Assumptions of physics: project overview

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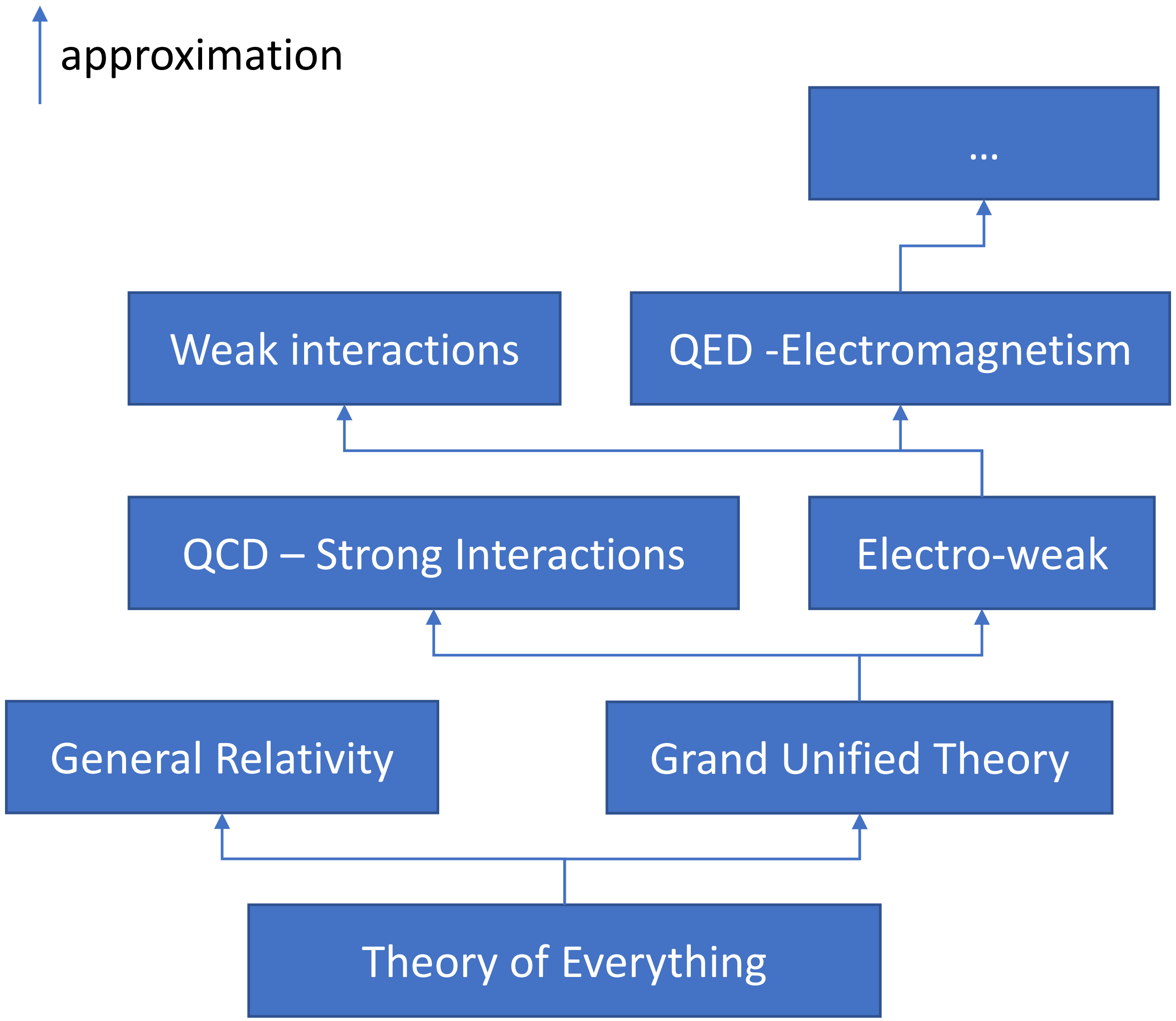
Assumptions of Physics

Typical view in the foundations of physics

- Start with the theory that describes "what really happens"
  - With the most complicated and most complete description
- Gradually derive other theories as approximations



The Assumptions of Physics project does not proceed in this manner

Understanding fundamental structures

Hamiltonian mechanics

Phase space (symplectic manifold) | Hamiltonian evolution

Differentiable manifold | Symplectic structure

Manifold | Differentiable structure

Topological space | Locally $\mathbb{R}^n$

We cannot truly understand this
if we don't first understand this

Even probability spaces
are not fundamental structures

Probability space

Set of points | σ-algebra | Measure

I.e. Before saying "there is a 50% chance to get tails" we need to define what tails, chance and 50% mean

- What are the "correct" axioms and definitions on which to build scientific theories? How can they be justified?

