Developing a general theory for statistical ensembles

Gabriele Carcassi, Christine A. Aidala, Tobias Thrien

Physics Department
University of Michigan

UNIVERSITY OF MICHIGAN

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Assumptions of Physics

Main goal of the program

Assumptions of Physics

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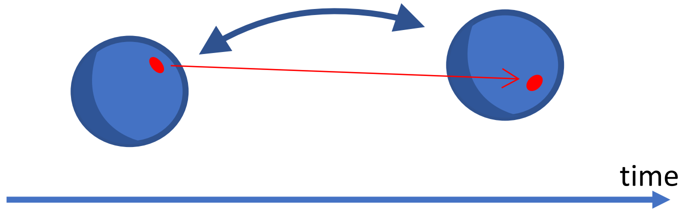
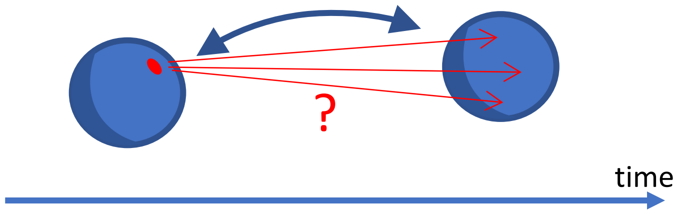
Identify a handful of physical starting points from which the basic laws can be rigorously derived

For example:

Infinitesimal reducibility ⇒ Classical state



Irreducibility ⇒ Quantum state



This also requires rederiving all mathematical structures from physical requirements

For example:

Science is evidence based ⇒ scientific theory must be characterized by experimentally verifiable statements ⇒ topologies and σ-algebras

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Assumptions of Physics

Break physics into different conditions and study their logical relationships

Reverse Physics

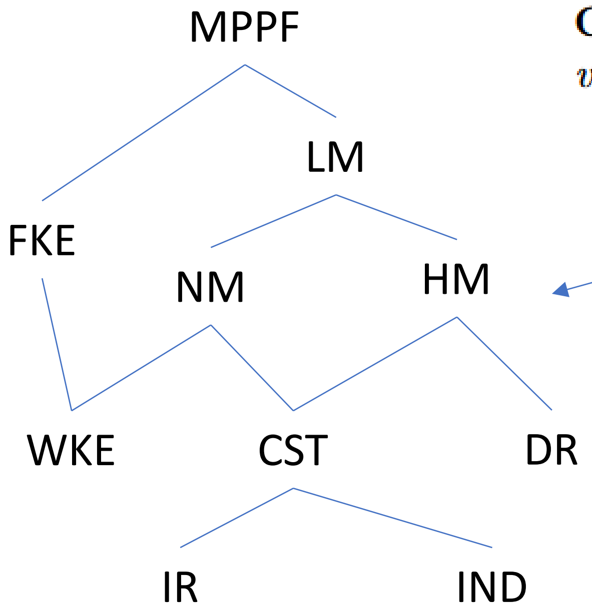
"When the math is derived from the right physical assumptions, the physical assumptions can be derived from the math."

Similar to what is done in the foundations of mathematics

Condition DR-SCEQ (Schrödinger equation). *The evolution follows the equation* $i\hbar\frac{d}{dt}\psi(t) = H\psi(t)$ *where* H *is a self-adjoint operator.*

$\Longleftrightarrow$

Condition DR-INFO (Information preservation). *The evolution is differentiable and preserves von Neumann entropy:* $S(\mathcal{T}(\rho)) = S(\rho)$ *for all* $\rho \in E(\mathcal{H})$.



Each assumption is an equivalence class of different conditions

Core assumptions for classical mechanics

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Assumptions of Physics

Base theory for all of physics

Collection of conditions that MUST be true in all physical theories

Condition [B]MUTEX (Mutual exclusivity conditions). *Let $\psi_i \in P(\mathcal{H})$ be a countable sequence of states. Then $p(\psi_i|\psi_j) = 0$ for all $i \neq j$ if and only if $S(\sum_i p_i \psi_i) = -\sum_i p_i \log p_i$ for all p_i such that $\sum_i p_i = 1$.*

Base theory ≠ Theory of everything

Reverse Physics only works on theories we have,
which is not enough!

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Assumptions of Physics

Start from mathematical axioms with a clear physical justification

Axioms must represent constitutive assumptions

Physical Mathematics

"When physical objects are mapped to the right mathematical objects, the physical specification maps to the mathematical definitions."

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called **mixing**, noted with the infix notation $p\mathsf{a} + \bar{p}\mathsf{b}$, with the following properties:*

- ***Continuity**: the map $+(p, \mathsf{a}, \mathsf{b}) \to p\mathsf{a} + \bar{p}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
- ***Identity**: $1\mathsf{a} + 0\mathsf{b} = \mathsf{a}$*
- ***Idempotence**: $p\mathsf{a} + \bar{p}\mathsf{a} = \mathsf{a}$ for all $p \in [0,1]$*
- ***Commutativity**: $p\mathsf{a} + \bar{p}\mathsf{b} = \bar{p}\mathsf{b} + p\mathsf{a}$ for all $p \in [0,1]$*
- ***Associativity**: $p_1\mathsf{e}_1 + \bar{p}_1\left(\left(\overline{\frac{p_3}{\bar{p}_1}}\right)\mathsf{e}_2 + \frac{p_3}{\bar{p}_1}\mathsf{e}_3\right) = \bar{p}_3\left(\frac{p_1}{\bar{p}_3}\mathsf{e}_1 + \left(\overline{\frac{p_1}{\bar{p}_3}}\right)\mathsf{e}_2\right) + p_3\mathsf{e}_3$ where $p_1 + p_3 \leq 1$*

Justification. This axiom captures the ability to create a mixture merely by selecting between the output of different processes. Let e_1 and e_2 be two ensembles that represent the output of two different processes P_1 and P_2. Let a selector S_p be a process that outputs two symbols, the first with probability p and the second with probability $\bar{p}$. Then we can create another process P that, depending on the selector, outputs either the output of P_1 or P_2. All possible preparations of such a procedure will form an ensemble. Therefore we are justified in equipping an ensemble space with a mixing operation that takes a real number from zero to one, and two ensembles.

