Assumptions of Physics

Bare Minimum: Linear Algebra

Category: Mathematics - Tags: Linear Algebra, Vector spaces

More bare minima at https://assumptionsofphysics.org/resources/bareminima

Abstract

A condensed overview of Linear Algebra. Bare minima are meant to give a rough overview, by no means complete, of the subject so that one at least knows what there is to know. It is mainly intended as background for those interested in participating in the Assumptions of Physics (https://assumptionsofphysics.org) project.

1 Introduction

Linear algebra studies linear relationships between spaces. Linear algebra is used throughout nearly every field of physics.

For details on linear algebra, see for example [1, 2].

2 Vector spaces

2.1 Basic definitions

Vector space

Definition 1 (Vector space)**.** *A* ***(real) vector space*** *is a set V equipped with a vector addition $+ : V \times V \to V$ and a scalar multiplication $\cdot : \mathbb{R} \times V \to V$ with the following properties for all $u, v, w \in V$ and $a, b \in \mathbb{R}$:*

Associativity of addition $u + (v + w) = (u + v) + w$
Existence of additive identity $v + \mathbf{0} = v$ *for some fixed* $\mathbf{0} \in V$
Existence of additive inverses $\exists -v \in V$ *s.t.* $v + (-v) = \mathbf{0}$
Commutativity of addition $v + u = u + v$
Distributivity over vector addition $a(v + u) = av + au$
Distributivity over scalar addition $(a + b)v = av + bv$
Compatibility of multiplication $a(bv) = (ab)v$
Identity element of scalar multiplication $1v = v$.

A ***vector*** *is an element of a vector space. A* ***scalar*** *is an element of* $\mathbb{R}$.

Remark. In general, vector spaces can be defined over arbitrary fields (i.e. sets equipped with addition, multiplication and their inverses), most notably $\mathbb{C}$. Unless otherwise stated, we will assume that all vector spaces are real.

TODO: add examples

Subspace

Definition 2 (Subspace)**.** *Given a vector space V, a subset of V is a* ***subspace*** *of V if it is a vector space with the same vector addition and scalar multiplication. A* ***proper subspace*** *is a subspace that is not equal to the vector space.*

Linear combination

Definition 3 (Linear combination)**.** *Let $\{v_1, v_2, \dots, v_n\}$ be a finite set of vectors in a vector space V. A* ***linear combination*** *is a vector of the form $a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ where $a_i \in \mathbb{R}$.*

Span: $\mathrm{span}(S)$

Definition 4 (Span)**.** *Let S be a nonempty set of vectors in a vector space V. The* ***span*** *of S, noted $\mathrm{span}(S)$, is the set of all linear combinations of vectors in S.*

Remark. When taking the span of an infinite set, we can only take the weighted sum of finitely many vectors. To make sense of an infinite sum $v_1 + v_2 + v_3 + \dots$ we would need additional structure on the space, such as a topology or inner product, which is outside of the scope of this bare minimum.

Proposition 5. *The span of any set is the smallest subspace containing that set.*

Remark. Let $v \in V$, $c \in \mathbb{R}$ and $A, B \subseteq V$, we define the following notation: $v + A = \{v + a \mid a \in A\}$, $cA = \{ca \mid a \in A\}$ and $A + B = \{a + b \mid a \in A, b \in B\}$.

Proposition 6 (Subspace properties)**.** *Let W, U be subspaces of a vector space V. Then:*

- *$W \cap U$ is a subspace of V*
- *$W + U$ is a subspace of V.*

Proposition 7. *The span of a set is a closure operation. The set of all subspaces, ordered by inclusion, forms a bounded complemented modular lattice that is also an $\bigcap$-structure where $A \vee B = A + B$, and $A \wedge B = A \cap B$.*

Linear independence, dependence

Definition 8 (Linear independence/dependence)**.** *A set of vectors S is **linearly independent** if for any distinct $v_1, v_2, \ldots, v_n \in S$, a linear combination $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = \mathbf{0}$ if and only if $a_1 = a_2 = \ldots = a_n = 0$. Otherwise, the vectors are said to be **linearly dependent**.*

Basis

Definition 9 (Basis)**.** *A **basis** , or Hamel basis, of a vector space V is a set $\mathcal{B} \subseteq V$ that is linearly independent and spans V. An **ordered basis** is a basis equipped with a linear order.*

Remark. Note that there are different notions of basis, such as Schauder basis for some Banach spaces and the orthogonal basis for inner product spaces. These coincide in the finite dimensional case, but they have radically different properties in the infinite dimensional case.