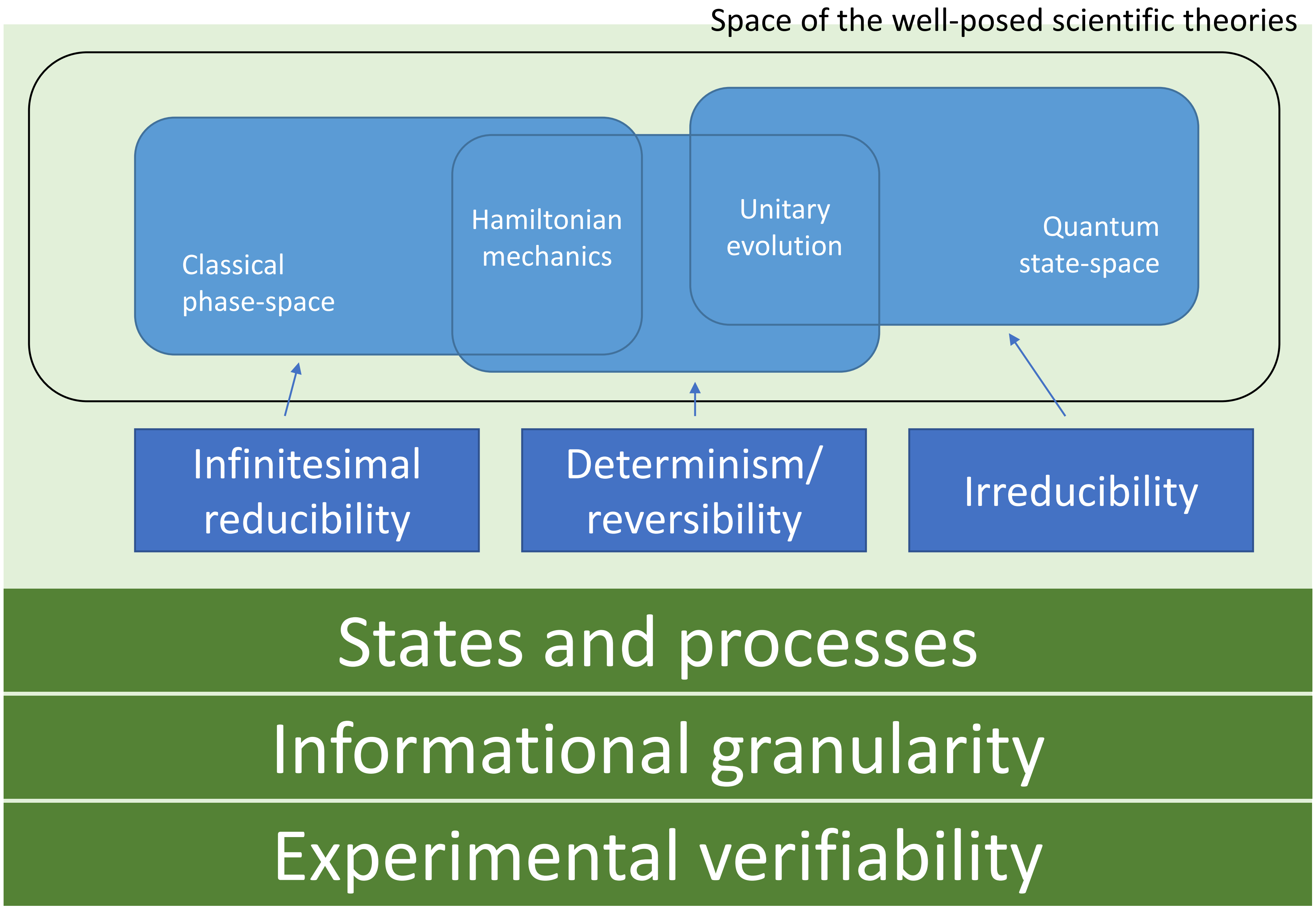
Physical theories

Specializations of the general theory under the different assumptions

Assumptions

General theory

Basic requirements and definitions valid in all theories



Assumptions of Physics

- Objectives:
  - Develop a mathematical framework that can serve as the foundation for all scientific theories (i.e. a mathematical theory about scientific theories)
  - Start from physical principles and assumptions and derive the math (not start from the math and add the physics later through an "interpretation")
  - Each mathematical object must have a clear physical meaning (no object is unphysical, can read math proofs as logical arguments on the physics)
  - Construct concepts and tools that span different disciplines (nature does not care about divisions in fields of knowledge)
  - Explore what happens when the assumptions fail, possibly leading to new physics ideas

The logic of experimental verifiability

Principle of scientific objectivity. Science is universal, non-contradictory and evidence based.

Science deals with well-posed sets of assertions (non-contradictory) that have a single truth value (universal) that can be defined/ascertained experimentally (evidence based)
⇒ **Verifiable statements**: assertions that can be experimentally verified in a finite time

Examples:

- The mass of the photon is less than 10^{-13} eV
- If the height of the mercury column is between 24 and 25 millimeters then its temperature is between 24 and 25 Celsius
- If I take 2 ± 0.01 Kg of Sodium-24 and wait 15 ± 0.01 hours there will be only 1 ± 0.01 Kg left

Counterexamples:

- Chocolate tastes good (not universal)
- It is immoral to kill one person to save ten (not universal and/or evidence-based)
- The number 4 is prime (not evidence-based)
- This statement is false (not non-contradictory)
- The mass of the photon is exactly 0 eV (not verifiable due to infinite precision)

We have to keep in mind that the meaning of the statements, their relationships and what truth values are allowed depends on context (e.g. premise, theory, etc…)

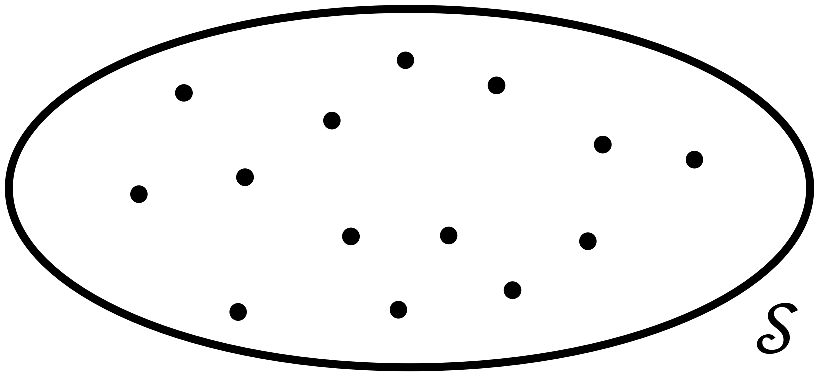
The mass of the electron is 511 ± 0.5 KeV

When measuring the mass, it is a verifiable hypothesis

When performing particle identification, it is assumed to be true

Axioms of logic

Axiom 1.2 (Axiom of context). *A **statement** s is an assertion that is either true or false. A **logical context** $\mathcal{S}$ is a collection of statements with well defined logical relationships. Formally, a logical context $\mathcal{S}$ is a collection of elements called statements upon which is defined a function* $\mathrm{truth} : \mathcal{S} \to \mathbb{B}$.



	s_1	s_2	s_3	…
a →	T	T	F	…
	T	F	T	…
	T	F	F	…

$\mathcal{A}_{\mathcal{S}}$

Axiom 1.4 (Axiom of possibility). *A **possible assignment** for a logical context $\mathcal{S}$ is a map $a : \mathcal{S} \to \mathbb{B}$ that assigns a truth value to each statement in a way consistent with the content of the statements. Formally, each logical context comes equipped with a set $\mathcal{A}_{\mathcal{S}} \subseteq \mathbb{B}^{\mathcal{S}}$ such that $\mathrm{truth} \in \mathcal{A}_{\mathcal{S}}$. A map $a : \mathcal{S} \to \mathbb{B}$ is a possible assignment for $\mathcal{S}$ if $a \in \mathcal{A}_{\mathcal{S}}$.*

