The Fundamental Connections Between Classical Hamiltonian Mechanics, Quantum Mechanics and Information Entropy

UNIVERSITY OF MICHIGAN

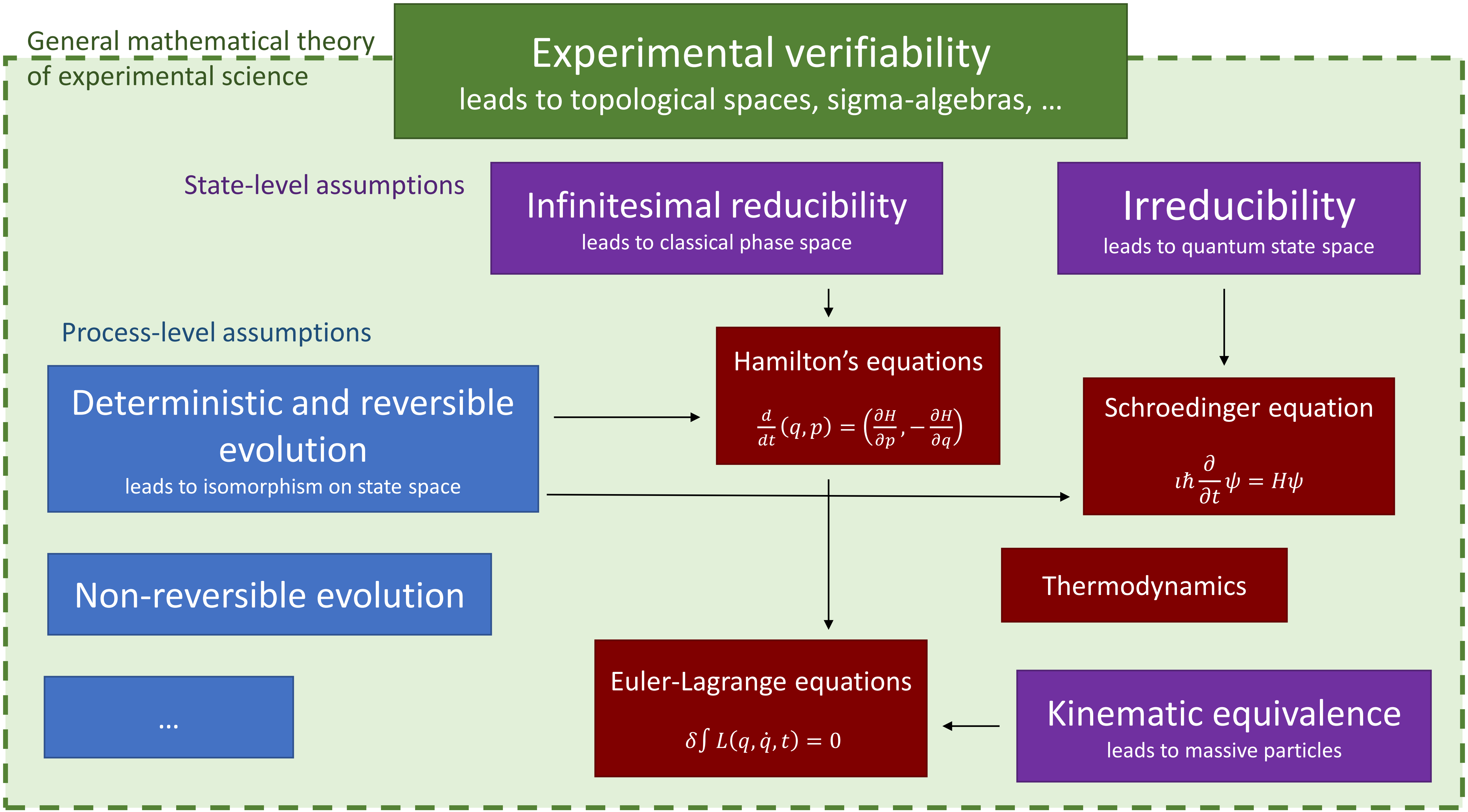
Gabriele Carcassi and Christine Aidala

May 31, 2019

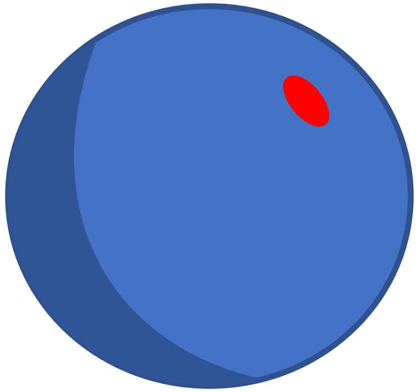
Quantum2019

Assumptions of Physics

- This talk is part of a broader project called Assumptions of Physics (see http://assumptionsofphysics.org/)
- The aim of the project is to find a handful of physical principles and assumptions from which the basic laws of physics can be derived
- To do that we want to develop a general mathematical theory of experimental science: the theory that studies scientific theories
  - A formal framework that forces us to clarify our assumptions
  - From those assumptions the mathematical objects are derived
  - Each mathematical object has a clear physical meaning and no object is unphysical
  - Gives us concepts and tools that span across different disciplines
  - Allows to explore what happens when the assumptions fail, possibly leading to new physics ideas


General mathematical theory of experimental science
Experimental verifiability
leads to topological spaces, sigma-algebras, …
State-level assumptions
Infinitesimal reducibility
leads to classical phase space
Irreducibility
leads to quantum state space
Process-level assumptions
Deterministic and reversible evolution
leads to isomorphism on state space
Hamilton's equations
$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$
Schroedinger equation
$\imath\hbar\frac{\partial}{\partial t}\psi = H\psi$
Non-reversible evolution
Thermodynamics
…
Euler-Lagrange equations
$\delta\int L(q,\dot{q},t) = 0$
Kinematic equivalence
leads to massive particles