Proposition 10. *Let V be a vector space with basis $\mathcal{B}$. Then for each vector $v \in V$ we can find finitely many base elements $\{e_i\}_{i=1}^n \subseteq \mathcal{B}$ and unique scalars $\{v^i\}_{i=1}^n \subseteq \mathbb{R}$ so that $v = \sum_{i=1}^n v^i e_i$.*

Proposition 11. *All bases of the same vector space have the same cardinality.*

Dimension: $\dim(V)$

Definition 12 (Dimension)**.** *Let V be a vector space with basis $\mathcal{B}$. The **dimension** of V, noted $\dim(V)$, is defined to be $|\mathcal{B}|$.*

Remark. To properly handle infinite dimensional vector spaces, we need topological vector spaces.

Definition 13. *Let V be a finite dimensional vector space and let $\mathcal{B} = \{e_i\}_{i=1}^n$ be a basis. Then each vector $v \in V$ can be expressed a unique linear combination $v = \sum_{i=1}^n v^i e_i$. With the Einstein summation convention, we can write $v = v^i e_i$, where the summation is implied. The vector $\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix} \in \mathbb{R}^n$ is the **coordinate vector** of v.*

Proposition 14 (Dimension of subspace sum)**.** *Let W, U be subspaces of a vector space V. Then $\dim(W + U) = \dim(W) + \dim(U) - \dim(W \cap U)$.*

2.2 Linear maps

Linear maps

Definition 15 (Linear maps)**.** *A **linear map** is a function $T : V \to W$ between two vector spaces such that the following properties hold:*

Additivity *$T(v + u) = T(v) + T(u)$ for all $v, u \in V$*
Homogeneity *$T(av) = aT(v)$ for all $v \in V$ and $a \in \mathbb{R}$.*

Linear map operations: $\mathcal{L}(V, W)$

Definition 16 (Linear map operations)**.** *We denote the set of all linear maps from V to W by $\mathcal{L}(V, W)$. We define the following operations between two linear maps:*

Addition *Noted $S+T$, the sum of $S,T \in \mathcal{L}(V,W)$ is defined by $(S+T)(v) = S(v)+T(v)$ for all $v \in V$.*

Scalar multiplication *Noted aT, the product of $a \in \mathbb{R}$ and $T \in \mathcal{L}(V,W)$ is defined by $(aT)(v) = a(T(v))$ for all $v \in V$.*

Product *Noted ST, the product of $S \in \mathcal{L}(U,W)$ and $T \in \mathcal{L}(V,U)$ is defined by the composition $(ST)(v) = S(T(v))$ for all $v \in V$.*

With addition and scalar multiplication, $\mathcal{L}(V,W)$ satisfies the definition of a vector space.

Identity: Id_V

Proposition 17. *Given a vector space V, the identity function Id_V is a linear map and it is the identity of the product of linear maps. That is, $T = T\,\mathrm{Id}_V = \mathrm{Id}_V\,T$ for all $T \in \mathcal{L}(V,V)$.*

Kernel: $\ker(T)$

Definition 18 (Kernel)**.** *The **kernel** of a linear transformation $T \in \mathcal{L}(V,W)$, noted $\ker(T) \subseteq V$, is the set of vectors in V that map to $\mathbf{0}$. That is, $\ker(T) = \{v \in V \mid T(v) = \mathbf{0}\}$.*

Image: $\mathrm{im}(T)$

Definition 19 (Image)**.** *The **image** of a linear transformation $T \in \mathcal{L}(V,W)$, noted $\mathrm{im}(T) \subseteq W$, is the set $\{T(v) \mid v \in V\}$.*