Axiom 1.9 (Axiom of closure). *We can always find a statement whose truth value arbitrarily depends on an arbitrary set of statements. Formally, let $S \subseteq \mathcal{S}$ be a set of statements and $f_{\mathbb{B}} : \mathbb{B}^{S} \to \mathbb{B}$ an arbitrary function from an assignment of S to a truth value. Then we can always find a statement $\bar{\mathsf{s}} \in \mathcal{S}$ that depends on S through $f_{\mathbb{B}}$.*

s_1	s_2	s_3	…
T	T	F	…
T	F	T	…
T	F	F	…

→ $f_{\mathbb{B}}$ →

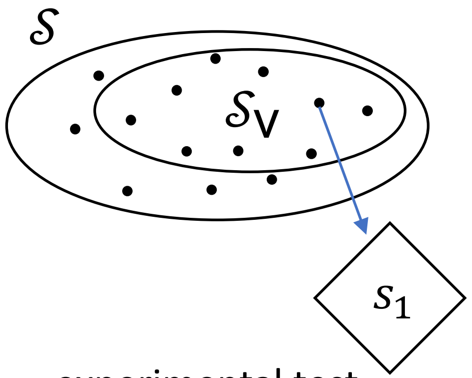
$f(s_1, s_2, s_3)$
T
T
F

Axioms of verifiability

Axiom 1.27 (Axiom of verifiability)**.** *A* ***verifiable statement*** *is a statement that, if true, can be shown to be so experimentally. Formally, each logical context $\mathcal{S}$ contains a set of statements $\mathcal{S}_v \subseteq \mathcal{S}$ whose elements are said to be verifiable. Moreover, we have the following properties:*

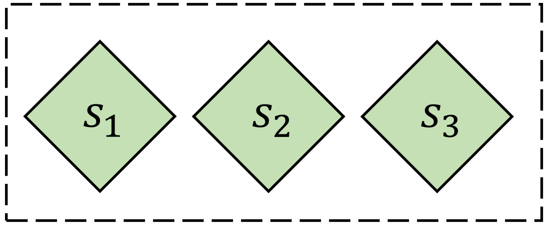
- *every certainty $\top \in \mathcal{S}$ is verifiable*
- *every impossibility $\bot \in \mathcal{S}$ is verifiable*
- *a statement equivalent to a verifiable statement is verifiable*

Remark. The **negation or logical NOT** of a verifiable statement is not necessarily a verifiable statement.



s_1	Test Result
T	SUCCESS (in finite time)
F	FAILURE (in finite time)
	UNDEFINED

$$\bigwedge_{i=i}^{n} s_i$$

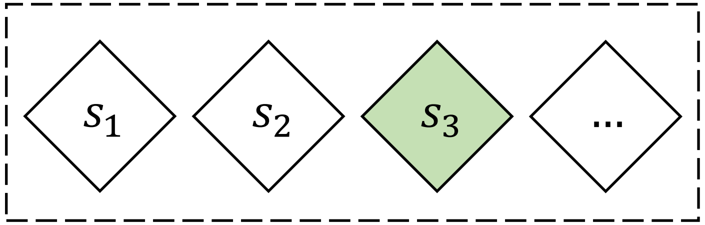


All tests must succeed

Axiom 1.31 (Axiom of finite conjunction verifiability)**.** *The conjunction of a finite collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^{n} \subseteq \mathcal{S}_v$ be a finite collection of verifiable statements. Then the conjunction $\bigwedge\limits_{i=1}^{n} s_i \in \mathcal{S}_v$ is a verifiable statement.*

Axiom 1.32 (Axiom of countable disjunction verifiability)**.** *The disjunction of a countable collection of verifiable statements is a verifiable statement. Formally, let $\{s_i\}_{i=1}^{\infty} \subseteq \mathcal{S}_v$ be a countable collection of verifiable statements. Then the disjunction $\bigvee\limits_{i=1}^{\infty} s_i \in \mathcal{S}_v$ is a verifiable statement.*

$$\bigvee_{i=1}^{\infty} s_i$$



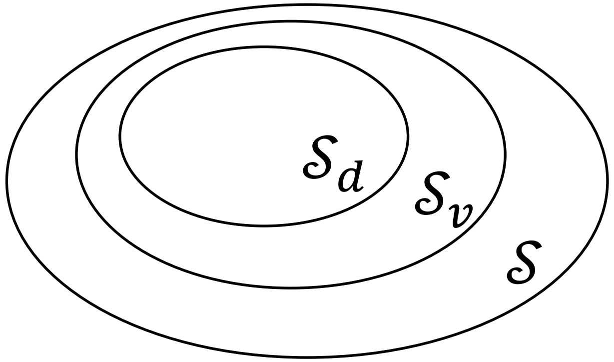
One successful test is sufficient

Properties of the logic system

Different algebras for the different types of statements

Operator	Gate	Statement	Verifiable Statement	Decidable Statement
Negation	NOT	allowed	disallowed	allowed
Conjunction	AND	arbitrary	finite	finite
Disjunction	OR	arbitrary	countable	finite

Table 1.3: Comparing algebras of statements.



(Different) notions of equivalences

Definition 1.15. *Two statements* s_1 *and* s_2 *are* ***equivalent*** $\mathsf{s}_1 \equiv \mathsf{s}_2$ *if they must be equally true or false simply because of their content. Formally,* $\mathsf{s}_1 \equiv \mathsf{s}_2$ *if and only if* $a(\mathsf{s}_1) = a(\mathsf{s}_2)$ *for all possible assignments* $a \in \mathcal{A}_{\mathcal{S}}$.

Are the same statement

"This animal is a bird" = "Questo animale e' un uccello"

Always have the same truth → "This animal is a bird" ≡ "This animal has feathers"

truth("This animal is a bird") = truth("That animal is a mammal")

Happen to have the same truth

Experimental domains (scientific models)

Start with a countable set of verifiable statements

From them generate all verifiable statements (close under finite AND countable OR)

Generate all meaningful predictions (close under negation as well)

Fill in all possible assignments

Basis $\mathcal{B}$				Experimental domain $\mathcal{D}_X$			Theoretical domain $\bar{\mathcal{D}}_X$		
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…
F	F	F	…	F	F	…	T	T	…
…	…	…	…	…	…	…	…	…	…
F	T	F	…	T	F	…	F	T	…
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~
…	…	…	…	…	…	…	…	…	…

Possibilities $X \subset \bar{\mathcal{D}}_X$

The points of the space (the possibilities, the distinguishable cases) are not given a priori but are constructed from the chosen verifiable statements

$x = \neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \cdots$

For each possible assignment we have a theoretical statement that is true only in that case. We call these statements possibilities of the domain.