Given that mixing represents an experimental relationship, and all experimental rela-

Informal intuitive requirement
(something that makes sense to a physicist or an engineer)

Formal definition
(something a mathematician will find precise)

Show that the formal definition follows from the intuitive requirement

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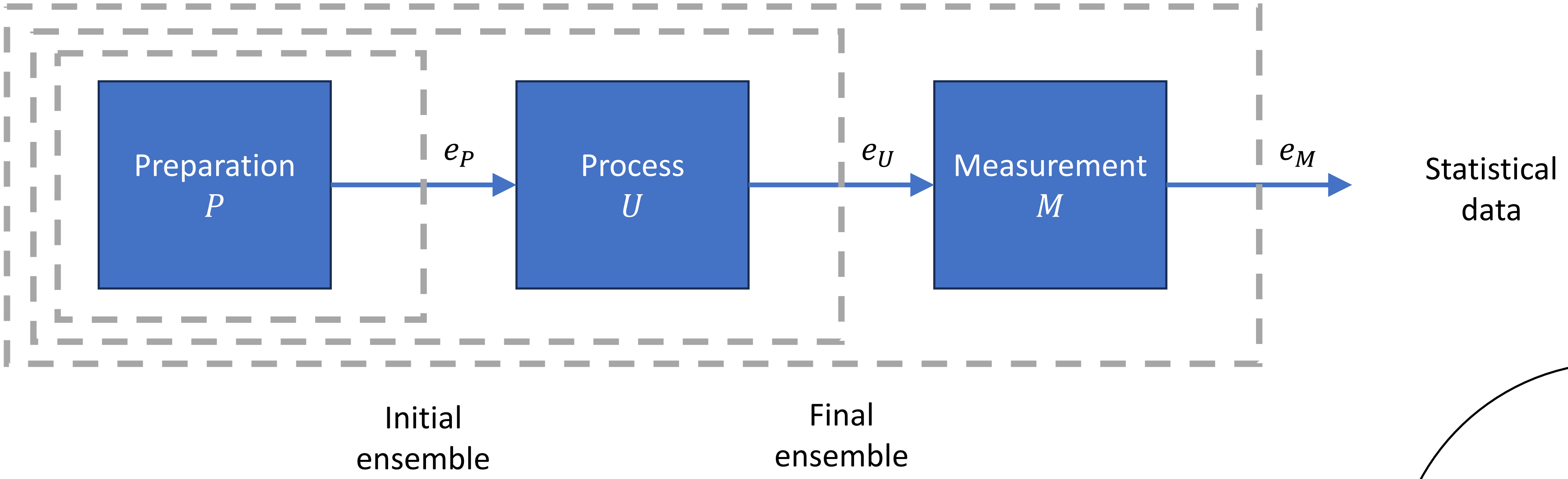
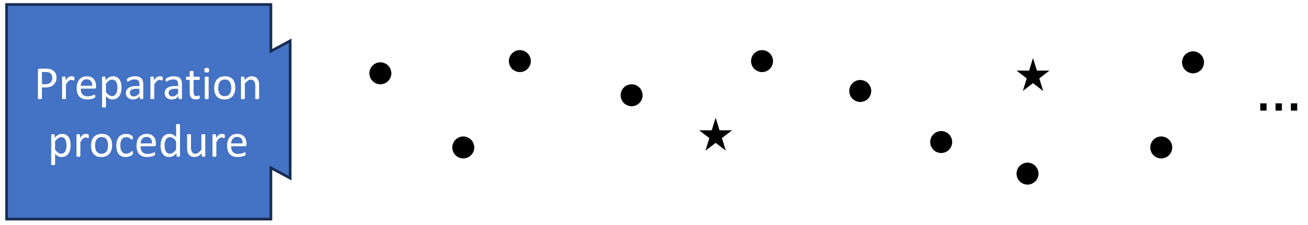
Assumptions of Physics

> **Principle of scientific reproducibility.** Scientific laws describe relationships that can always be experimentally reproduced.

⇒ Scientific laws are relationships between ensembles





Ensemble of outputs

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Minimal requirements for an ensemble space

- Experimental verifiability
  - Verifiable statement: statement with a test that terminates successfully in finite time if and only if the statement is true ⇒ topology and σ-algebra
- Statistical mixing
  - $pa + (1 - p)b$ ⇒ Convex structure
- Ensemble variability
  - Outputs of preparations may not be identical ⇒ entropy, geometric structures
- Ensemble equilibria
  - Ensembles are symmetries of processes ⇒ Lie algebra, Poisson structure

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Assumptions of Physics

Goals for the workshop

- What is the relationship between probability as used in physics and as used in a more generalized sense?
  - Probability is/can be applied as an inference tool to non-repeatable events (e.g. political elections)
  - Probability can be defined on spaces that lack physical structures (e.g. we need a symplectic structure to define classical phase space and Hamiltonian evolution, but not to define probability measures)
    - Can the converse happen?
  - Conjecture: if probabilities are defined in terms of bets, and we are applying those bets to physical systems, is the physical knowledge of the state space EXACTLY the knowledge of all the sure bets?
  - Given that the entropy plays a fundamental role in characterizing ensemble spaces, is that knowledge equivalent to the entropic structure?

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Assumptions of Physics

Goals for the workshop

- Can we use non-additive measures to characterize statistical ensembles when mutual exclusivity cannot be assumed?
  - In Kolmogorov probability, additivity is equivalent to assuming that all elements of the sample space are mutually exclusive
  - In classical mechanics, additivity of the Liouville measure (i.e. count of states) is equivalent to assuming states are mutually exclusive
  - Sub-additive measures may be used to describe the case where mutual exclusivity is lost: there is some "overlap" even between the "pure" ensembles
  - What integral/derivative can we use to generalize probability calculus?

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Assumptions of Physics

Goals for the workshop

- How can we generalize the Lie algebraic structures of classical (Poisson brackets) and quantum (commutators) mechanics, and what independent structures are they describing?
  - What is the right Poisson structure to put on the space of ensembles so that the standard ones are recovered in the classical and quantum case?
  - Should we understand the elements of the structure as observables (i.e. linear functionals) or as generators (i.e. derivatives of one parameter groups)?
  - What is the right product for the generalized Poisson structure?

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Assumptions of Physics

Goals for the workshop

- How does all of this work in infinite dimensional spaces and field theories?
  - What is the right probability space for function spaces? Are there constraints on the function space?
  - The linear and entropic structures do not seem to be enough to characterize infinite dimensional spaces
    - $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^n)$ are isomorphic as LCTVS with an inner product
  - The Lie algebras seem to be necessary
    - The number of independent generators for $\frac{[X^i, P_j]}{\imath\hbar} = \delta^i_j I$ depend on the dimension, and a linear operation cannot change the number of generators
  - Is there a way to understand the Lie algebra as properties of the probability calculus, and relate them to rules of inference?

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Basic structures

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	State space	Ensembles
Classical discrete	$X = \{x_1, x_2, \dots\}$	$\mathcal{E} = \{ p_i \mid \sum_i p_i = 1\}$
Classical continuum	Phase space Symplectic manifold $X = \{\mathbb{R}^{2n}, \omega\}$	$\mathcal{E} = \{ \rho \in L^1(\mathbb{R}^{2n}) \mid \int \rho dq^n dp_n = 1\}$
Quantum mechanics	Projective complex Hilbert space $X = P(\mathcal{H})$	$\mathcal{E} = \{$ positive semi-definite Hermitian with $\mathrm{tr}(\rho) = 1$ $\}$

Concrete examples

with some caveats

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Assumptions of Physics

Topology and convex structure

Experimental verifiability justifies ensemble spaces to be T_0 second-countable topological spaces

Axiom 1.4 (Axiom of ensemble). *The state of a system is given by an* ***ensemble***, *which represents the collection of all possible outputs of a preparation procedure for a physical system. The set of all possible ensembles for a physical system is its* ***ensemble space***. *Formally, an ensemble space is a T_0 second countable topological space where each element is called an ensemble.*

Statistical mixing justifies ensemble spaces to have a continuous convex structure

Axiom 1.7 (Axiom of mixture). *The statistical mixture of two ensembles is an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with an operation $+ : [0,1] \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ called* ***mixing***, *noted with the infix notation $\mathsf{pa} + \bar{p}\mathsf{b}$, with the following properties:*

