕ mapping M into V such that

1 ϕ *is one-to-one into the cone K generating V;*

2 $\phi(x\alpha y) = \alpha\phi(x) + (1-\alpha)\phi(y)$ *for* $\alpha \in (0,1)$.

3 $\phi(M) = \mathscr{C}(\phi(M))$, *i.e.,* $\phi(M)$ *is convex.*

4 $V = \mathscr{V}(\phi(M))$

5 $0 \notin \mathscr{H}(\phi(M))$

where $\mathscr{C}$ is the convex-hull, $\mathscr{V}$ is the span and $\mathscr{H}$ are affine combinations.

Any mixture space can be embedded into a convex subset of a vector space

Proof of the mixture space theorem

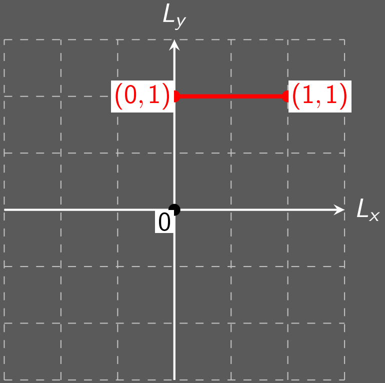
- Embed the mixture space into a cone K
- Build a vector space $V = span(K - K)$

Result: the mixture space is now embedded into a convex subset of a vector space.

Example coin: p is probability of Heads

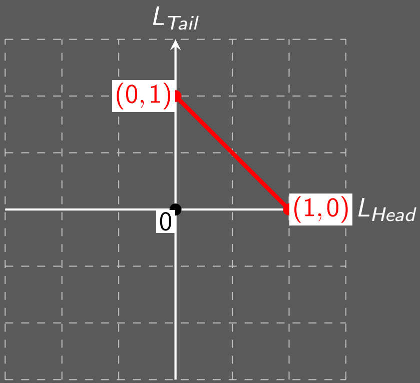
$M = \{p \in [0,1]\}$

Every mixture space can be embedded in a vector space.



$M = \{[p, 1-p] \mid p \in [0,1]\}$

Some mixture spaces are the bases of symmetric cones



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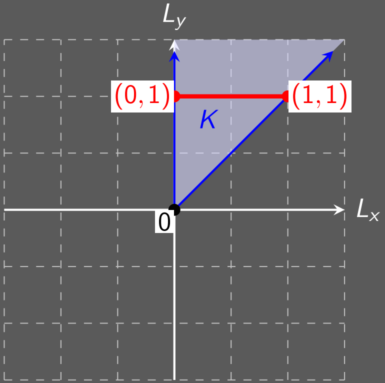

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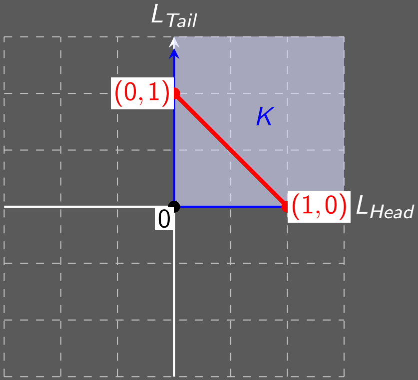

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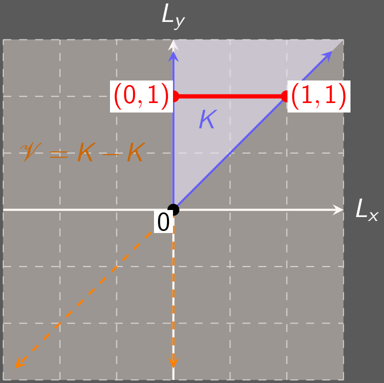

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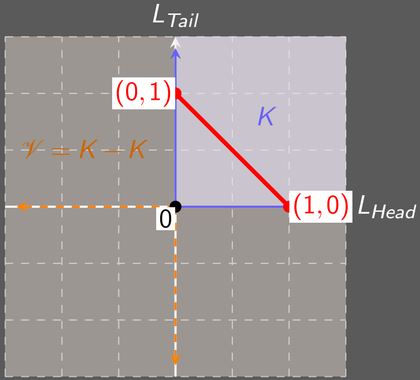

Duality

For any vector space V over $\mathbb{R}$, its dual is defined as the set of all linear functionals on V, that is

$$V^\circ = \{g : V \to \mathbb{R} \mid g \text{ is linear}\}$$

Given a cone $K \subset V$, its dual cone:

$$K^\circ = \{g \in V^\circ \mid \langle g, k\rangle \geq 0 \quad \forall k \in K\}$$

Then one can prove that if K is a proper cone then K° is also proper (convex, closed, pointed and generating cone).