Topologies and σ-algebras

Each column (statement) is also a set of possibilities

$$s = \bigvee_{x \in U} x$$

Topologies (needed for manifold/geometric spaces) and σ-algebras (needed for integration and probability spaces) naturally arise from requiring experimental verifiability

Finite AND and countable OR become finite intersection and countable union

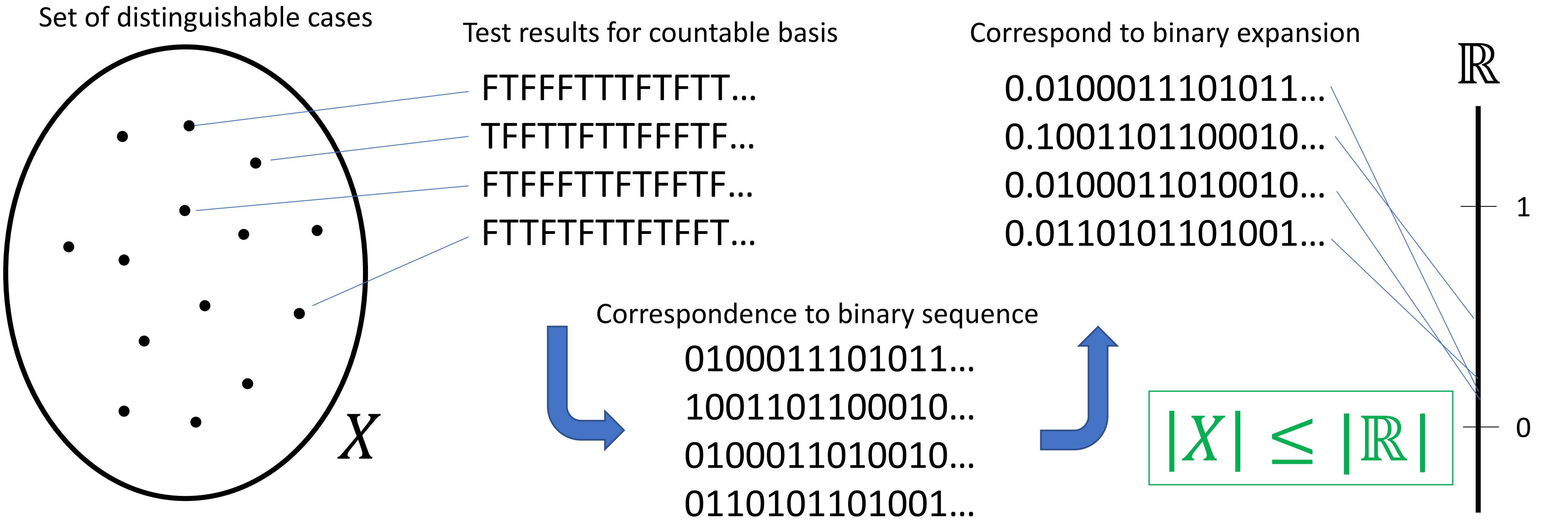
Negation and countable AND become complement and countable union

Basis $\mathcal{B}$				Experimental domain $\mathcal{D}_X$			Theoretical domain $\overline{\mathcal{D}_X}$			
e_1	e_2	e_3	…	$s_1 = e_1 \vee e_2$	$s_2 = e_1 \wedge e_3$	…	$\overline{s_1} = e_1 \vee \neg e_2$	$\overline{s_2} = \neg e_1$	…	Possibilities X
F	F	F	…	F	F	…	T	T	…	
…	…	…	…	…	…	…	…	…	…	
F	T	F	…	T	F	…	F	T	…	
~~T~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	~~T~~	~~F~~	~~…~~	
…	…	…	…	…	…	…	…	…	…	

The experimental domain $\mathcal{D}_X$ induces a topology on the possibilities X.

The theoretical domain $\overline{\mathcal{D}_X}$ induces a (Borel) σ-algebra

Maximum cardinality of distinguishable cases



- Sets with greater cardinality (e.g. the set of all discontinuous functions from $\mathbb{R}$ to $\mathbb{R}$) cannot represent physical objects
- Issues about higher infinities (e.g. large cardinals) are not relevant, but those surrounding the continuum hypothesis may be

Topologies and σ-algebras

All definitions and all proofs about these structures have precise physical meaning in this context

s_1	Test Result
T	SUCCESS (in finite time)
T	UNDEFINED
F	UNDEFINED
F	FAILURE (in finite time)

$int(A)$ corresponds to the verifiable part of a statement

∂A corresponds to the undecidable part of a statement

$ext(A)$ corresponds to the falsifiable part of a statement

If $U \subseteq X$ is an open set then "x is in U" is a verifiable statement (e.g. "the mass of the electron is 511 ± 0.5 KeV")

If V $\subseteq X$ is a closed set then "x is in V" is a falsifiable statement (e.g. "the mass of the electron is exactly 511 KeV")

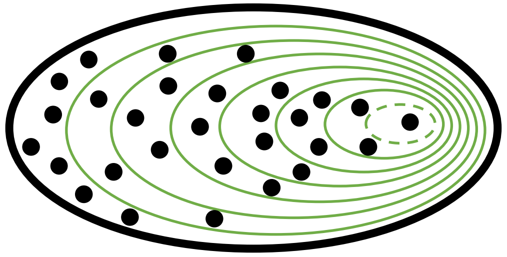
If $A \subseteq X$ is a Borel set then "x is in A" is a theoretical statement: a test can be created, though we have no guarantee of termination (e.g. "the mass of the electron in KeV is a rational number" is undecidable, the test will never terminate)

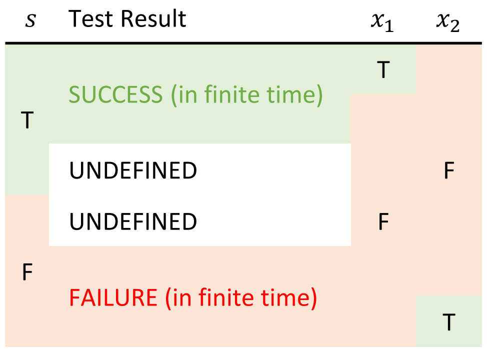
Topologies and σ-algebras each capture part of the formal structure

For us, they are part of a single unified structure

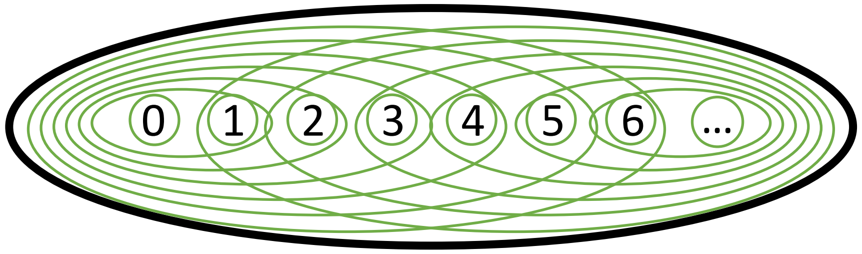
Physical meaning of separation axioms

- All topologies are Kolmogorov (i.e. T_0)
  - Possibilities are experimentally well-defined
    i.e. possibilities constructible from a base by countable AND/OR and NOT (singletons in the σ-algebra)
- The topology is T_1 if all possibilities are approximately verifiable
  - Possibilities are the limit of a sequence of verifiable statements
    i.e. possibilities are the countable conjunction of verifiable statements
- The topology is Hausdorff (i.e. T_2) if all possibilities are pairwise experimentally distinguishable
  - Given two possibilities, we can find a test that confirms one and excludes the other
  - i.e. for any $x_1, x_2 \in X$ there is a statement $s \in \bar{\mathcal{D}}_X$ such that $x_1 \preccurlyeq ver(s)$ and $x_2 \preccurlyeq fal(s)$