- ***Continuity***: *the map $+(p, \mathsf{a}, \mathsf{b}) \to p\mathsf{a} + \bar{p}\mathsf{b}$ is continuous (with respect to the product topology of $[0,1] \times \mathcal{E} \times \mathcal{E}$)*
- ***Identity***: $1\mathsf{a} + 0\mathsf{b} = \mathsf{a}$
- ***Idempotence***: $p\mathsf{a} + \bar{p}\mathsf{a} = \mathsf{a}$ *for all* $p \in [0,1]$
- ***Commutativity***: $p\mathsf{a} + \bar{p}\mathsf{b} = \bar{p}\mathsf{b} + p\mathsf{a}$ *for all* $p \in [0,1]$
- ***Associativity***: $p_1\mathsf{e}_1 + \bar{p}_1\left(\frac{p_2}{\bar{p}_1}\mathsf{e}_2 + \frac{p_3}{\bar{p}_1}\mathsf{e}_3\right) = \bar{p}_3\left(\frac{p_1}{\bar{p}_3}\mathsf{e}_1 + \frac{p_2}{\bar{p}_3}\mathsf{e}_2\right) + p_3\mathsf{e}_3$ *where* $p_1, p_3 \in [0,1)$ *and* $p_1 + p_3 \leq 1$ *and* $p_2 = 1 - p_1 - p_3$

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Topological convex spaces (TCS)

- Modern analysis is based on Topological Vector Spaces (TVS) but there is no work on Topological Convex Spaces (TCS). A lot of our problems are understanding how to relate the two.
- The algebraic structure only defines finite convex combinations $\sum_{i=1}^{n} p_i e_i$. The topology defined infinite convex combinations $\sum_{i=1}^{\infty} p_i e_i$in terms of limits of convergent sequences.
- Open problem: define a notion of convex integral $\int_{\mathcal{E}} e dp$

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Assumptions of Physics

Some terminology

$$e = \sum_i p_i a_i$$

mixture mixing coefficient components

$$\bar{p} = 1 - p$$

$$a = pc + \bar{p}d \quad b = \lambda c + \bar{\lambda} e$$

have a common component

$$a ⫪ b$$

separate, don't have a common component

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Assumptions of Physics

More terminology

- An extreme point e is a point that can only be expressed as a trivial mixture (i.e. $e = pa + \bar{p}b \Rightarrow a = b = e$)
- A set A is
  - Convex if it is closed under finite convex combinations (i.e. contains all its mixtures)
  - σ-convex if it is closed under infinite convex combinations (i.e. contains all infinite mixtures)
  - Closed and convex if it is convex and topologically closed (i.e. it contains all mixtures and their limits)
  - Componential if it contains all the components of its elements
  - Facial if it is componential and convex
- These definitions recover standard notions of convex geometry

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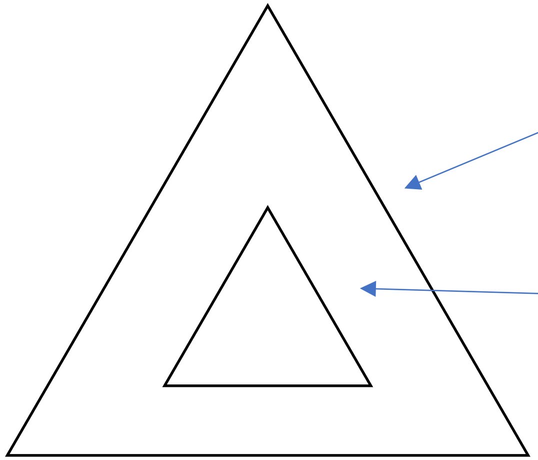
Limitations of TCS for probability/physics

- This structure is not enough for physics:
  - The space of probability measures over $\mathbb{R}^2$ is the same regardless of what volume measure (i.e. state count) is chosen: no unique Hamiltonian/statistical mechanics
- TCS can't tell if extreme points are mutually exclusive



Set of all probability distributions over 3 mutually exclusive events

Subset of distributions: vertices are NOT mutually exclusive

Also, expectation values are NOT enough to recover mutual exclusivity

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Assumptions of Physics

Entropy definition

Axiom 1.58 (Axiom of entropy). *Every ensemble is associated with an* ***entropy*** *which quantifies the variability of the instances within an ensemble. Formally, an ensemble space $\mathcal{E}$ is equipped with a function $S : \mathcal{E} \to \mathbb{R}$, defined up to a positive multiplicative constant representing the unit numerical value. The entropy has the following properties:*

- ***Continuity***[a]
- ***Strict concavity****:* $S(p\mathsf{a} + \bar{p}\mathsf{b}) \geq pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$ *with the equality holding if and only if* $\mathsf{a} = \mathsf{b}$
- ***Upper variability bound****: there exists a universal function* $I(p, \bar{p})$ *(i.e. the same for all ensemble spaces) such that* $S(p\mathsf{a} + \bar{p}\mathsf{b}) \leq I(p, \bar{p}) + pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$*; if the equality holds,* a *and* b *are* ***mutually exclusive*** *or* ***orthogonal****, noted* $\mathsf{a} \perp \mathsf{b}$
- ***Mixtures preserve orthogonality****:*[b] $\mathsf{a} \perp \mathsf{b}$ *and* $\mathsf{a} \perp \mathsf{c}$ *if and only if* $\mathsf{a} \perp p\mathsf{b} + \bar{p}\mathsf{c}$ *for any* $p \in (0, 1)$

Small ensemble changes, small variability changes

Average variability cannot decrease during mixtures

Maximal increase with mutual exclusivity

Statistical mixing preserves mutual exclusivity

May not be ALL the properties we need… but enough for many things

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Assumptions of Physics

Entropy upper bound uniqueness

Theorem 1.62 (Uniqueness of entropy). *If there exists an ensemble space with an infinite set of orthogonal ensembles, then the entropy of the coefficients $I(p, \bar{p})$ is the Shannon entropy. That is, $I(p, \bar{p}) = -\kappa (p \log p + \bar{p} \log \bar{p})$ where $\kappa > 0$ is the arbitrary multiplicative constant for the entropy. For a mixture of arbitrarily many elements, $I(\{p_i\}) = -\kappa \sum_i p_i \log p_i$.*

Shannon entropy is recovered as maximal increase

We have a Shannon-type entropy without having defined what the actual states are, only their maximum increase during mixing.

Typical derivations of entropy use product spaces, knowledge of the "points," etc… This proof only uses decomposability over mutually exclusive preparations.