Unit:

$$g = \mathbb{1} \text{ such that } \langle \mathbb{1}, k\rangle = 1 \quad \forall k \in \phi(M)$$

order and order-unit

$$g \succ h \text{ if } g - h \in K^\circ \qquad \lambda\mathbb{1} \succ g \text{ for some } \lambda > 0$$

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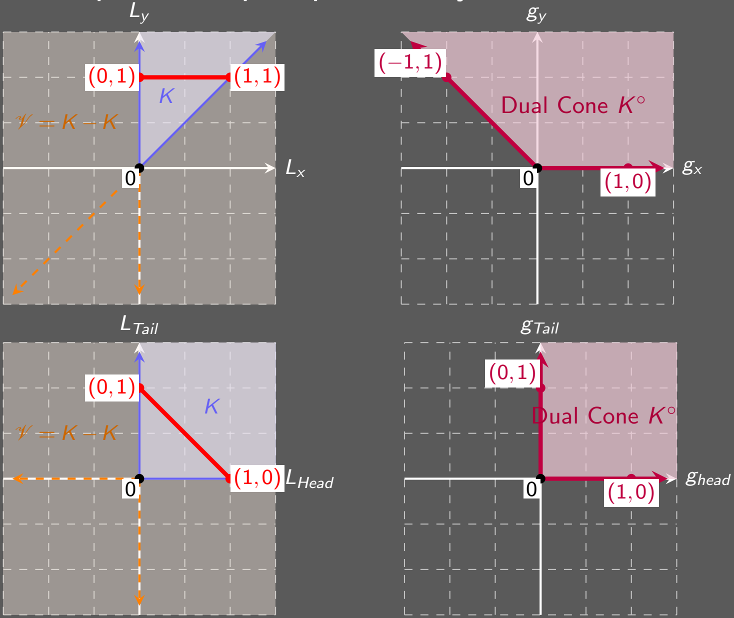

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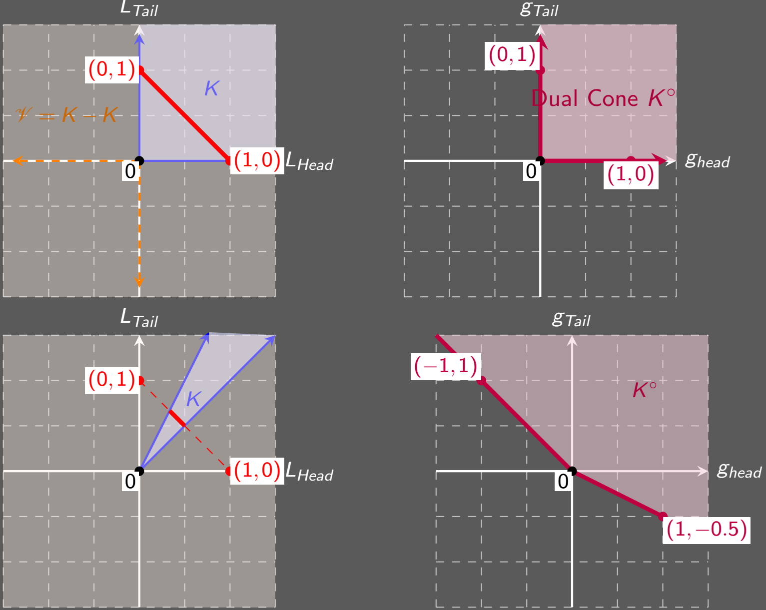


Interpretation

- Preparation $\rightarrow$ Mixture space $\rightarrow$ convex subset of the vector space
- Duality $\rightarrow$ utility $\rightarrow$ Effect/Measurement

Why symmetric cones?

- Large automorphism group
- Homogeneous
- Self-dual (for a suitable inner product)
- Spectral decomposition (Jordan algebra)
- Canonical determinant and logarithm

Consequences: the transformations that preserve the positive cone (automorphisms) allows one to define trasformations that preserve the embedded mixture space (e.g., permutations, unitary operators).