Assumption of infinitesimal reducibility



- Reducible: the state of the whole is equivalent to the state of the parts
- Infinitesimally: each part can be subdivided into parts indefinitely
- The state of the system is a distribution over the state of the infinitesimal parts (i.e. particles)

$$\rho: \mathcal{S} \to \mathbb{R}$$

- If particle states are identified by a set of continuous quantities ξ^a

$$\rho\left(\mathscr{s}(\xi^a)\right) = \rho(\xi^a)$$

$$I[\rho] = -\int_{\mathcal{S}} \rho \log(\rho) d\xi^a$$

Distributions and change of variables

- In general, density and information entropy are not invariant under change of variables

$$\rho(\xi^a) = \left|\frac{\partial \xi^a}{\partial \hat{\xi}^b}\right| \rho\left(\hat{\xi}^b\right)$$

$$-\int_{\mathcal{S}} \rho \log(\rho) d\xi^a = -\int_{\mathcal{S}} \rho \log(\rho) d\hat{\xi}^b - \int_{\mathcal{S}} \rho \log\left|\frac{\partial \xi^a}{\partial \hat{\xi}^b}\right| d\hat{\xi}^b$$

- Note that they are both invariant if and only if $\left|\frac{\partial \xi^a}{\partial \hat{\xi}^b}\right| = 1$
- Yet, since the state is invariant under coordinate transformation $\mathscr{s}(\xi^a) = \mathscr{s}\left(\hat{\xi}^b\right)$, we should also have $\rho\left(\mathscr{s}(\xi^a)\right) = \rho\left(\mathscr{s}\left(\hat{\xi}^b\right)\right)$

How can this work?

Invariant distributions

- It can only work if $\xi^a = (q^i, k_i)$ and k_i uses inverse units of q^i
- That way $dq^i dk_i = d\hat{q}^j d\hat{k}_j$ is a pure number and is invariant under coordinate transformations and we have

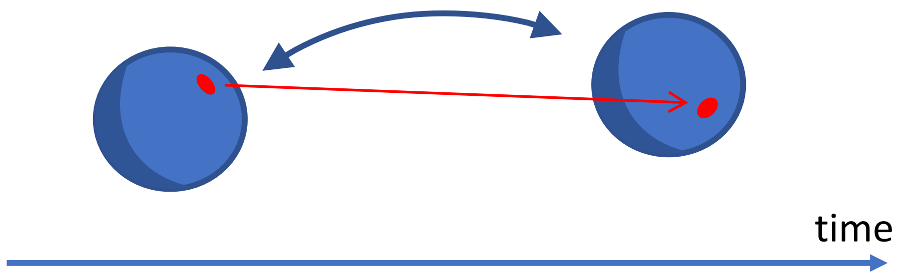
$$\rho(q^i, k_i) = \rho(\hat{q}^j, \hat{k}_j)$$

$$-\int_{\mathcal{S}} \rho \log(\rho) dq^i dk_i = -\int_{\mathcal{S}} \rho \log(\rho) d\hat{q}^j d\hat{k}_j$$