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Assumptions of Physics

Constraints from the entropy

- Entropy forces the topology to be Hausdorff
  - Open problem: does it force it to be metrizable?
- Entropy forces the ensemble space to be cancellative
  - Affine combinations, if exist, are unique
  - Ensemble spaces embed in vector spaces
- A set A is
  - Affine if it is closed under affine combinations
  - Closed and affine if it is affine and topologically closed
- Can define
  - Affine independence (no elements of a set A can be expressed as an affine combination of other elements of A)
  - Dimension (number of affine elements that can generate the set)

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Assumptions of Physics

Constraints from the entropy

- Finite-dimensional affine subsets (still finalizing proofs)
  - Embed continuously in TVS
  - Are bounded
- Infinite-dimensional affine subsets
  - May not embed continuously in TVS (strawman counterexample found)
  - Will not be bounded (physically meaningful counterexample)

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Assumptions of Physics

Statistical quantities (affine functionals)

- A statistical property is a continuous affine map $F : \mathcal{E} \to \mathcal{Q}$ where $\mathcal{Q}$ is a convex space
  - That is, $F(pa + \bar{p}b) = pF(a) + \bar{p}F(b)$
- A statistical quantity is a continuous affine map $F : \mathcal{E} \to \mathbb{R}$
  - The space of statistical quantities it the topological dual
  - Not all statistical quantities are expectations of a random variable/observable
    - Open problem: how do we characterize the difference?
- Quantifiable ensemble space: there is a family F_i of statistical quantities that generate the topology
  - Since the topology is Hausdorff, they separate points
  - Conjecture: an ensemble space embeds continuously in a LCTVS iff it is a quantifiable ensemble space

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Geometry from the entropy

- The quantity $MS(a,b) = S\left(\frac{1}{2}a + \frac{1}{2}b\right) - \frac{1}{2}S(a) - \frac{1}{2}S(b)$ is a pseudo-distance (no triangle inequality). It recovers classical and quantum JSD. Its open balsl generate, in the classical case, the total variation/L^1 topology
  - Warning: that topology does not guarantee continuity of the entropy in infinite dimensional spaces!
- The quantity $g|_e(da, db) = -\left.\frac{\partial^2 S}{\partial e^2}\right|_e (da, db)$ is a symmetric, positive-definite function of two variations (within a face where e is a relative internal point) and recovers classical Fisher-Rao and quantum BKM metrics

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Assumptions of Physics

Two ways to define "non-overlapping" ensembles

Separate ensembles

$$\mathsf{a} \;⫪\; \mathsf{b}$$

No "common component" $\mathsf{c} \in \mathcal{E}$

such that
$$\mathsf{a} = p_1\mathsf{c} + \bar{p}_1\mathsf{e}_1$$
$$\mathsf{b} = p_2\mathsf{c} + \bar{p}_2\mathsf{e}_2$$

$$\Leftarrow$$

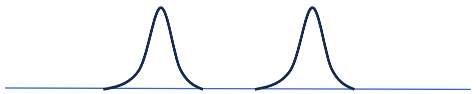
Orthogonal ensembles

$$\mathsf{a} \perp \mathsf{b}$$

Saturate upper entropy bound

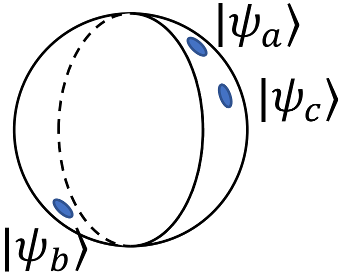
$$S(p\mathsf{a} + \bar{p}\mathsf{b}) = I(p, \bar{p}) + pS(\mathsf{a}) + \bar{p}S(\mathsf{b})$$

Coincide in classical ensemble spaces



disjoint support

Different in quantum ensemble spaces



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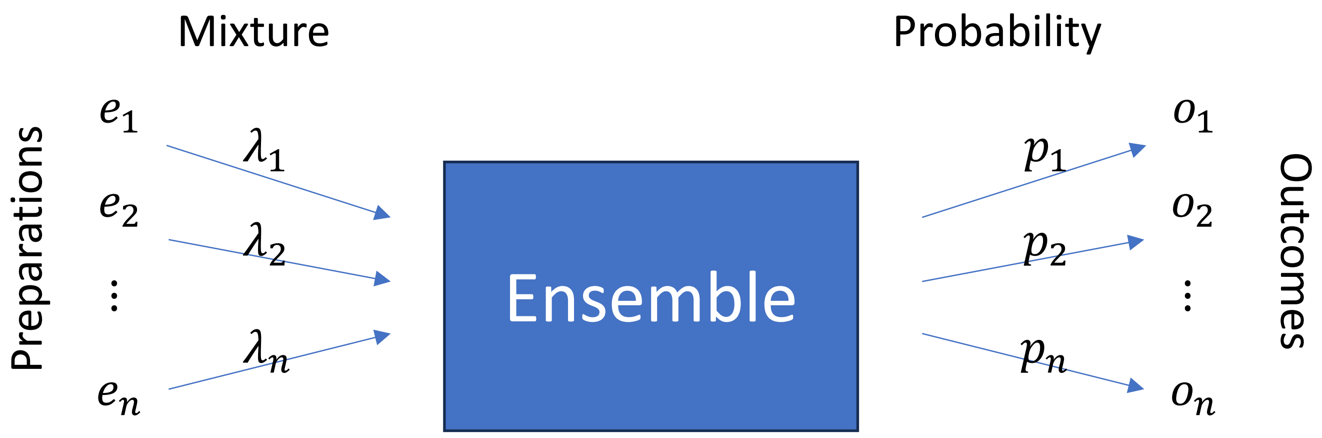
Assumptions of Physics

Mixture coefficient vs probability



In general, mixtures of preparations and probability of outcomes do NOT coincide

$$\text{e.g. } p\left(z^+ \middle| \frac{1}{2} z^+ + \frac{1}{2} x^+\right) \neq \frac{1}{2}$$

They coincide only if the ensembles are mutually exclusive

⇒ Entropy is necessary to transform mixture coefficients and expectation values into probability of outcomes (i.e. frequencies of outputs of some process)

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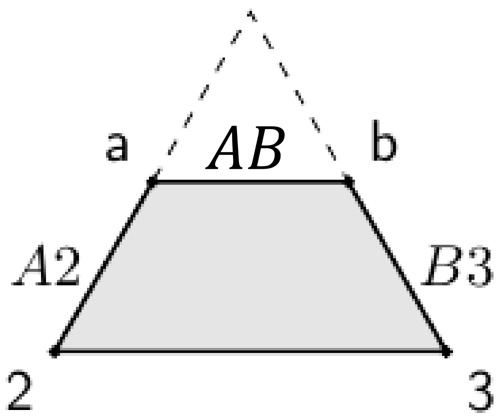
Assumptions of Physics