Jordan, von Neumann and Wigner (1934)

Theorem

Every symmetric cone is the unique (up to order) orthogonal direct sum of nontrivial irreducible symmetric cones, and every nontrivial irreducible symmetric cone is Jordan-isomorphic to a member of one of the five distinct families

Lorentz: *The n-dimensional Lorentz cones*

Sym: *The* $n \times n$ *real symmetric PSD cones*

Herm: *The* $n \times n$ *complex Hermitian PSD cones*

Quat: *The* $n \times n$ *quaternion Hermitian PSD cones*

Albert algebra *The cone of squares in the Albert algebra*

Finite dimensional QM

- Embedded mixture space: $M = \{\rho \in H^{n \times n} \mid \rho > 0, \quad Tr(\rho) = 1\}$
- $K = K^{\circ} = \{A \in H^{n \times n} \mid A \geq 0\}$ symmetric PSD cone
- Inner product: $\langle A, B \rangle = Tr(AB)$
- Homogeneous transformations of the positive cone: $RAR^{\dagger}$ where $A \in K$.
- Unitary transformations: $U \rho U^{\dagger}$ where $\rho \in M$
- Logarithm: $\log A$ for $A \in K$.
- Von-Neumann entropy: $-Tr(\rho \log \rho)$

Some parts have been left out

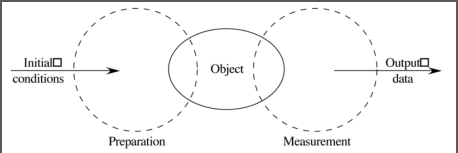
Tensor product and entanglement (the real divide).

Why am I not satisfied by this construction?

- It does not answer why QT is not classical.
- It compares QT with CPT with a finite possibility space:
  - "QT mixture space includes the CPT mixture space (diagonal matrices)".
  - "CPT is the only theory in the framework with unique decompositions of all mixed states."
  - "In contrast, when the set of pure states is an overcomplete basis there is linear dependence amongst the pure states"

Why should we compare finite dimensional QT with CPT with finite possibility spaces?
ρ is more similar to a covariance matrix than to a PMFs.

Experiment



$$g(prep, measur) \to \mathbb{R}$$

$$g_{meas}(x_1, \ldots, x_n) \to \mathbb{R}$$

Classical probability

- Possibility space Ω of the variables $x_1, \ldots, x_n$.
- Observables/Gambles $g : \Omega \to \mathbb{R}$ (bounded)

An agent expresses her beliefs about the experiment as a set of gambles

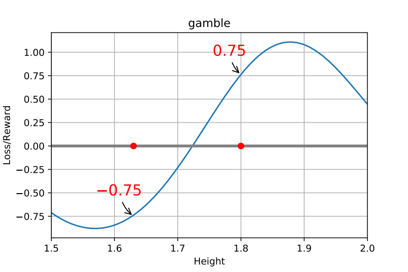
$$\mathscr{G} = \{g_1, g_2, \ldots\}$$

the agent is willing to accept.

How can we enforce rational behaviour?

How tall was Einstein? Do you want to bet on it?

- Possibility space $\Omega = [1.5, 2]m$ (or $[4.92, 6.56]feet$)
- Gambles $g : \Omega \to \mathbb{R}$ (bounded)



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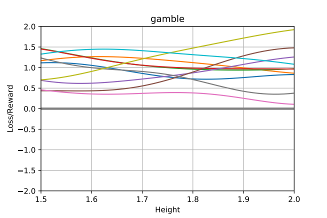

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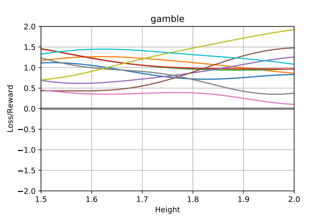

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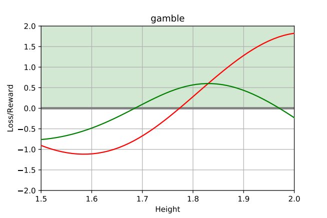