Proposition 20. *Given $T \in \mathcal{L}(V,W)$, then $\ker(T)$ is a subspace of V, and $\mathrm{im}(T)$ is a subspace of W.*

Proposition 21. *A linear map $T \in \mathcal{L}(V,W)$ is injective if and only if $\ker(T) = \{\mathbf{0}\}$. It is surjective if and only if $\mathrm{im}(T) = W$.*

Proposition 22 (Fundamental theorem of linear maps)**.** *Given $T \in \mathcal{L}(V,W)$, if V is finite dimensional, then $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$.*

Inverse: T^{-1}

Definition 23 (Inverse)**.** *Let $T \in \mathcal{L}(V,W)$. We say T is **invertible** if there exists $T^{-1} \in \mathcal{L}(W,V)$, called **inverse**, such that $ST = \mathrm{Id}_V$ and $TS = \mathrm{Id}_W$.*

Proposition 24. *A linear map is invertible if and only if it is bijective.*

Isomorphism and isomorphic

Definition 25 (Isomorphism and isomorphic)**.** *An invertible linear map is called an **isomorphism**. If there exists an isomorphism $T \in \mathcal{L}(V,W)$, we say V and W are **isomorphic**.*

Proposition 26. *Two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.*

2.3 Product and quotient spaces

Product space: $V_1 \times V_2$

Definition 27 (Product space)**.** *Let $V_1, V_2, \ldots, V_n$ be vector spaces. The **product space**, noted $V_1 \times V_2 \times \ldots \times V_n$, is the set $\{(v_1,\ldots,v_n) \mid v_1 \in V_1, \ldots, v_n \in V_n\}$ equipped with the following operations:*

Addition $(v_1,v_2,\ldots,v_n) + (w_1,w_2,\ldots,w_n) = (v_1+w_1, v_2+w_2, \ldots, v_n+w_n)$ *for all* $(v_1,v_2,\ldots,v_n),(w_1,w_2,\ldots,w_n) \in V_1 \times \ldots \times V_n$

Scalar multiplication $c(v_1,v_2,\ldots,v_n) = (cv_1,cv_2,\ldots,cv_n)$ *for all* $(v_1,v_2,\ldots,v_n) \in V_1 \times \ldots \times V_n$ *and* $c \in \mathbb{R}$.

With these operations, $V_1 \times \ldots \times V_n$ satisfies the definition of a vector space.

Quotient space: $V_{/U}$

Definition 28 (Quotient space and quotient map)**.** *Let V be a vector space with a subspace U. Then the **quotient space** of V over U, noted $V_{/U}$, is the set $\{v+U \mid v \in V\}$ equipped with the following operations:*

Addition $(v_1+U)+(v_2+U) = (v_1+v_2)+U$ *for all* $(v_1+U),(v_2+U) \in V_{/U}$

Scalar multiplication $c(v+U) = (cv)+U$ *for all* $v+U \in V_{/U}$ *and for all* $c \in \mathbb{R}$.

With these operations, $V_{/U}$ satisfies the definition of a vector space. The ***quotient map*** *$q : V \to V_{/U}$ is defined by $q(v) = v + U$ for all $v \in V$ and is a linear map.*

Proposition 29 (Dimension of product and quotient spaces)**.** *Let $V_1 \times V_2 \times \ldots \times V_n$ be a product space of finite dimensional vector spaces, and let $V_{/U}$ be a quotient space of a finite dimensional vector space. Then:*

- $\dim(V_1 \times V_2 \times \ldots \times V_n) = \dim(V_1) + \dim(V_2) + \ldots + \dim(V_n)$
- $\dim(V_{/U}) = \dim(V) - \dim(U)$.