Orthogonal and separate subspaces

- Separateness and orthogonality are symmetric and irreflexive
- Given a symmetric irreflexive relationship R, we can define the R-complement
$$A^R = \{e \in \mathcal{E} | \forall a \in A, aRe\}$$
(e.g. all ensembles that are orthogonal to all elements of the original set)
- An R-subspace is a set such that $\left(A^R\right)^R = A$
(e.g. it's equal to the orthogonal complement of the orthogonal complement)
- This allows us to define separate and orthogonal subspaces



$$\{2\}^{\perp} = B3$$
$$\{2\}^{\pitchfork} = B3 \cup AB$$

Gabriele Carcassi - University of Michigan

Order structures – lattices

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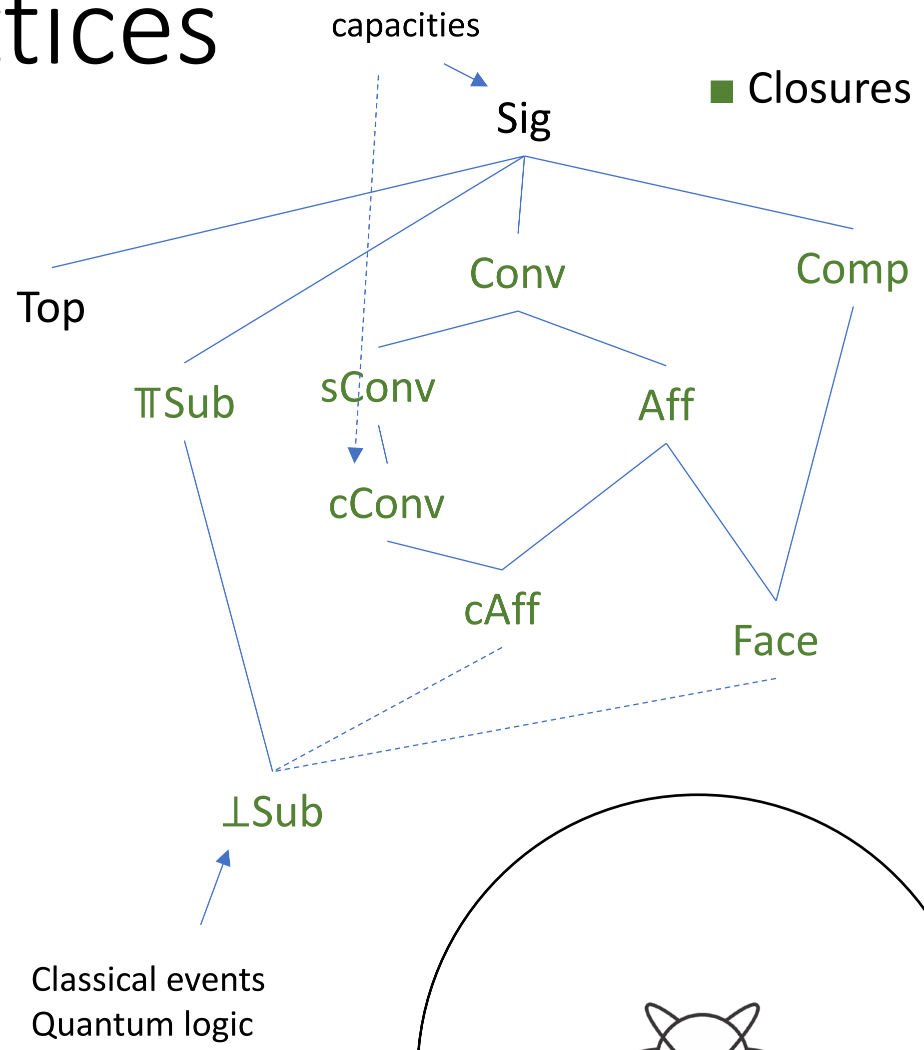
Assumptions of Physics

Several useful lattices

- Experimental verifiability
  - Topology on the ensemble space (verifiable statements)
    - "This coin is fair within .1%" "The average energy is 5 ± 0.01 eV" …
  - σ-algebra on the ensemble space (statements associated with a test)
    - "This coin is perfectly fair" "The average energy is a rational number in eV" …
- Mixing
  - Convex sets (closed under finite mixtures)
  - σ-convex sets (closed under infinite mixtures – possibly convex integrals)
  - Closed convex sets (closed under mixtures and limits)
- "Unmixing"
  - Affine sets (closed under finite mixing/removal)
  - Closed affine sets (closed under mixing/removal and limits)
- Components
  - Componential sets (contain all components of their elements)
  - Facial sets (contain all components and mixtures)
- Subspaces (orthogonality space constructions)
  - Separate (no common component) subspaces (contain all separate to its separate – $V = (V^{⫪})^{⫪}$)
  - Orthogonal (mutually exclusive) subspaces (contain all orthogonal to its orthogonal – $V = (V^{\perp})^{\perp}$)



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Assumptions of Physics

Subspace lattices in classical/quantum probability

- In classical probability, ⊥Sub and ⫪Sub are equivalent to the lattice of supports for probability distributions that are absolutely continuous with respect to the counting/Lebesgue/Liouville measure.
  - ⊥Sub=⫪Sub $\equiv L = \{A \in \Sigma_X | \mu(A) < \infty\}_{/\sim}$ where $A \sim \mathrm{B}$ iff $\mu(A \triangle B) = 0$
  - X is σ-finite $\Leftrightarrow L = (\Sigma_X)_{/\sim} \Rightarrow L$ generates Σ_X
- In fact, ⊥Sub equal to ⫪Sub if and only if $\mathcal{E}$ is a classical space
- In quantum probability, ⊥Sub and ⫪Sub are NOT equivalent.
  ⫪Sub is the lattice of closed subspaces (i.e. quantum logic lattice)

Beyond classical/quantum probability

- In general, ⊥Sub and ⫪Sub are not equivalent, and ⫪Sub is not orthomodular
  - Conjecture: ⫪Sub is orthomodular if the space is orthogonally decomposable (i.e. every decomposable ensemble is orthogonally decomposable)
- Currently, there is no guarantee that orthogonal elements even exist. In that case, ⫪Sub may be trivial. No “quantum logic type” lattice would exist.

Measures

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State counting measure

Hilbert space dimensionality

Probability measure

State capacity

Fraction capacity

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Assumptions of Physics

Probability measure and fraction capacity

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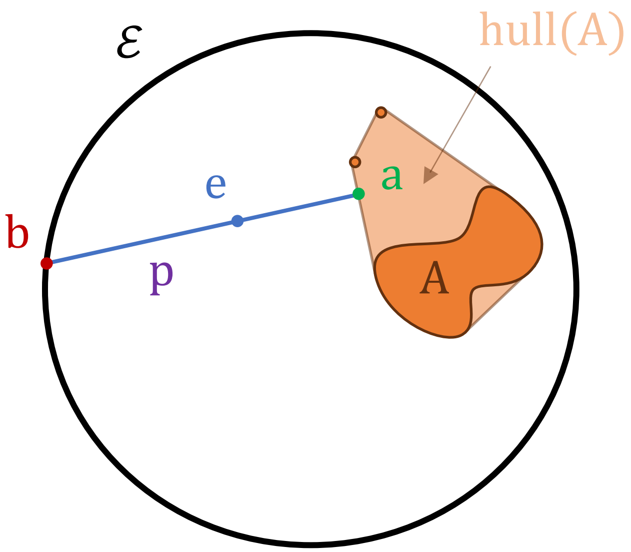
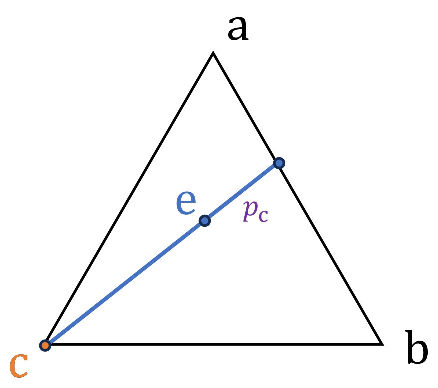
Assumptions of Physics