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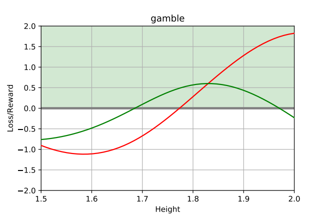

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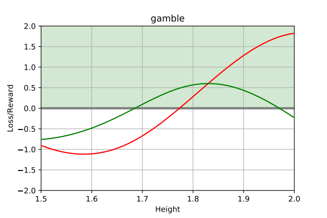

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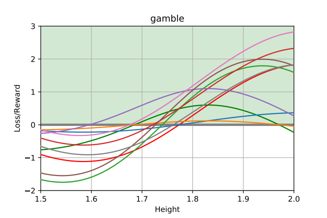

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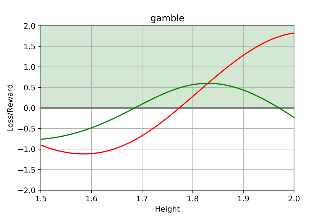

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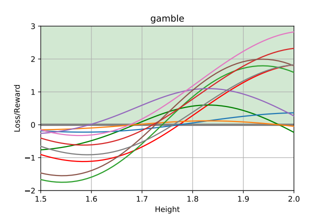

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$$\text{posi}(\mathscr{A}) = \left\{ \sum_i \lambda_i g_i : \lambda_i \geq 0,\ g_i \in \mathscr{A} \right\}$$

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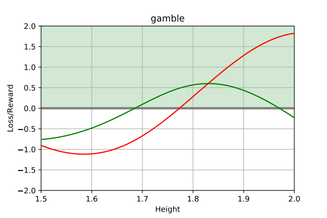

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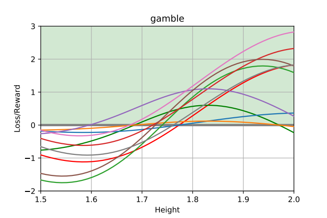

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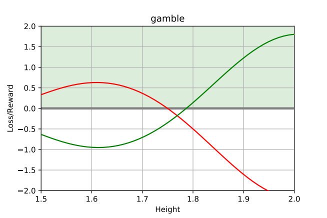

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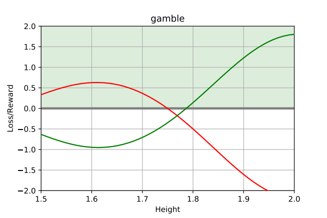

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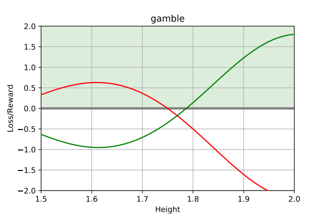

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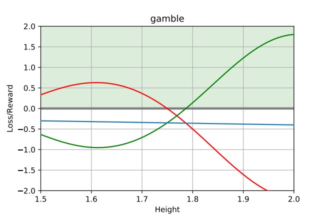


Linearity

Logical consistency and Mathematical duality

You are a rational agent if

A0 (*Tautologies*): accepts all nonnegative gambles

A1 (*Deductive closure*): The set of gambles you accept is closed under any nonnegative combination

A2 (*Coherence*): The set of gambles you accept does not include a negative function, that is **no Dutch book** can be done against her

A set of gambles $\mathcal{K}$ that satisfies A0–A2 is a closed convex cone

Mathematical duality

Assume $\mathcal{K}$ satisfies A0–A2. We can give $\mathcal{K} \subset \mathscr{L}$ a probabilistic interpretation under some topological assumptions.
The dual of $\mathcal{K}$ is defined as:

$$\mathcal{K}^{\circ} = \{L \in \mathsf{S} \mid L(g) \geq 0, \ \forall g \in \mathcal{K}\}, \tag{1}$$

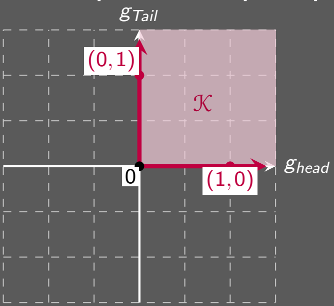
where $\mathsf{S} = \{L \in \mathscr{L}^{*} \mid L(1) = 1, \ \ L(h) \geq 0 \ \ \forall h \in \mathscr{L}^{\geq}\}$ is the set of (belief) *states*.