- Invariant densities, invariant information entropy, phase-space: requiring one requires the others

Deterministic and reversible evolution

- Deterministic and reversible evolution can be defined in two ways:
  - The part of the system in one state is found in one and only one future state $\rho_t\big(\mathcal{s}(t)\big) = \rho_{t+\Delta t}\big(\mathcal{s}(t+\Delta t)\big)$
  - The information needed to describe the system does not change $I[\rho_t] = I[\rho_{t+\Delta t}]$



- These two definitions are identical since density and information entropy are preserved in the same circumstances
- Both definitions lead to the preservation of the geometry of phase-space (i.e. symplectomorphism) and coincide with Hamiltonian evolution
- Density conservation, information entropy conservation, Hamiltonian evolution: requiring one requires the others

Deterministic and reversible evolution

- Deterministic and reversible evolution (i.e. state densities are mapped one-to-one)
- System isolation (i.e. the state of the system does not depend on anything else)
- Conservation of information entropy (i.e. the information required to describe the system does not change in time)
- Conservation of energy (i.e. Hamiltonian evolution)

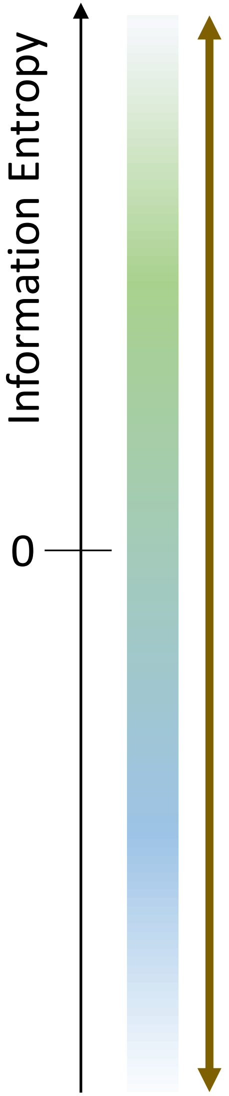
- These four concepts are the same concept from different angles

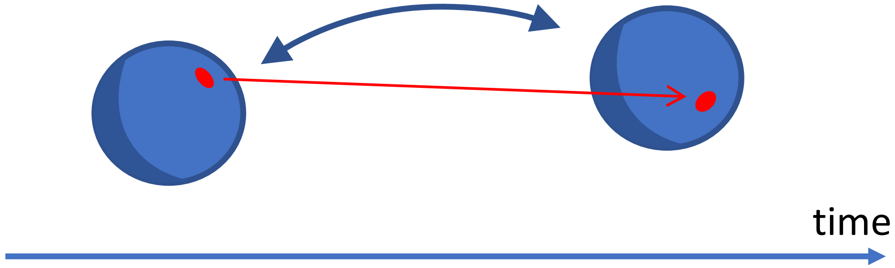
Classical uncertainty

- Note:
  - information entropy is invariant during the evolution
$$I[\rho_{t+\Delta t}] = I_0 = I[\rho_t]$$
  - with fixed entropy, the gaussian distribution minimizes the spread
$$I_G = 2\pi e \, \sigma_q \sigma_k$$
- Therefore:
$$\sigma_q \sigma_k \geq \frac{\exp(I_0)}{2\pi e}$$

