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Non-uniqueness of the moment problem for Schwartz functions

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Category: Quantum mechanics - Tags: Schwartz space, moment problem

Abstract

We construct explicit counterexamples to show that the quantum moment problem does not always admit a unique solution in Schwartz, and therefore in L^2. In particular, we show that there are infinitely many different states that have the same expectation values for all moments of position and momentum.

1 Introduction

As in classical probability, we can ask whether each state can be identified by its moments of polynomials in X and P. As not all elements of L^2 have finite moments, we construct the counterexamples in the Schwartz space, which is a strict subset of L^2 where all moments are finite.

2 Two Schwartz functions with the same moments

Theorem 1. *Schwartz functions are not uniquely determined by their moments of position and momentum. In other words, there exist two (linearly independent) Schwartz functions $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R})$ such that for all $m, n \in \mathbb{N}_0$: $\langle\psi_1|X^m P^n|\psi_1\rangle = \langle\psi_2|X^m P^n|\psi_2\rangle$.*

Proof. Take two functions $f, g \in \mathcal{S}(\mathbb{R})$ with compact disjoint supports (see Figure 1 for an example). Define the family of functions $\psi_\theta = f + e^{i\theta} g$. The expectation value of $X^m P^n$ is given by

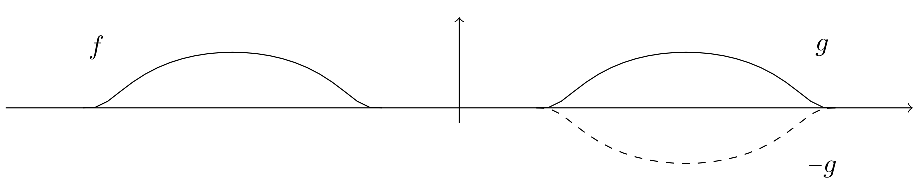

Figure 1: Two bump functions with disjoint support.

$$
\begin{aligned}
\langle\psi_\theta|X^m(-i\hbar\partial_x)^n|\psi_\theta\rangle &= \langle f + e^{i\theta}g|X^m(-i\hbar\partial_x)^n|f + e^{i\theta}g\rangle \\
&= \langle f|X^m(-i\hbar\partial_x)^n|f\rangle \\
&\quad + e^{+i\theta}\langle f|X^m(-i\hbar\partial_x)^n|g\rangle \\
&\quad + e^{-i\theta}\langle g|X^m(-i\hbar\partial_x)^n|f\rangle \\
&\quad + \underbrace{e^{-i\theta}e^{+i\theta}}_{1}\langle g|X^m(-i\hbar\partial_x)^n|g\rangle
\end{aligned}
\tag{2}
$$

Note that the functions $x^m\partial_x^n f$ and $x^m\partial_x^n g$ will have the same or smaller support of f and g respectively. Therefore $x^m\partial_x^n f$ and g will have disjoint support and are therefore orthogonal: $\langle f|X^m(-i\hbar\partial_x)^n g\rangle = 0$. Similarly, $\langle g|X^m(-i\hbar\partial_x)^n f\rangle = 0$. This means that the mixed terms in the original expression vanish and we have

$$
\langle\psi_\theta|X^m(-i\hbar\partial_x)^n|\psi_\theta\rangle = \langle f|X^m(-i\hbar\partial_x)^n|f\rangle + \langle g|X^m(-i\hbar\partial_x)^n|g\rangle \tag{3}
$$

which does not depend on θ. Since f and g are linearly independent (even orthogonal), each θ defines a different physical state (element in the projective space). □

Remark. A point of confusion is that the topology of the Schwartz space can be equivalently generated by semi-norms of the type $\|f\|^{L^2}_{\alpha,\beta} = \left(\int_{\mathbb{R}} \left|x^\alpha\,\partial^\beta f(x)\right|^2 dx\right)^{1/2}$. Ignoring the absolute value, these look very close to the expectation of the moments, and since the topology is Hausdorff, one may erroneously expect the moment problem to be solvable. The problem is that the topology is generated by the open balls of the distances $\|f-g\|^{L^2}_{\alpha,\beta}$. The moments, then, can only give information about all distances to 0, and not to all other functions.

3 Conclusion

We conclude that moments of polynomials in X and P are not enough to identify a quantum mechanical state. The key obstruction is that polynomials of X and P are local operators[1] and can never detect phase differences between spatially separated parts. Consequently any space that contains functions with compact support will suffer the same consequences.

[1] The crucial property of not increasing the domain is one of the required properties for Peetre's theorem, which characterize differential operators.