Examples



Standard topology on integers

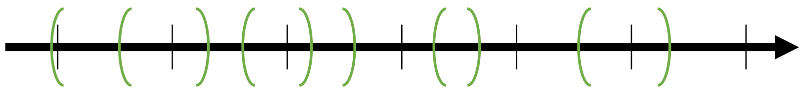
Decidable domain (all statements are decidable)

Discrete topology (every set is clopen); topology and σ-algebra both coincide with the power set

Standard topology on the reals

Finite precision measurements (open intervals are verifiable)

Topology generated by open intervals (coincides with order and metric topology); separable, complete, connected (no clopen sets except full and empty set); σ-algebra is the Borel algebra (strict subset of power set)

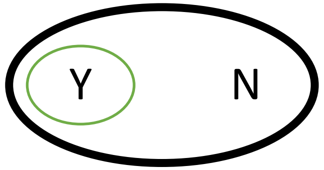


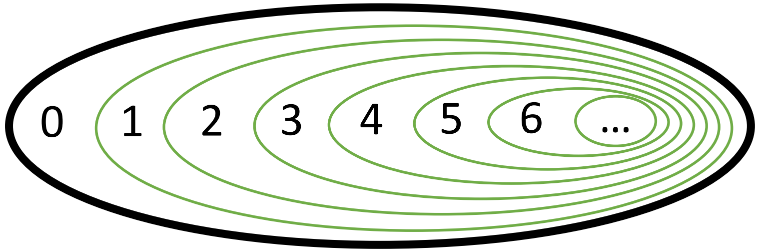
Examples

Does extra-terrestrial life exist?

Semi-decidable question

Topology $\{\emptyset, \{Y\}, \{Y, N\}\}$ is strictly T_0; σ-algebra is the power set





How many leptons (electron-like particles) are there? (through direct observation)

Can only measure lower bound (e.g. "there are at least i")

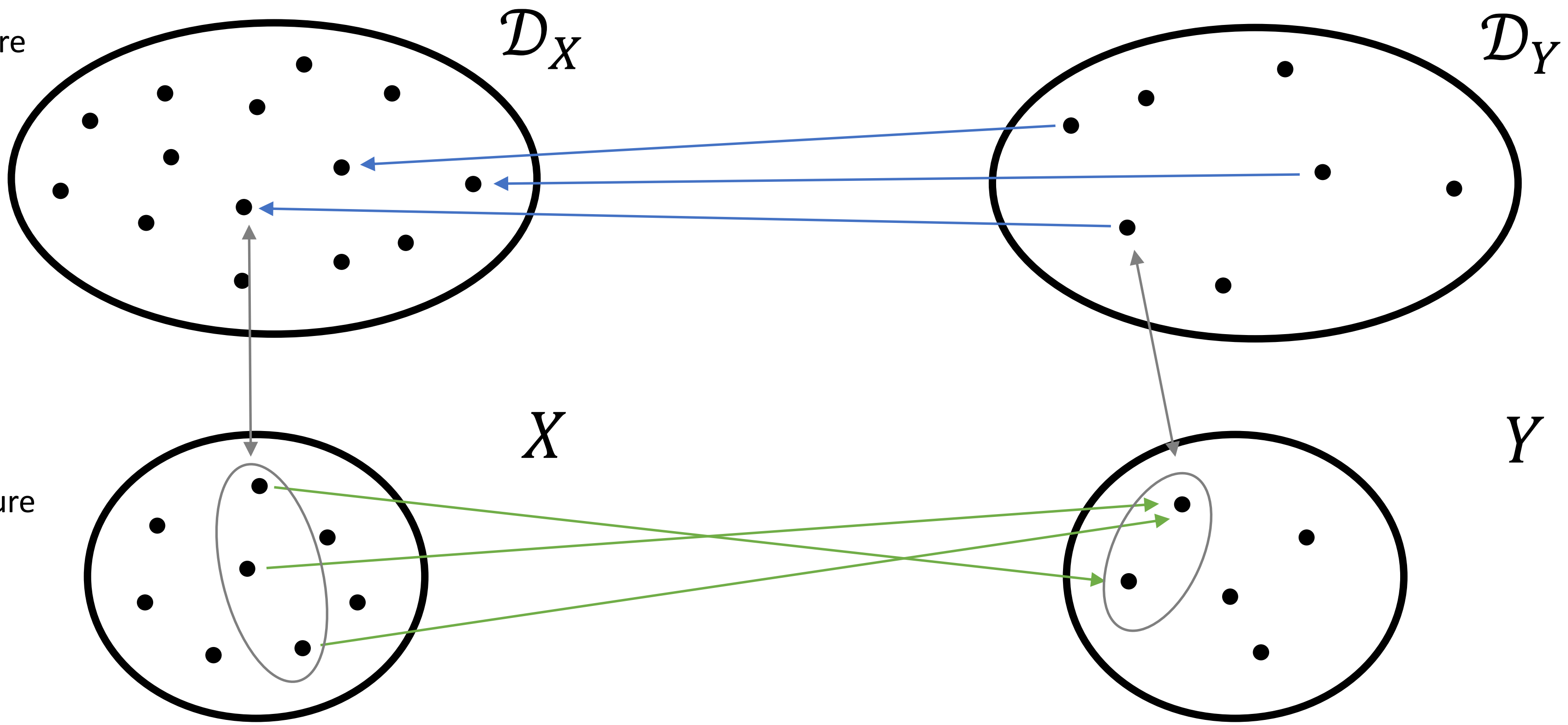
Topology contains empty set and $\{i, i+1, i+2, \dots\}$ for all i; strictly T_0; σ-algebra is the power set

Inference/causal relationships and continuity

An inference relationship is a map $r: \mathcal{D}_Y \to \mathcal{D}_X$ such that $r(s) \equiv s$

e.g. the water temperature is between 0 and 0.52 Celsius or between 7.6 and 9.12 Celsius

e.g. the water density is between 999.8 and 999.9 kg/m^3



e.g. the water temperature is exactly 4 Celsius

e.g. the water density is exactly 1 kg/m^3

A causal relationship is a map $f: X \to Y$ such that $x \preccurlyeq f(x)$

Two general and important results:

1) Two domains admit an inference relationship if and only if they admit a causal relationship

2) The causal relationship must be a continuous map in the natural topology

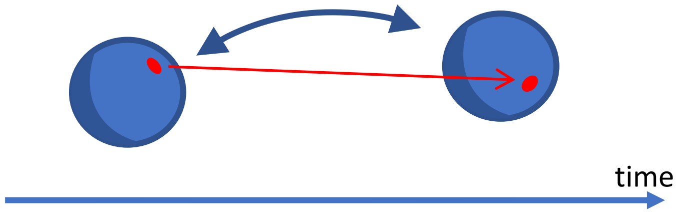
Takeaway

- A “science first” formal structure is possible
  - Physically meaningful, mathematically precise, philosophically consistent
  - Precise science/math dictionary
  - “Well-behaved” mathematical objects are really “well-defined” physical objects
- Experimental verifiability is the basis for scientifically well-defined objects
  - Topologies and σ-algebras arise from scientific epistemological requirements, not from ontological features of the universe
  - Most other structures used in science (differential geometry, measure theory, probability theory, Lie algebras, …) are based on topologies and σ-algebras
- No progress in the foundations of physics is possible without proper understanding of these connections

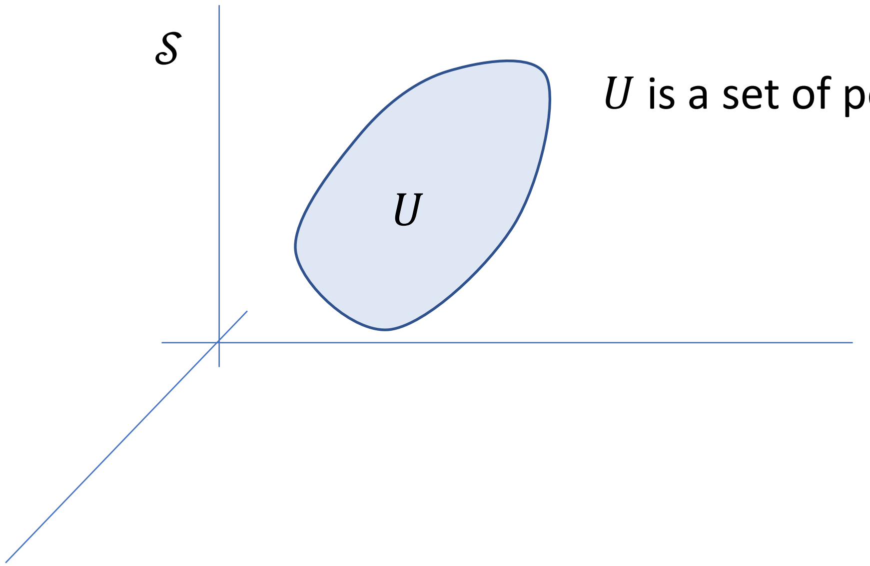
The assumptions of classical mechanics

Assumption of infinitesimal reducibility

The system is reducible to its parts: giving the state of the whole is equivalent to giving the state of the parts. The system can be subdivided indefinitely.



$\mathcal{S}$ is the state of the infinitesimal parts (i.e. particles)



U is a set of possible states for the particles

V is a set of possible amounts of materials

We'll have statements of the form "the amount of material found in U is within the range V"

Density over states

The state of the whole is given by a distribution over the state of the infinitesimal parts (i.e. particles)

$$\int_U \rho d\mathcal{S}$$

Fraction of the system in a region U

$$\rho : \mathcal{S} \to \mathbb{R}$$

Density depends on the state; unit is [amount]/[states]

This presents a puzzle:

$$\rho\big(\mathcal{s}(\xi^a)\big) = \rho(\xi^a)$$

Under a change of variables $\hat{\xi}^b = \hat{\xi}^b(\xi^a)$ $\quad \mathcal{s}(\xi^a) = \mathcal{s}\big(\hat{\xi}^b\big)$

we have $\rho(\xi^a) = \left|\frac{\partial \xi^a}{\partial \hat{\xi}^b}\right| \rho\big(\hat{\xi}^b\big)$ $\quad \rho\big(\mathcal{s}(\xi^a)\big) = \rho\left(\mathcal{s}\big(\hat{\xi}^b\big)\right)$

How can ρ both change as a density and be an invariant?

Units

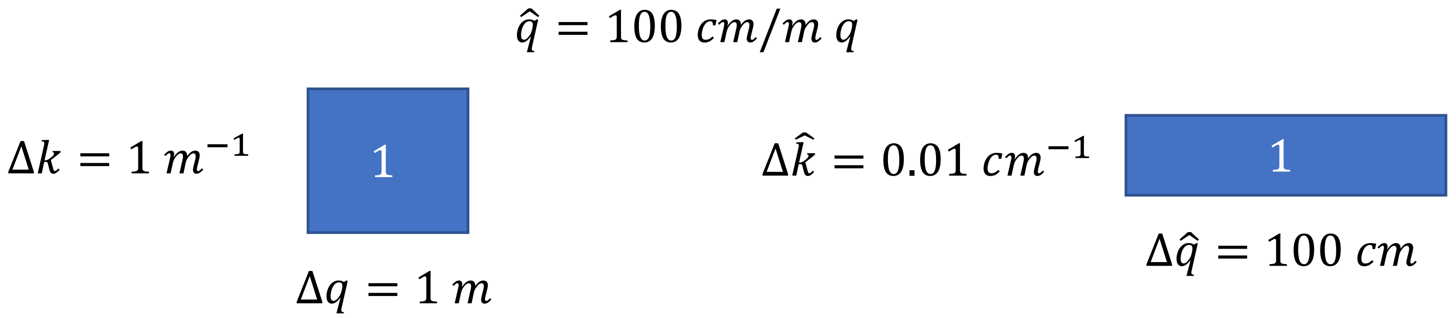
- When we write $\int_U \rho(s)d\mathcal{S}$, ρ is expressed in units of [amount]/[states]
- When we write $\int_U \rho(\xi^a)d\xi^n$, ρ is expressed in units of [amount]/[ξ^1]…[ξ^n]
- It seems we need to characterize the role of units

- The units of some variables depend on the units of others
  - E.g. the unit for velocity $v = dx/dt$ along a direction x depends on the unit for distance along that direction and time; the unit for entropy $dS = dQ/T$ depends on the unit for energy and temperature
- Within the state variables ξ^a, we identify the unit variables q^i as those that define the unit system
  - A change of units $\hat{q}^j = \hat{q}^j(q^i)$ must induce a unique transformation $\hat{\xi}^b = \hat{\xi}^b(\xi^a)$ on all variables

Phase space (symplectic manifold)