Classical state

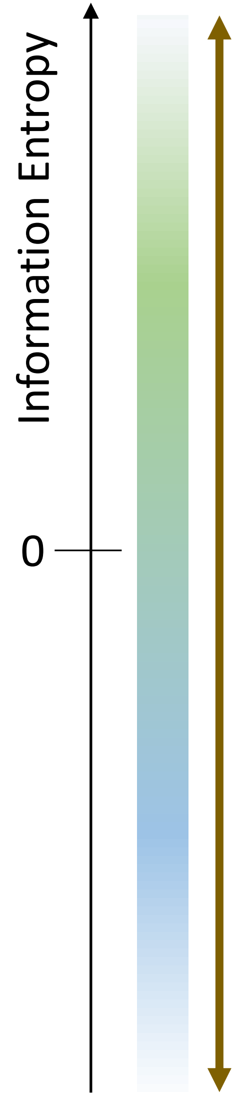
$$\rho(q^i, k_i)$$





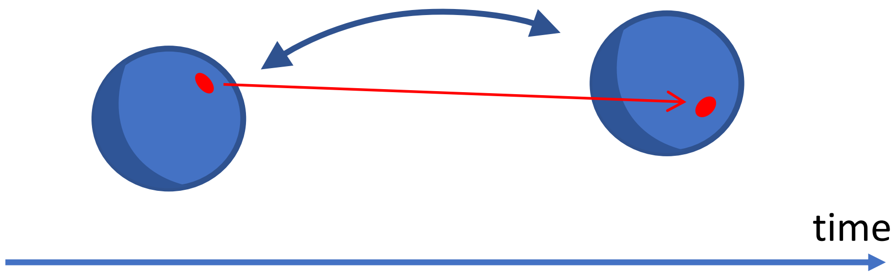
We always have access to the internal dynamics

Any initial value for information entropy is allowed: we can study arbitrarily small parts



Classical state

$$\rho(q^i, k_i)$$

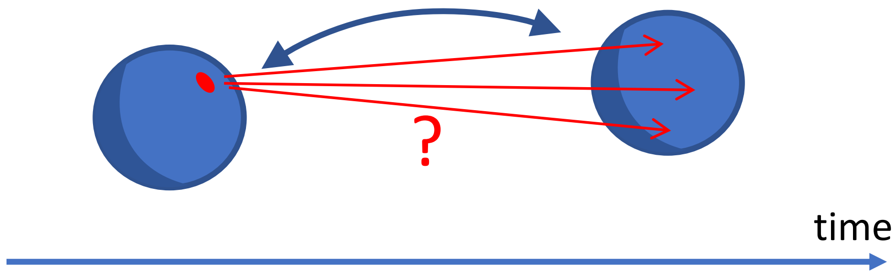


We always have access to the internal dynamics

Any initial value for information entropy is allowed: we can study arbitrarily small parts

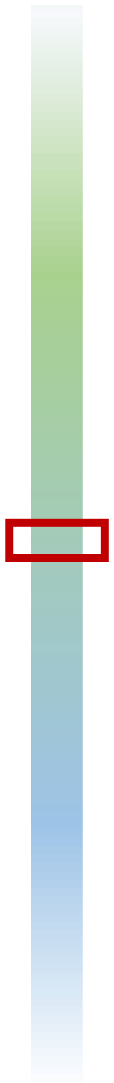
Quantum state

$$\psi(q^i)$$



We have no access to the internal dynamics

All pure quantum states have the same information entropy (i.e. zero): no description for parts