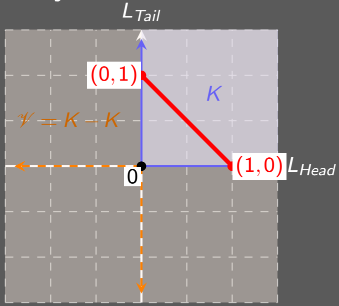
To $\mathcal{K}^{\circ}$ we can then associate its extension $\mathcal{K}^{\bullet}$ in $\mathscr{M}$, that is, the set of all probability charges on Ω extending an element in $\mathcal{K}^{\circ}$, leading to:

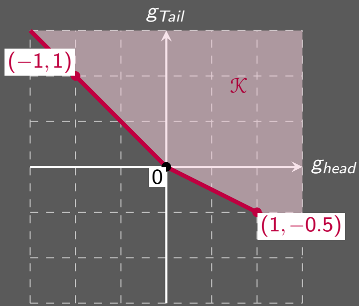
$$\mathcal{K}^{\bullet} = \left\{\mu \in \mathsf{P} \middle| \int_{\Omega} g(\omega) d\mu(\omega) \geq 0, \ \forall g \in \mathscr{G}\right\}, \tag{2}$$

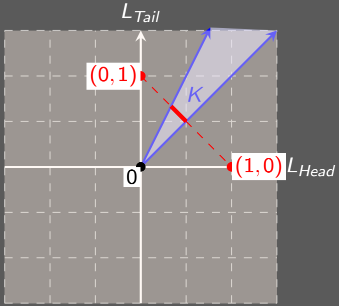
where $\mathsf{P} = \{\mu \in \mathscr{M} \mid \inf \mu \geq 0, \ \int_{\omega} d\mu(\omega) = 1\}$ is the set of all probability charges in Ω, and $\mathscr{M}$ the set of all charges on Ω.

Example coin: p is probability of Heads

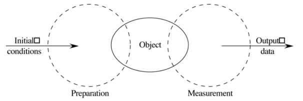








How does the physicist choose the class of g?



$$g_{meas}(x_1, \ldots, x_n) \to \mathbb{R}$$

How can you satisfy A0–A2? You must solve:

$$\exists \lambda_i \geq 0 \text{ such that } -1 - \sum_{i=1}^{|\mathscr{G}|} \lambda_i g_i(x) \geq 0 \qquad \forall x \in \Omega$$

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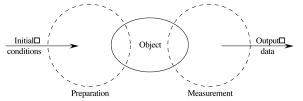

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Assessing the nonnegativity/negativity of a **generic gamble** when Ω is infinite is **undecideable**.

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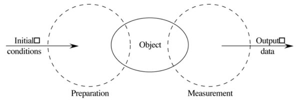

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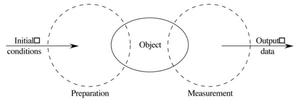

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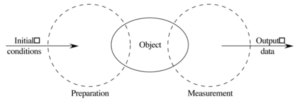

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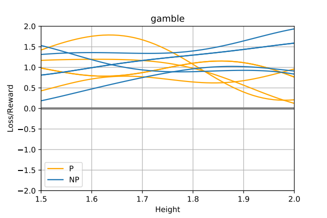
CPT, when Ω is infinite, is either **undecideable** or **NP-hard**!

Computational Rationality

A gamble is `nonnegative/negative` *if it is nonnegative/negative and its nonnegativity/negativity can be assessed efficiently (in* **PTIME***).*



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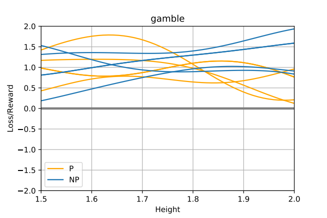

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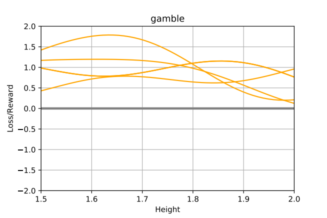

B1: **Tautologies** if $g \in \Sigma^{\geq}$ then $g \in \mathscr{G}$

Computational rationality axioms and dual

You are a computationally rational agent if

B0 (*Tautologies*) : Accepts all `nonnegative` gambles [if $g \geq 0$ and assessable in **PTIME** then $g \in \mathscr{C}$]

B1 (*Deductive closure*): Same as A1.