- The structure of phase space is exactly what is needed to define invariant densities over particle states

The product $\Delta q \Delta k$ is invariant

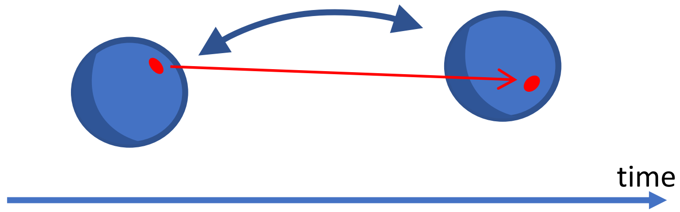


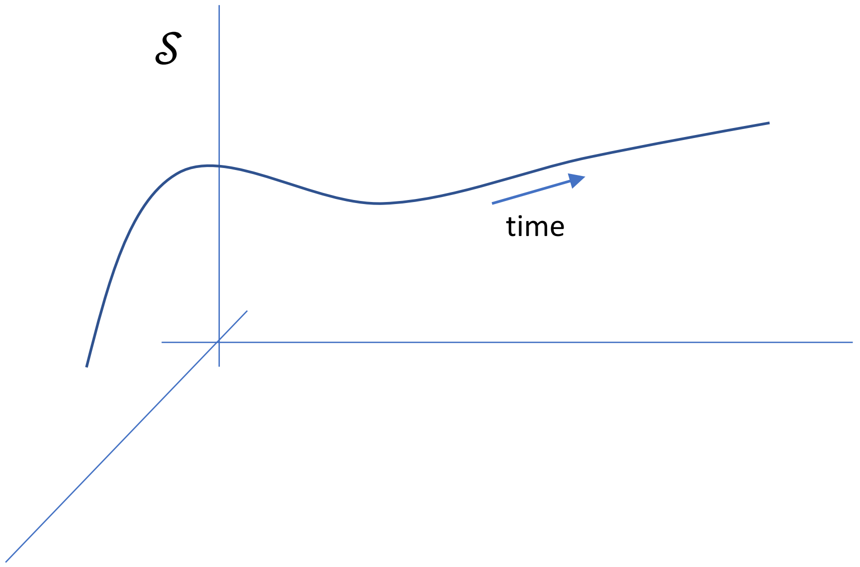
  - For a single degree of freedom (i.e. one independent unit variable)
$$dS = \hbar dq dk = \hbar d\hat{q} d\hat{k}$$
  - For n independent degrees of freedom
$$dS = \hbar^n dq^n dk_n = \hbar^n d\hat{q}^n d\hat{k}_n$$

- Canonical variables are those that allow us to express density in the correct units over each independent degree of freedom

Assumption of deterministic and reversible evolution

Given the state of the system at one time, we are able to predict the state at future times (determinism) and reconstruct (reversibility) the state at past times.





Dynamical system $s_t \mapsto s_{t+\Delta t}$

Not enough!

All and only the particles from s_t must be found in $s_{t+\Delta t}$: $\rho(s_t, t) = \rho(s_{t+\Delta t}, t + \Delta t)$

Independent degrees of freedom must be mapped to independent degrees of freedom

$\Rightarrow$ Hamiltonian mechanics (symplectic structure must be preserved)