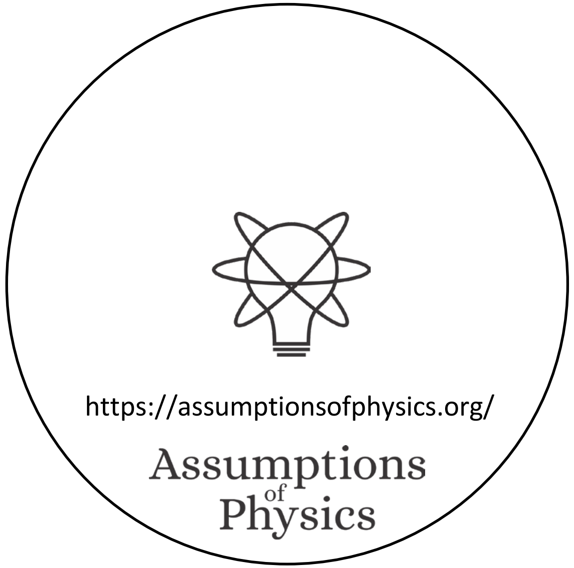
Fraction capacity

- Fix a target ensemble e and let A be a set of ensembles.
- What is the biggest fraction of e that can be expressed as a convex combination of elements of A?

$$\mathrm{fcap}_e(A) = \sup(\{p \in [0,1] \mid \exists a \in \mathrm{hull}(A), b \in \mathcal{E} \text{ s.t. } e = pa + (1-p)b\})$$







Properties of fraction capacity

- $\mathrm{fcap}_{\mathrm{e}} : \Sigma_{\mathcal{E}} \to [0,1]$ is
  - **increasing** : $A \subseteq B \Rightarrow \mathrm{fcap}_{\mathrm{e}}(A) \leq \mathrm{fcap}_{\mathrm{e}}(\mathrm{B})$
  - **subadditive**: $\mathrm{fcap}_{\mathrm{e}}(A \cup B) \leq \mathrm{fcap}_{\mathrm{e}}(A) + \mathrm{fcap}_{\mathrm{e}}(\mathrm{B})$
    - **additive** for orthogonal/mutually exclusive elements:
      if $A \perp B$ and $e \in \mathrm{hull}(A \cup B)$ then $\mathrm{fcap}_{e}(A \cup B) = \mathrm{fcap}_{e}(A) + \mathrm{fcap}_{e}(B)$
  - **continuous from below**
  - σ-subadditive?

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Assumptions of Physics

Connection to Choquet theory

- Choquet theory:
  - Suppose E is a compact convex subset of a LCTVS V. Let $X \subseteq E$ be the set of extreme points. Then every probability measure represents an element of E. That is, there exists an $e \in E$ such that $F(e) = \int_E F dp$ for all continuous linear functionals F.
  - First theorem. For every $e \in E$ there exists a probability measure p on X that represents e
  - Second theorem. The probability measure is unique if and only if E is a simplex
- Connection between fraction capacity and Choquet theory:
  - If $\mathrm{M_e}$ is the set of all (additive) probability measures on X that represent e, then $\mathrm{fcap_e}|_X(A) = \sup\{p(A) | \, p \in \mathrm{M_e}\}$
  - For a simplex there is only one and additivity is recovered

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Assumptions of Physics

Generalization of Choquet theory

- We need to generalize Choquet theory since
  - Ensemble spaces are not, in general, compact.
    - E.g. classical mechanics (probability measure of phase space)
  - Extreme points may not exist in the ensemble space
    - E.g. Dirac measure would have $-\infty$ entropy
- How do we get the “extreme points” and their σ-algebra?
  - Use the limit values of linear functionals?
  - Or should we simply use lattices?

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Assumptions of Physics

Generalization of Choquet theory

- How can we reformulate the idea of a simplex in a more general case?

Corollary 1.319. *The $\mathbin{\top\!\!\top}$-subspaces are facial if and only if mixtures preserve separateness. That is, the following are equivalent:*

1. *$\mathcal{E}$ is monodecomposable*
2. *mixtures preserve separateness in $\mathcal{E}$*
3. *every $\mathbin{\top\!\!\top}$-subspace is convex*
4. *every $\mathbin{\top\!\!\top}$-subspace is facial*

If $e = pa_1 + \bar{p}a_2 = \lambda b_1 + \bar{\lambda} b_2$, then a_1 must have a common component with either b_1 or b_2

If $e \mathbin{\top\!\!\top} a$ and $e \mathbin{\top\!\!\top} b$, then $e \mathbin{\top\!\!\top} pa + \bar{p}b$

Preliminary result

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Assumptions of Physics

Connection to imprecise probability?

- In the case where extreme points are definable, each ensemble can be understood as a collection of precise probabilities
  - Is this the upper probability in imprecise probability?
- Can we say that, in the classical case, a state corresponds to a single precise probability, while in quantum mechanics it corresponds to an imprecise probability, where the indeterminancy is exactly the impossibility of recovering the precise preparation procedure?

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Assumptions of Physics

Connection to imprecise probability?

- Differently from imprecise probability, the different probability measures represent the same outcome. That is, there is no observable/random variable/process that can tell the difference
  - The ensemble is exactly the equivalence class of preparation probabilities that always give the same outcome
- Can we say, in some precise sense, that the quantum indeterminacy is exactly that lost information?

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Assumptions of Physics

State counting measure and state capacity

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State capacity

- How much is the "spread" of a target ensemble e?
- Must be a function f of the entropy with the following properties:

Small change of entropy gives small change in spread

Spread must be positive

More spread, more entropy

- **Continuity**
- **Positivity**: $f : \mathbb{R} \to (0, +\infty)$
- **Increasing**: $x \leq y \implies f(x) \leq f(y)$
- **Subadditivity**: $f(S(p\mathsf{a} + \bar{p}\mathsf{b})) \leq f(S(\mathsf{a})) + f(S(\mathsf{b}))$
- **Saturable additivity**: if $\mathsf{a} \perp \mathsf{b}$ then we can find $p_* \in [0,1]$ such that $f(S(p_*\mathsf{a} + \bar{p}_*\mathsf{b})) = f(S(\mathsf{a})) + f(S(\mathsf{b}))$

Spread can at most sum, and it must sum under orthogonality

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Properties of the state capacity

- (Preliminary) The "spread," or ensemble volume, must be the exponential of the entropy

> **Proposition 1.183** (Uniqueness of exponential entropy). *Let $f : \mathbb{R} \to (0, +\infty)$ be a continuous increasing function such that $f(S(p\mathsf{a} + \bar{p}\mathsf{b})) \leq f(S(\mathsf{a})) + f(S(\mathsf{b}))$ for all $\mathsf{a}, \mathsf{b} \in \mathcal{E}$ and $p \in [0,1]$, with equality for some $p \in (0,1)$ whenever $\mathsf{a} \perp \mathsf{b}$. Then $f(S) = c \cdot 2^S$ for some $c > 0$.*

- State capacity of A is defined as
$\mathrm{scap}(A) = \sup\left(2^{S(\mathrm{hull}(A))} \cup \{0\}\right)$

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State capacity

- $\text{scap}: \Sigma_{\mathcal{E}} \to [0, \infty]$ is
  - **Increasing**: $A \subseteq B \Rightarrow \text{scap}(A) \leq \text{scap}(\text{B})$
  - **subadditive**: $\text{scap}(A \cup B) \leq \text{scap}(A) + \text{scap}(\text{B})$
    - **additive** for orthogonal/mutually exclusive elements: if $A \perp B$ then $\text{scap}(A \cup B) = \text{scap}(A) + \text{scap}(B)$
  - **continuous from below**
  - σ-subadditive?
- Recovers the counting measure and the Liouville measure in the classical case
  - That is, $\text{scap}(A) = \mu(A)$ if $A \in \perp\text{Sub}$
- Recovers the dimensionality of the subspace in the quantum case
  - That is, $\text{scap}(A) = \dim(A)$ if $A \in \perp\text{Sub}$

Non-additive probability calculus?