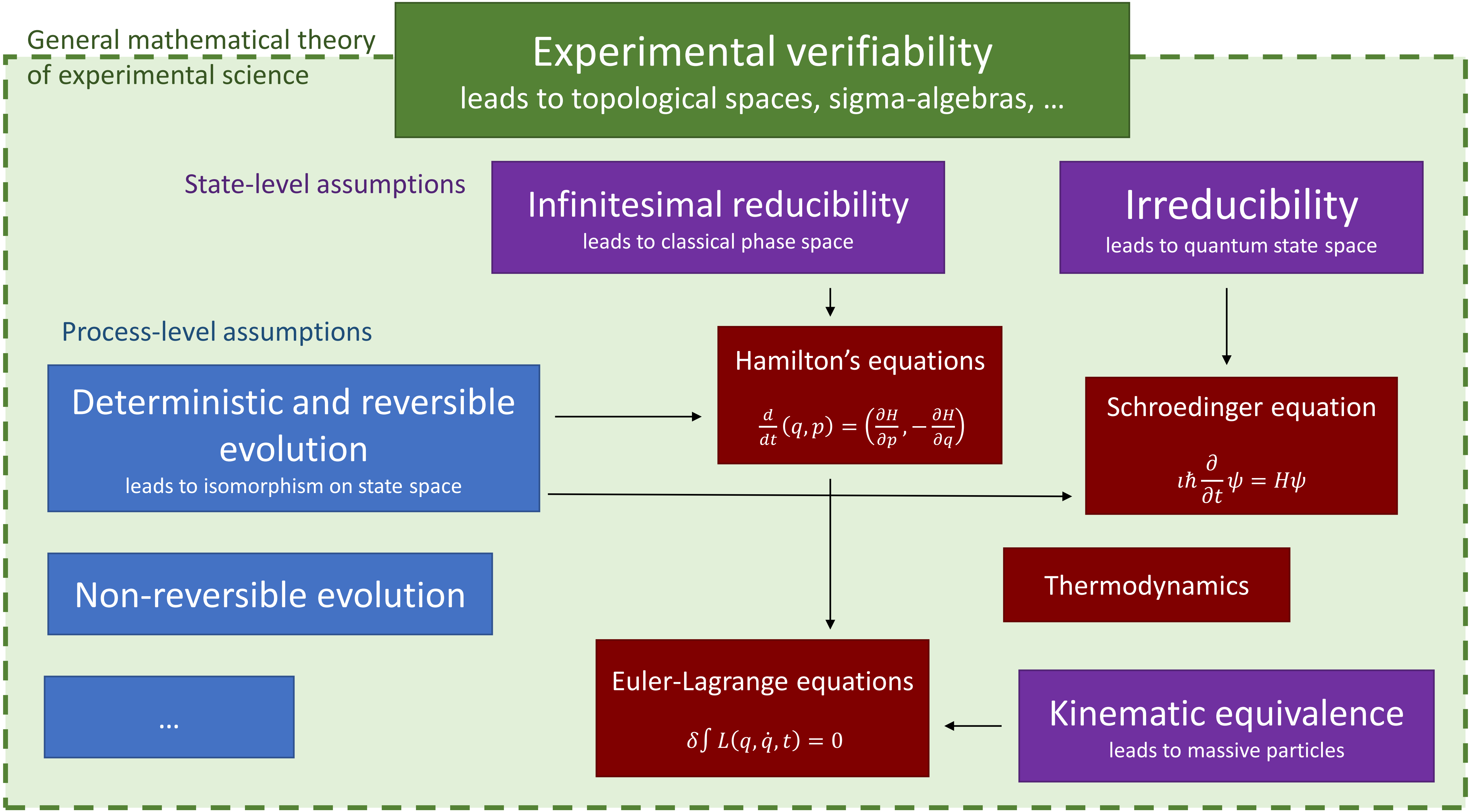


Information about the internal dynamics

- Consider a muon
  - while we can’t predict when and how, it will (most likely) decay into an electron and two neutrinos
  - if we assume it’s the internal dynamics that causes the decay, then information about the internal dynamics is mapped to the state of the decay products
- It is difficult to imagine that any process where the particle number changes (e.g. absorption, emission, decay) will not “expose” or “hide” information about the internal dynamics
  - More or less information is now accessible
- Neither classical mechanics nor quantum mechanics would be able to properly describe this case

Conclusions

- Even in physics, we can proceed deductively, from physical assumptions to mathematical equations
- Deterministic and reversible evolution, system isolation, conservation of information entropy and conservation of energy are different aspects of the same concept
  - If you want to understand when each one is violated, you probably need to understand how all are violated
- The "only" difference between a classical and a quantum system is what one can tell about the parts of the system
  - In classical mechanics everything; in quantum mechanics nothing
- Physical theories are about large systems being conceptually divided into small parts, not about small parts being combined into large systems
  - We start from large systems (i.e. planets, balls) and can progressively study the internal dynamics in terms of smaller and smaller systems (i.e. molecules, atoms, fundamental particles). The game stops not because there is no more internal dynamics but because it is not (yet?) accessible.


General mathematical theory of experimental science
Experimental verifiability
leads to topological spaces, sigma-algebras, …
State-level assumptions
Infinitesimal reducibility
leads to classical phase space
Irreducibility
leads to quantum state space
Process-level assumptions
Deterministic and reversible evolution
leads to isomorphism on state space
Hamilton's equations
$\frac{d}{dt}(q,p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$
Schroedinger equation
$\imath\hbar\frac{\partial}{\partial t}\psi = H\psi$
Non-reversible evolution
Thermodynamics
…
Euler-Lagrange equations
$\delta\int L(q,\dot{q},t) = 0$
Kinematic equivalence
leads to massive particles

For more information

- [**From physical assumptions to classical and quantum Hamiltonian and Lagrangian particle mechanics**](#)
  Gabriele Carcassi *et al* 2018 *J. Phys. Commun.* **2** 045026
- Topology and Experimental Distinguishability
  Christine A. Aidala, Gabriele Carcassi, and Mark J. Greenfield, *Top. Proc.* **54** (2019) pp. 271-282
- Assumptions of Physics project website:
  http://assumptionsofphysics.org/