3 Matrices

3.1 Basic definitions

Matrices: $\mathbb{R}^{m\times n}$

Definition 30 (Matrix)**.** *A* ***(real)*** $\mathbf{m \times n}$ ***matrix*** *A is a function $A : [1, m]_{\mathbb{N}} \times [1, n]_{\mathbb{N}} \to \mathbb{R}$. We can view A as a rectangular array with m rows and n columns:*

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \tag{31}$$

The notation a_{ij} or $(A)_{ij}$ refers to the entry in row i and column j. We denote the set of all (real) $m \times n$ matrices by $\mathbb{R}^{m\times n}$.

Remark. The indexes may be up or down depending on what the matrices represent.

Similarly to vector spaces, matrices can be defined over arbitrary fields. For example, a complex $m \times n$ matrix A is a function $A : [1, m]_{\mathbb{N}} \times [1, n]_{\mathbb{N}} \to \mathbb{C}$. The set of all complex $m \times n$ matrices is noted $\mathbb{C}^{m\times n}$.

Proposition 32 (Matrix operations)**.** *Let $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times o}$ and $d \in \mathbb{R}$. Then the* ***matrix sum*** *$A + B$ is given by:*

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \ldots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{bmatrix} \tag{33}$$

That is, $(A + B)_{ij} = (A)_{ij} + (B)_{ij}$. The ***scalar multiplication*** *dA is given by:*

$$d\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} da_{11} & \ldots & da_{1n} \\ \vdots & \ddots & \vdots \\ da_{m1} & \ldots & da_{mn} \end{bmatrix} \tag{34}$$

That is, $(dA)_{ij} = d(A)_{ij}$. The ***matrix multiplication*** *AC is the $m \times o$ matrix given by $(AC)_{ij} = \sum_{k=1}^{n} a_{ik}c_{kj}$.*

Identity matrix: I_n

Definition 35 (Identity matrix)**.** *The* $\mathbf{n \times n}$ ***identity matrix*** *is the $n \times n$ matrix I_n such that $AI_n = I_nA = A$ for all $A \in \mathbb{R}^{n\times n}$. It is given by:*

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \tag{36}$$

Proposition 37. *Let $\{e_i \mid i \in I\}$ be an ordered basis for V and $\{w_i \mid i \in I\}$ be a set of vectors in W indexed by I. There exists a unique linear map $T \in \mathcal{L}(V, W)$ such that $T(e_i) = w_i$ for each $i \in I$.*

Matrix of a linear map: $[T]_{\mathcal{B}_1}^{\mathcal{B}_2}$

Definition 38 (Matrix of a linear map)**.** *Let $\mathcal{B}_1 = \{e_1, \ldots, e_n\}$ be an ordered basis of V, and let $\mathcal{B}_2 = \{e'_1, \ldots, e'_m\}$ be an ordered basis of W. Let $T \in \mathcal{L}(V, W)$. The* ***matrix of*** *T with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$, noted $[T]_{\mathcal{B}_1}^{\mathcal{B}_2}$, is the $m \times n$ matrix so that $T(e_i) = \sum_{j=1}^{m}([T]_{\mathcal{B}_1}^{\mathcal{B}_2})_{ji}e'_j$.*

Column rank and row rank

Definition 39 (Column rank and row rank)**.** *Let $A \in \mathbb{R}^{m\times n}$. Then the **column rank** of A is the dimension of the span of the columns of A in $\mathbb{R}^m$, and the **row rank** of A is the dimension of the span of the rows of A in $\mathbb{R}^n$.*

Rank

Proposition 40 (Rank)**.** *Let $A \in \mathbb{R}^{m\times n}$. Then the row rank of A equals the column rank of A. The **rank** of A is defined to be the column rank, or equivalently the row rank.*

Transpose, conjugate transpose: $A^T, A^\dagger$

Definition 41 (Transpose, conjugate transpose)**.** *Let $A \in \mathbb{C}^{m\times n}$. The **transpose** of A, noted A^T, is the $n \times m$ matrix obtained by swapping the rows and columns of A. That is, $(A^T)_{ij} = (A)_{ji}$. The **conjugate transpose** of A, noted $A^\dagger$, is the $n \times m$ matrix obtained by taking the complex conjugate of each entry in A^T. That is, $(A^\dagger)_{ij} = \overline{(A^T)_{ij}}$.*