B2 (*P-coherent*): The set of gambles you accept does not include a `negative` function [if $g < 0$ and assessable in **PTIME** then $g \notin \mathscr{C}$]

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The dual of $\mathcal{C}$ is defined as:

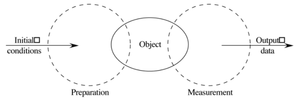
$$\mathcal{C}^\circ = \{L \in \mathsf{S} \mid L(g) \geq 0, \ \forall g \in \mathcal{C}\}, \tag{3}$$

where $\mathsf{S} = \{L \in \mathscr{L}^* \mid L(1) = 1, \ \ L(h) \geq 0 \ \ \forall h \in \Sigma^{\geq}\}$ is the set of (belief) *states*.
Equivalently, its extension to charges:

$$\mathcal{C}^\bullet = \left\{\mu \in \mathsf{S} \middle| \int_\Omega g(\omega) d\mu(\omega) \geq 0, \ \forall g \in \mathcal{C}\right\}, \tag{4}$$

where $\mathsf{S} = \{\mu \in \mathscr{M} \mid \ \int_\omega d\mu(\omega) = 1\}$ is the set of all charges.

Assessing rationality/irrationality can be done in **PTIME**



$$g_{meas}(x_1, \ldots, x_n) \to \mathbb{R}$$

Checking A2 for $\mathcal{K}$:

rationality $$-1 - \sum_{i=1}^{|\mathscr{G}|} \lambda_i g_i \in \mathscr{L}^{\geq}$$

Checking B2 for $\mathcal{C}$:

computational rationality $$-1 - \sum_{i=1}^{|\mathscr{G}|} \lambda_i g_i \in \Sigma^{\geq}$$

The latter can be computed in polynomial time.

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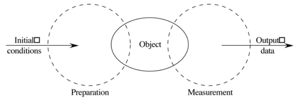

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The latter can be computed in polynomial time.

Being computationally irrational implies being irrational
Being computationally rational doesn't imply being rational

The Weirdness theorem

Theorem

Assume that $\Sigma^{\geq}$ includes all positive constant gambles and that it is closed (in $\mathscr{L}$). Denote by $\Sigma^{<}$ the interior of $-\Sigma^{\geq}$. Let $\mathcal{C} \subseteq \mathscr{L}$ be set of desirable gambles satisfying B0–B2. The following statements are equivalent:

- $\mathcal{C}$ *includes a negative gamble that is not in* $\Sigma^{<}$.
- $\mathrm{posi}(\mathscr{L}^{\geq} \cup \mathscr{G})$ *is incoherent, and thus* $\mathscr{P}$ *is empty.*
- $\mathcal{C}^{\circ}$ *is not (the restriction to* $\mathscr{L}$ *of) a closed convex set of mixtures of classical evaluation functionals.*
- *The extension* $\mathcal{C}^{\bullet}$ *of* $\mathcal{C}^{\circ}$ *in the space* $\mathscr{M}$ *of all charges in* Ω *includes only non-probabilistic charges (those with some negative value).*

QT is an instance of computational rationality

For a single particle (n-level system), the possibility space (phase-space) is

$$\overline{\mathbb{C}}^n := \{x \in \mathbb{C}^n : \; x^\dagger x = 1\}.$$

Stern-Gerlach: $\tilde{x} \in \overline{\mathbb{C}}^n$ represents the "direction" of the spin.

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For m particles, the possibility space is the Cartesian product $\Omega = \times_{j=1}^m \overline{\mathbb{C}}^{n_j}$.
The gambles are **polynomials**

$$g(x_1, \ldots, x_m) \in \operatorname{span}\left(\prod_{i=1}^m g(x_i)\right) = (\otimes_{j=1}^m x_j)^\dagger G (\otimes_{j=1}^m x_j) \quad G \text{ is Hermitian.}$$

Evaluating the nonnegativity of g is **NP-hard**.