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Probability space: $(X, \Sigma_X, p: \Sigma_X \to [0,1])$

Sample space
What can happen

Events
What can be tested

Probability measure
Probability of positive outcome of a test

Probability in classical mechanics

State count: $\mu: \Sigma_X \to [0, +\infty]$

Count of states corresponding to each event

Absolutely continuous
Zero states $\Rightarrow$ zero probability

$$\Rightarrow \rho = \frac{dp}{d\mu}$$

Probability density
Radon-Nikodym derivative

Random variable: $f: X \to \mathbb{R}$

Assigns a value to every possible outcome

$$\Rightarrow E[f] = \int f \rho d\mu = \int f dp$$

Expectation
Average value of the random variable

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??? space: $(X, \Sigma_X, \mathrm{fcap}_e : \Sigma_X \to [0,1])$

Extreme points
Ideal preparations

Events
Sets of ideal preps

Fraction capacity
Fraction that can be prepared from a set of preparations

State count: $\mathrm{scap} : \Sigma_X \to [0, +\infty]$

Count of states corresponding to each event

Non-additive probability in general

Absolutely continuous
Zero states $\Rightarrow$ zero probability

$$\Rightarrow \rho = \frac{d\mathrm{fcap}_e}{d\mathrm{scap}}$$

Probability density
??? derivative

Statistical variable: $f : \mathcal{E} \to \mathbb{R}$

Assigns a value to ensembles and extreme points

$$\Rightarrow f(e) = (E) \int f \rho \, d\mathrm{scap} = (E) \int f \, d\mathrm{fcap}_e$$

Expectation

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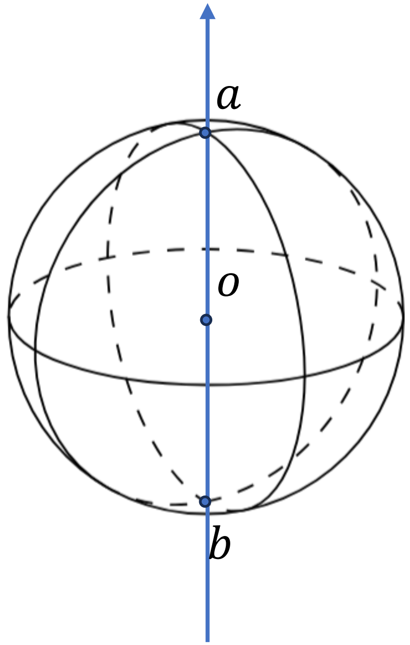
What integral?

- On a simplex, the integral must recover the standard integral
  - I.e. it must reduce to the Riemann integral in the additive case
- This rules out the **Sugeno** integral
- Quantum mechanics provides a counterexample for the **Choquet** integral
  - Simplest example: two-state system (e.g., spin ½, $c_+|\uparrow\rangle + c_-|\downarrow\rangle$) → Bloch ball
  - Any observable/linear functional F is defined by its values (a and b) at two points along an axis
    - e.g., $S[|\uparrow\rangle] = +\frac{1}{2}$ and $S[|\downarrow\rangle] = -\frac{1}{2}$
    - Everything else is a linear/affine combination of these

      ⇒ center o gets assigned $\frac{1}{2}a + \frac{1}{2}b$
  - The integral gives $\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}(b - a)\ln 2$



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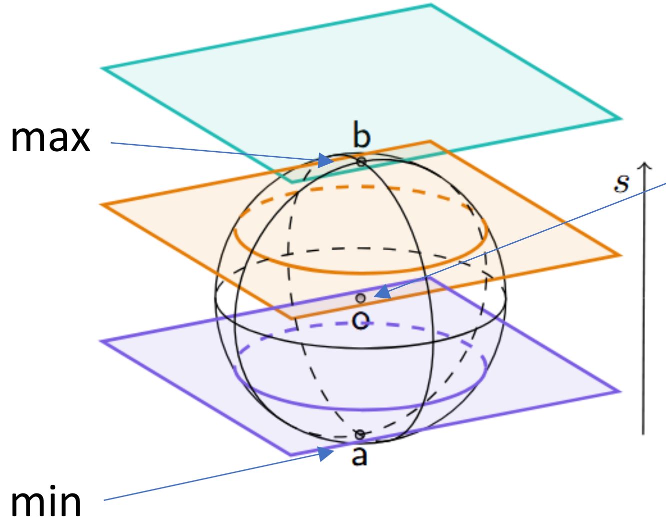
Assumptions of Physics

Counterexample for the Choquet integral

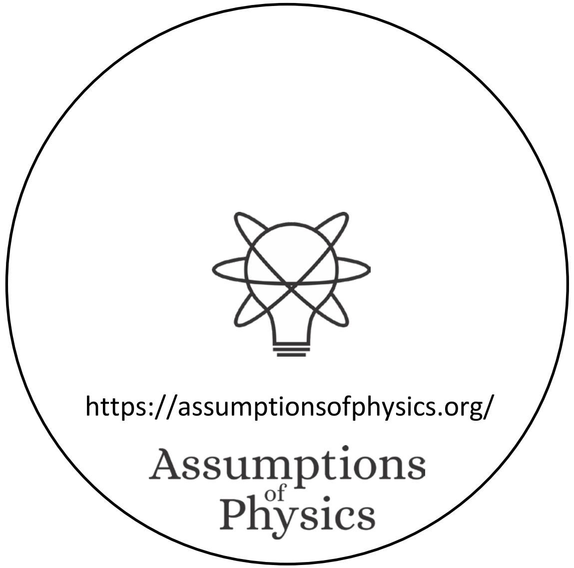
$$(C)\int F d\mathrm{fcap}_{\mathsf{o}} := \int_{-\infty}^{0} (\mathrm{fcap}_{\mathsf{o}}(\{x|F(x) \geq s\}) - \mathrm{fcap}_{\mathsf{o}}(X))\, ds$$
$$+ \int_{0}^{\infty} \mathrm{fcap}_{\mathsf{o}}(\{x|F(x) \geq s\})\, ds.$$