Invertible and inverse: A^{-1}

Definition 42 (Invertible matrix)**.** *A matrix $A \in \mathbb{R}^{n\times n}$ is **invertible** if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$. If A is invertible, we denote its **inverse** by A^{-1}. If A is not invertible, then we say A is **singular**.*

Change of basis

Proposition 43 (Change of basis)**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered bases of V. Then $[T]_{\mathcal{B}_2}^{\mathcal{B}_2} = [\mathrm{Id}]_{\mathcal{B}_1}^{\mathcal{B}_2}[T]_{\mathcal{B}_1}^{\mathcal{B}_1}[\mathrm{Id}]_{\mathcal{B}_2}^{\mathcal{B}_1}$. Additionally, $[\mathrm{Id}]_{\mathcal{B}_1}^{\mathcal{B}_2} = ([\mathrm{Id}]_{\mathcal{B}_2}^{\mathcal{B}_1})^{-1}$.*

Definition 44 (Types of square matrices)**.** *Let $A \in \mathbb{C}^{n\times n}$. Then A is said to be:*

Upper triangular *if all entries below the diagonal are 0*
Lower triangular *if all entries above the diagonal are 0*
Diagonal *if all entries outside of the diagonal are 0*
Symmetric *if $A^T = A$*
Hermitian *if $A^\dagger = A$.*

Trace: $\mathrm{tr}(A)$

Definition 45 (Trace)**.** *The **trace** of a square matrix is a function $\mathrm{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$.*

Proposition 46 (Trace properties)**.** *Let $A, B \in \mathbb{R}^{n\times n}$. The trace has the following properties:*

- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $\mathrm{tr}(A) = \mathrm{tr}(A^T)$

Commute

Definition 47 (Commute)**.** *Two linear transformations $T, S \in \mathcal{L}(V, V)$ **commute** if $ST = TS$. Two matrices $A, B \in \mathbb{C}^{n\times n}$ **commute** if $AB = BA$.*

3.2 The determinant

Remark. Recall that a **permutation** of the set $[1, n]_{\mathbb{N}}$ is a bijective function $\sigma : [1, n]_{\mathbb{N}} \to [1, n]_{\mathbb{N}}$. The set of all permutations on $[1, n]_{\mathbb{N}}$ is noted by S_n. A **transposition** is a permutation that swaps two elements only. The **sign** of a permutation σ, noted $\mathrm{sgn}(\sigma)$, is defined to be +1 if the permutation can be formed by the composition of an even number of transpositions, and −1 otherwise.

Determinant: $|A|$

Definition 48. *Given a matrix $A \in \mathbb{R}^{n\times n}$, the **determinant** of A, noted $|A|$ or $\det(A)$, is defined to be*

$$|A| = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \tag{49}$$

Proposition 50. *Let $T \in \mathcal{L}(V, V)$, where V is finite dimensional. For any two bases $\mathcal{B}_1, \mathcal{B}_2$ of V, $\det([T]_{\mathcal{B}_1}) = \det([T]_{\mathcal{B}_2})$. We define the determinant of a linear transformation T to be $\det([T]_{\mathcal{B}_1})$, where $\mathcal{B}_1$ is any basis of V.*

Proposition 51 (Determinant properties)**.** *Let $A, B \in \mathbb{R}^{n\times n}$. Then the determinant has the following properties:*

- A *is triangular* $\implies |A| = \prod_{i=1}^n a_{ii}$
- $|AB| = |A||B|$
- A *is invertible* $\iff |A| \neq 0$
- A *is invertible* $\implies |A^{-1}| = \frac{1}{|A|}$
- $|A^T| = |A|$

3.3 Eigenvalues and eigenvectors

Eigenvalues and eigenvectors

Definition 52 (Eigenvalues and eigenvectors)**.** *Let* $A \in \mathbb{C}^{n\times n}$*. An* ***eigenvalue*** *of* A *is a scalar* $\lambda \in \mathbb{C}$ *such that there exists a non-zero* $v \in \mathbb{C}^n$ *so that* $Av = \lambda v$*. In this case,* v *is called an* ***eigenvector*** *of* A *corresponding to* λ*.*