Hamiltonian mechanics for one degree of freedom

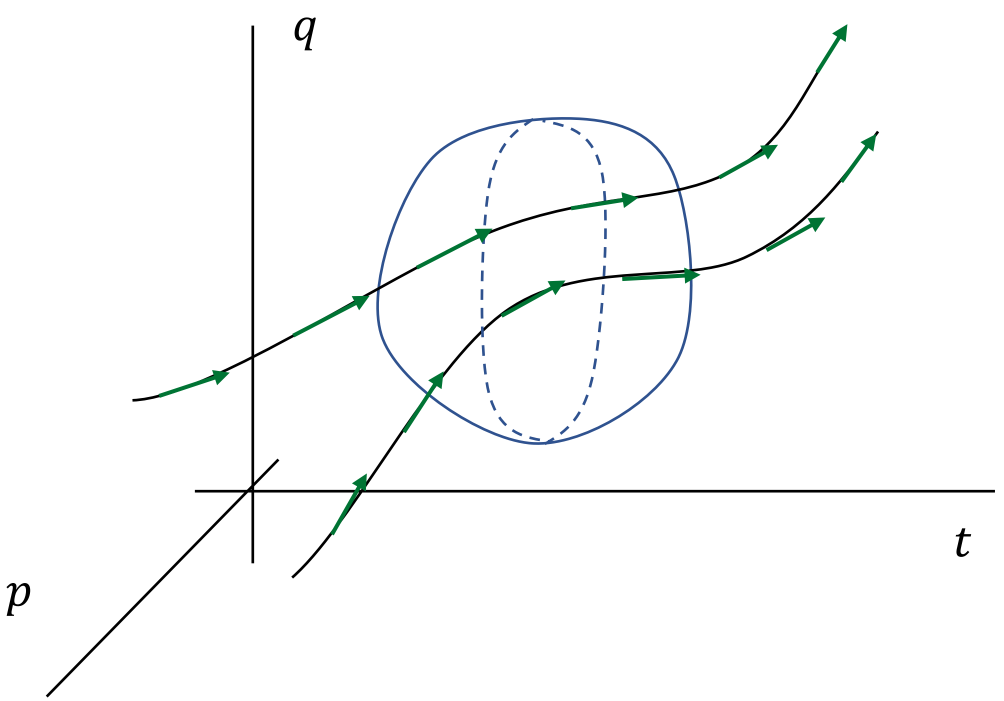
Displacement along the trajectory

$$\vec{S} = \left(\frac{dq}{dt}, \frac{dp}{dt}, \frac{dt}{dt}\right)$$

Deterministic and reversible:
flux over a closed surface is zero

$$div(\vec{S}) = 0$$

$$\vec{S} = -curl(\vec{\theta})$$



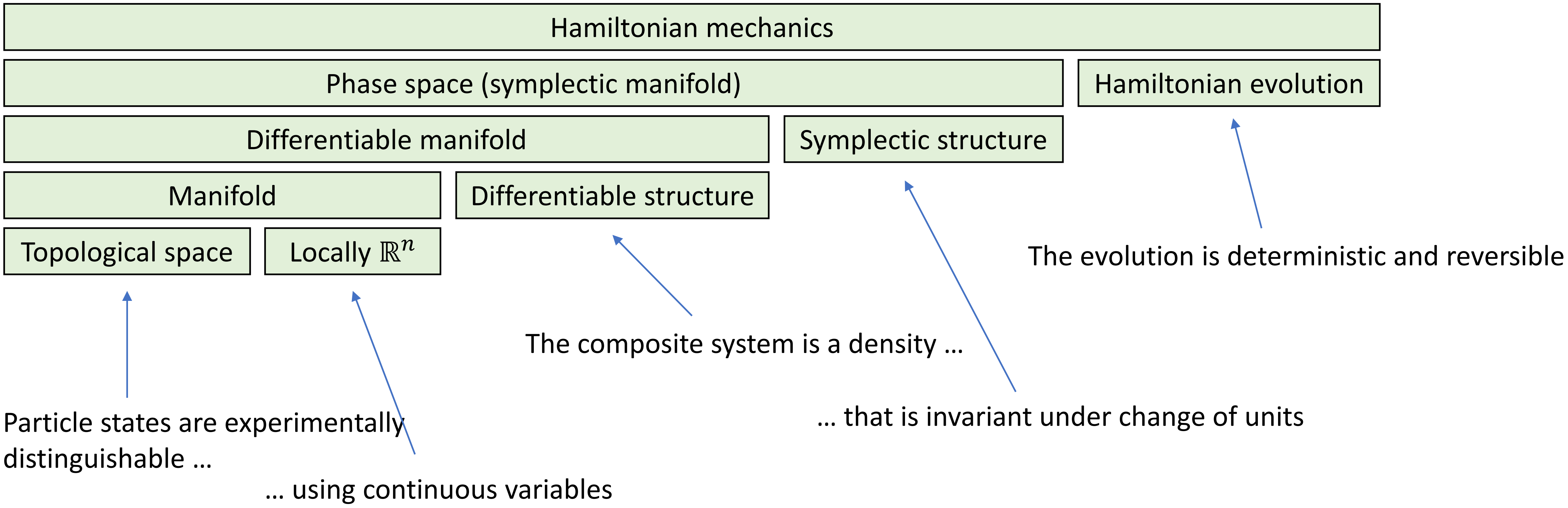
Because $\frac{dt}{dt} = 1$ we can choose a gauge such that:

$$\vec{\theta} = (p, 0, -H(q,p))$$

This recovers Hamilton's equations

$$\vec{S} = \left(\frac{dq}{dt}, \frac{dp}{dt}, \frac{dt}{dt}\right) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}, 1\right) = -curl(\vec{\theta})$$

Understanding Hamiltonian mechanics



Each mathematical structure is linked to a specific physical requirement

Possible contributions

Bare minima

- Project is very interdisciplinary and requires knowledge from different areas of math, physics and engineering
- We want to create a series of short (12-16 pages) articles that give the basic definitions and main results of each field: the bare minimum one needs to know
  - E.g. Set theory: https://assumptionsofphysics.org/resources/bareminima/SetTheory.pdf
- Hourly work

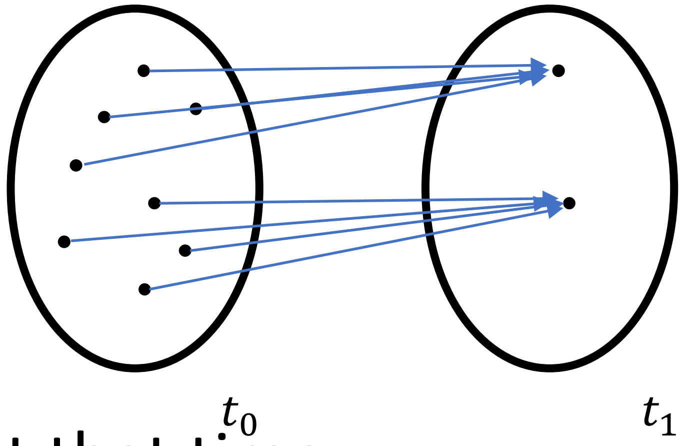
Other small tasks (hourly work)

- There are a number of smaller questions it would be nice to settle
  - Is every Heyting algebra embeddable in a Boolean algebra?
    - To make sure we are not ruling anything out
  - Finalize last few details in our basic structures
    - Find the "correct" morphism that gives us the right product, …
  - Study a Gaussian peak under linear Hamiltonian flow
    - To generalize the "classical uncertainty relationship"
  - Special relativity from densities
  - Look for hints of general relativity in the extended phase space
    - See if the link between symplectic form, metric and vector potential leads to relationships to the curvature
  - Characterize quantum projections as processes with constraints that maximize entropy
  - Analyze the relationship between linearity of mixed (classical mixtures) and pure states (quantum superposition) in quantum mechanics
    - Can superposition be fully characterized by "aliasing" of mixed states?
- Some of these questions may be already solved in the literature, some may be hard
- Helping to formalize/organize the questions is also useful

A new foundation for measures and geometry

- We saw that topological and σ-algebraic structures come from experimental verifiability: how do we recover the rest?
- Note that measures and metrics are used to give a “size” to sets
- In physics, conceptually, we start with the ability to compare sizes (this is bigger than this); we then construct measurement scales to give numerical values
- The idea is to provide a foundation for measure theory and geometry in the same way: we have the lattice of all possible descriptions (our σ-algebra); we add a preorder that tells us whether one description is “finer” (i.e. more refined) than another; pick a unit and construct a “measure” that respects the order and that is linear under “disjoint addition”
- The goal is to find a set of sufficient physically justifiable conditions for which such measures can be constructed
- For preliminary work, see https://assumptionsofphysics.org/resources/blueprints/InformationGranularity.pdf

Physical entropy as counting evolutions



- The idea is to define how to “count” (in a measure theoretic sense) the possible evolutions of a system; we define the process entropy as the logarithm of that count
- State entropy becomes the process entropy associated to all possible evolutions that “pass” through that state at that time
- Equilibrium states concentrate the evolutions and therefore they maximize the process entropy
- The goal is, with similar considerations, to rederive the basic laws of thermodynamics in the most possible general setting, and recover the standard formula for entropy (i.e. Gibbs, log of count of states, …) in specific cases
- For preliminary work, see https://assumptionsofphysics.org/resources/blueprints/ProcessEntropy.pdf

Other bigger tasks

- Find a reformulation of quantum mechanics that fits better in the framework
  - Projective spaces? Use mixed states as prime object? Algebraic?
- Find a set of physical motivations to introduce differentiability and differential forms
  - General idea is to describe linear quantities associated to k-dimensional submanifolds (rough ideas in Bachelor's thesis https://assumptionsofphysics.org/Thesis-Johnson-DifferentialGeometry.pdf)
- …

Final thought

*Prima dovete capire le cose **nel piccolo**, e poi generalizzare*

> *First you have to understand the simple case, and then generalize*

*Se non avete capito **nel piccolo**, capirete ancora meno quando generalizzate*

> *If you haven't understood the simple case, you will understand even less when you generalize*