Want to integrate the fraction capacity of the center of the sphere

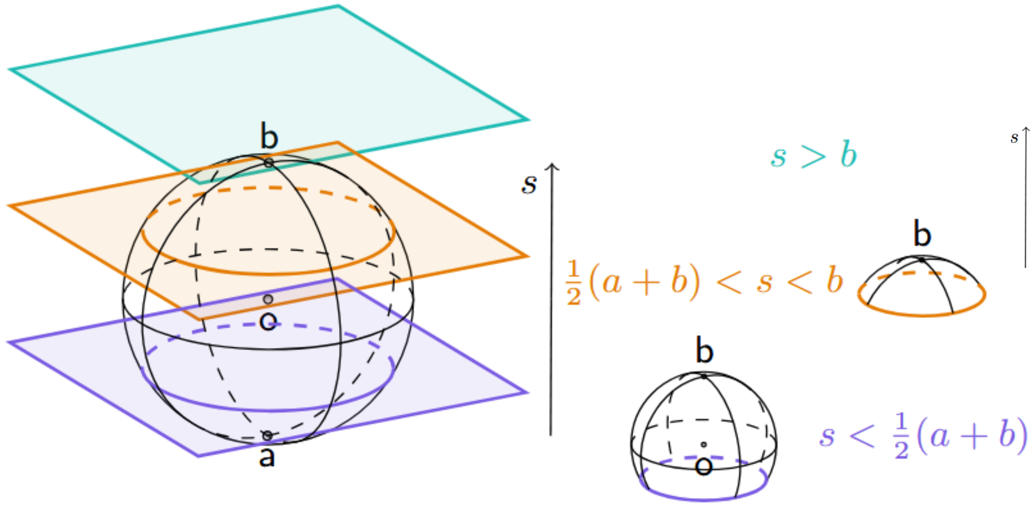
s is a linear function in the vertical direction

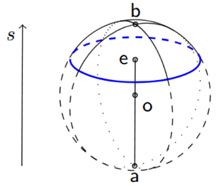


Sets of extreme points that have a value greater than a given s



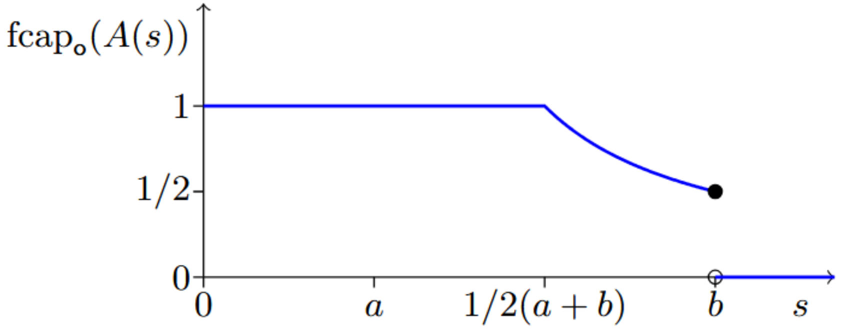
Counterexample for the Choquet integral





$\mathsf{o} = \frac{\overline{\mathsf{oa}}}{\overline{\mathsf{ea}}}\mathsf{e} + \frac{\overline{\mathsf{oe}}}{\overline{\mathsf{ea}}}\mathsf{a}$

$$\mathrm{fcap}_{\mathsf{o}}(A(s)) = \begin{cases} 1 & \text{if } s \leq \frac{1}{2}a + \frac{1}{2}b \\ \frac{1}{2}\frac{b-a}{s-a} & \text{if } \frac{1}{2}a + \frac{1}{2}b < s \leq b \\ 0 & \text{if } s > b. \end{cases}$$



$$(C)\int F d\mathrm{fcap}_{\mathsf{o}} := \int_{-\infty}^{0} (\mathrm{fcap}_{\mathsf{o}}(A(s)) - \mathrm{fcap}_{\mathsf{o}}(X))\, ds + \int_{0}^{\infty} \mathrm{fcap}_{\mathsf{o}}(A(s))\, ds$$

$$= \int_{-\infty}^{0} (1-1)ds + \int_{0}^{\frac{1}{2}a+\frac{1}{2}b} 1ds + \int_{\frac{1}{2}a+\frac{1}{2}b}^{b} \frac{1}{2}\frac{b-a}{s-a}ds + \int_{b}^{\infty} 0ds$$

$$= [s]_{0}^{\frac{1}{2}a+\frac{1}{2}b} + \frac{1}{2}(b-a)\left[\ln(s-a)\right]_{\frac{1}{2}a+\frac{1}{2}b}^{b}$$

$$= \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}(b-a)\left[\ln(b-a) - \ln\left(\frac{1}{2}a + \frac{1}{2}b - a\right)\right]$$

$$= \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}(b-a)\ln(2) \longleftarrow \text{Should be } \frac{1}{2}a + \frac{1}{2}b!$$

Choquet integral does not work

What integral does?

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What derivative?

- Even less clear…

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Lie algebras

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Lie algebras in physics

$$s(t) \to s(t + dt)$$

H - generator of the transformation

t - parameter of the generated transformation

$$\frac{df}{dt} = \{f, H\} \qquad \frac{dO}{dt} = \frac{[O, H]}{\imath\hbar}$$

A skew-symmetric bracket that tells you how a variable/observable changed during the transformation

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Assumptions of Physics

Lie algebras in physics

$$F(e) = \int f \rho_e d\mu$$

Classical case

$$F(e) = tr[O_f \rho_e]$$

Quantum case

Define bracket as $\{F, G\}_{\mathcal{E}}(e) =$

$$\int \{f, g\}_P \rho_e d\mu$$

$$tr\left[\frac{[O_f, O_g]}{\imath\hbar}\rho_e\right]$$

Same Lie algebra in classical/quantum mechanics

$$\{X, P\}_{\mathcal{E}} = I$$

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$$\{F, G\}_{\mathcal{E}}(e) =$$

$$\int \{f, g\}_P \rho_e d\mu \qquad tr\left[[O_f, O_g]\rho_e\right]$$

- *Bilinearity*,

  $$[ax + by, z] = a[x, z] + b[y, z],$$
  $$[z, ax + by] = a[z, x] + b[z, y]$$

  for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.
- The *Alternating* property,

  $$[x, x] = 0$$

  for all x in $\mathfrak{g}$.
- The *Jacobi identity*,

  $$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

  for all x, y, z in $\mathfrak{g}$.

It’s a Lie algebra!

Since both integral and trace are linear operators, properties are inherited from Poisson bracket/commutator

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$$\{F, G\}_{\mathcal{E}}(e) =$$

$$\int \{f, g\}_P \rho_e d\mu \qquad\qquad tr\left[[O_f, O_g]\rho_e\right]$$

The Poisson bracket acts as a derivation of the associative product ·,

Is it a Poisson bracket?

What would the product be?

It's not the product of statistical variables because $E[fg] \neq E[f]E[g]$

Can it be defined in terms of the generator of the composed transformation?

Can we define non-commutativity of variables based on non-commutativity of composed transformations? Can we find a relationship between the expectation of the product and the commutativity, or lack thereof?

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What new axiom is needed (if at all)?

$$\frac{df}{dt} = \{f, H\} \qquad \frac{dO}{dt} = \frac{[O, H]}{\imath\hbar}$$

These are both deterministic and reversible transformations: they leave the entropy unchanged

Also, this structure guarantees us that there are always stationary ensembles under each deterministic and reversible transformation (e.g. Boltzmann distributions)

Can we turn it around? We already assumed ensembles must be equilibria to have an entropy well-defined. Is assuming that every ensemble is a stationary state of some time evolution enough?

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