QT is an instance of computational rationality

We thus redefine the meaning of being nonnegative as follows:

A gamble $g(x_1, \ldots, x_m)$ is said to be `nonnegative`*, resp.* `negative`*, if the matrix G is positive-semi, resp. negative, definite.*

We can then prove a duality between

(a) sets of gambles satisfying B0–B2 and

(b) density matrices:

$$\mathcal{Q} = \{\rho \in \mathcal{S} \mid Tr(G\rho) \geq 0, \quad \forall g \in \mathscr{G}\},$$

where $\mathcal{S} = \{\rho \in \mathscr{H}^{n \times n} \mid \rho \geq 0, \quad Tr(\rho) = 1\}$ is the set of all density matrices, $L(g) = Tr(G\rho)$ and $\rho = L((\otimes_{j=1}^{m} x_j)(\otimes_{j=1}^{m} x_j)^{\dagger})$ is a generalised moment matrix.

Example

Consider the case $n = m = 2$, then

$$L\left((x_1 \otimes x_2)^{\dagger} G(x_1 \otimes x_2)\right) = Tr\left(GL\left((\otimes_{j=1}^{2} x_j)(\otimes_{j=1}^{2} x_j)^{\dagger}\right)\right)$$

this follows by the linearity of the trace operator. Note that,

$$L\left((\otimes_{j=1}^{2} x_j)(\otimes_{j=1}^{2} x_j)^{\dagger}\right) = L\left(\begin{bmatrix} x_{11}x_{11}^{\dagger}x_{21}x_{21}^{\dagger} & x_{11}^{\dagger}x_{12}x_{21}x_{21}^{\dagger} & x_{11}x_{11}^{\dagger}x_{21}^{\dagger}x_{22} & x_{11}^{\dagger}x_{12}x_{21}^{\dagger}x_{22} \\ x_{11}x_{12}^{\dagger}x_{21}x_{21}^{\dagger} & x_{12}x_{12}^{\dagger}x_{21}x_{21}^{\dagger} & x_{11}x_{12}^{\dagger}x_{21}^{\dagger}x_{22} & x_{12}x_{12}^{\dagger}x_{21}^{\dagger}x_{22} \\ x_{11}x_{11}^{\dagger}x_{21}x_{22}^{\dagger} & x_{11}^{\dagger}x_{12}x_{21}x_{22}^{\dagger} & x_{11}x_{11}^{\dagger}x_{22}x_{22}^{\dagger} & x_{11}^{\dagger}x_{12}x_{22}x_{22}^{\dagger} \\ x_{11}x_{12}^{\dagger}x_{21}x_{22}^{\dagger} & x_{12}x_{12}^{\dagger}x_{21}x_{22}^{\dagger} & x_{11}x_{12}^{\dagger}x_{22}x_{22}^{\dagger} & x_{12}x_{12}^{\dagger}x_{22}x_{22}^{\dagger} \end{bmatrix}\right)$$

we can then define:

$$\rho := L\left((\otimes_{j=1}^{2} x_j)(\otimes_{j=1}^{2} x_j)^{\dagger}\right) = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12}^{\dagger} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13}^{\dagger} & \rho_{23}^{\dagger} & \rho_{33} & \rho_{34} \\ \rho_{14}^{\dagger} & \rho_{24}^{\dagger} & \rho_{34}^{\dagger} & \rho_{44} \end{bmatrix},$$

$\rho_{11} = L(x_{11}x_{11}^{\dagger}x_{21}x_{21}^{\dagger}) \in \mathbb{C}$, $\rho_{12} = L(x_{11}^{\dagger}x_{12}x_{21}x_{21}^{\dagger}) \in \mathbb{C}$, etc.

Some interesting facts

For the single particle case:

$$\rho = L\left(xx^\dagger\right) = \int_{x\in\Omega} xx^\dagger d\mu(x)$$

therefore, ρ is a moment matrix (covariance matrix) as μ is a probability charge.

For the $m > 1$ case:

$$\rho = L\left((\otimes_{j=1}^{2} x_j)(\otimes_{j=1}^{2} x_j)^\dagger\right) = \int_{x\in\Omega} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1^\dagger & x_2^\dagger \end{bmatrix} d\mu(x_1, x_2)$$

then $\mu(x_1, x_2)$ is just a charge (it may have negative values). This follows by the Weirdness theorem

$$-1 - \sum_{i=1}^{|\mathscr{G}|} \lambda_i g_i \in \mathscr{L}^{\geq} \quad vs. \quad -1 - \sum_{i=1}^{|\mathscr{G}|} \lambda_i g_i \in \Sigma^{\geq}$$

Non-additive probability

Claim 2: CPT is the only theory in the framework with unique decompositions of all mixed states.