Proposition 53 (Eigenspace)**.** *Given* $A \in \mathbb{C}^{n\times n}$ *and an eigenvalue* λ *of* A*, the set of eigenvectors corresponding to* λ *along with* $\mathbf{0} \in \mathbb{C}^n$ *constitutes a subspace of* $\mathbb{C}^n$*, called the* ***eigenspace*** *of* λ*.*

Proposition 54. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\lambda \in \mathbb{C}$*.* λ *is an eigenvalue of* A *if and only if* $\det(\lambda I - A) = 0$*. The polynomial function* $\det(xI - A)$ *is called the* ***characteristic polynomial*** *of* A*.*

Algebraic, geometric multiplicity

Definition 55 (Algebraic, geometric multiplicity)**.** *Let* $A \in \mathbb{C}^{n\times n}$ *with an eigenvalue* λ*. The* ***algebraic multiplicity*** *of* λ *is the multiplicity of* λ *as a root of the characteristic polynomial. The* ***geometric multiplicity*** *of* λ *is the dimension of the eigenspace corresponding to* λ*.*

Proposition 56. *Let* $A \in \mathbb{C}^{n\times n}$ *with eigenvalues* $\lambda_1, \dots, \lambda_n$ *(with algebraic multiplicity). Then:*

- $|A| = \prod_{i=1}^n \lambda_i$
- $\text{tr}(A) = \sum_{i=1}^n \lambda_i$.

4 Dual spaces

Dual space: V^*

Definition 57 (Dual space)**.** *The* ***dual space*** *of a vector space* V*, noted* V^**, is the vector space* $\mathcal{L}(V, \mathbb{R})$*. A* ***linear functional*** *is an element of the dual space.*

Dual basis: $\mathcal{B}^*$

Definition 58 (Dual basis)**.** *Let* $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ *be an ordered basis of* V*. The* ***dual basis*** *of* V^* *with respect to* $\mathcal{B}$*, noted* $\mathcal{B}^*$*, is the ordered basis* $\{e^1, e^2, \dots, e^n\}$ *of* V^* *where each* $e^j \in \mathcal{B}^*$ *is uniquely defined by* $e^j(e_i) = \delta_{ji}$ *for all* $i \in [1, n]$*.*

Dual map: T^*

Definition 59 (Dual map)**.** *Let* $T \in \mathcal{L}(V, W)$*. Then the* ***dual map*** *of* T *is the linear map* $T^* \in \mathcal{L}(W^*, V^*)$ *uniquely defined by* $T^*(f) = f \circ T$ *for all* $f \in W^*$*.*

Proposition 60. *Let* $T \in \mathcal{L}(V, W)$*,* $\mathcal{B}_1$ *be an ordered basis of* V *and* $\mathcal{B}_2$ *be an ordered basis of* W*. Then* $[T^*]_{\mathcal{B}_2^*}^{\mathcal{B}_1^*} = ([T]_{\mathcal{B}_1}^{\mathcal{B}_2})^T$*.*

Annihilator: S^0

Definition 61 (Annihilator)**.** *Let* $S \subseteq V$*. The* ***annihilator*** *of* S*, noted* $S^0 \subseteq V^*$*, is the set* $S^0 = \{f \in V^* \mid f(S) = \{0\}\}$*.*

Proposition 62 (Annihilator properties)**.** *Let* $S \subseteq V$*. The annihilator satisfies the following properties:*

- S^0 *is a subspace of* V^*
- *If* $\dim(V) < \infty$ *and* S *is a subspace, then* $\dim(S^0) = \dim(V) - \dim(S)$*.*

References

[1] Sheldon Axler. *Linear Algebra Done Right.* Springer, 4 edition, 2024.

[2] Steven Roman. *Advanced Linear Algebra.* Springer, 3 edition, 2008.