Claim 3: In contrast, when the set of pure states is an overcomplete basis there is linear dependence amongst the pure states

These claims are not true for CPT!

For a single particle, for a given ρ, there may exist two probability distributions $\mu_1(x) \neq \mu_2(x)$ such that

$$\rho = \int_{x \in \Omega} xx^\dagger d\mu_1(x) = \int_{x \in \Omega} xx^\dagger d\mu_2(x).$$

What is a Bell/CHSH experiment?

A CHSH experiment is a preparation procedure for which a `negative` but non `negative` gamble has positive expectation:

$$h(x,y) = (x\otimes y)^{\dagger}(G_{\alpha_1}\otimes(G_{\beta_2}-G_{\beta_2}))(x\otimes y)+(x\otimes y)^{\dagger}(G_{\alpha_2}\otimes(G_{\beta_1}+G_{\beta_2}))(x\otimes y)\leq 2$$

therefore, $h(x,y)-2-\varepsilon<0$, but when we perform the experiment and we measure the expectation of $h(x,y)$:

$$\begin{aligned}
&L_{ent}(h(x,y)-2-\varepsilon)\\
&=-2-\varepsilon+Tr(\rho_{ent}((G_{\alpha_1}\otimes(G_{\beta_2}-G_{\beta_2}))+(G_{\alpha_2}\otimes(G_{\beta_1}+G_{\beta_2})))\\
&=-2-\varepsilon+2\sqrt{2}\geq 0.
\end{aligned}$$

Being computationally rational doesn't imply being rational

Thank you for your attention

- A Benavoli, A Facchini, M Zaffalon “**Connecting classical finite exchangeability to quantum theory and indistinguishability**”
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  Physical Review A 94.4 (2016).
- “A Gleason-type theorem for any dimension based on a gambling formulation of Quantum Mechanics.” Found. of Physics (2017) 47: 991–1002.
- “Quantum rational preferences and desirability.” Proc. of the 1st International Workshop on “Imperfect Decision Makers: Admitting Real-World Rationality”, NIPS 2016.

The crux: the tensor product

Cone of real symmetric PSD matrices

$$\begin{bmatrix} x_1 & x_2 \\ & x_3 \end{bmatrix} \otimes \begin{bmatrix} x_4 & x_5 \\ & x_6 \end{bmatrix} \neq \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ & z_5 & z_6 & z_7 \\ & & z_8 & z_9 \\ & & & z_{10} \end{bmatrix}$$

dim of $\otimes$ is $3 \times 3 = 9$, dimension of 4×4 PSD matrices is 10.
Cone of complex Hermitian PSD matrices

$$\begin{bmatrix} x_1 & x_2 + \imath a_2 \\ & x_3 \end{bmatrix} \otimes \begin{bmatrix} x_4 & x_5 + \imath a_5 \\ & x_6 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 + \imath b_2 & z_3 + \imath b_3 & z_4 + \imath b_4 \\ & z_5 & z_6 + \imath b_6 & z_7 + \imath b_7 \\ & & z_8 & z_9 + \imath b_9 \\ & & & z_{10} \end{bmatrix}$$

dim of $\otimes$ is $4 \times 4 = 16$, dimension of 4×4 PSD matrices is 16.

Unitary evolution

At time t_0, the Subject declares which gambles G_0 she accepts. Bookie can perform the experiment at any time $t_1 \geq t_0$. After having performed the experiment, Bookie determines Subject's payoff by "moving time-forward" Subject's gambles ($\mathscr{G}_0 \overset{\phi}{\hookrightarrow} \mathscr{G}_1$), using a map $\phi(\cdot, t_1, t_0)$ from $\mathcal{C}$ to itself that satisfies the following properties:

(i) $\phi(\cdot, t_0, t_0)$ is the identity map;

(ii) $\phi(\cdot, t_1, t_0)$ is onto;

(iii) $\phi(\Sigma^{\geq}, t_1, t_0) = \Sigma^{\geq}$;

(iv) $\phi(\cdot, t_1, t_0)$ is linear and constant preserving.

Channels generalise the above desideratum but with completely positive maps which allows one to model 'dissipation'.