Topology and experimental distinguishability

Gabriele Carcassi
Christine A Aidala
David J Baker
Mark J Greenfield

University of Michigan

Motivation for this work

- This work has its root in an effort to better understand fundamental physics in general and classical Hamiltonian/Lagrangian particle mechanics in particular
  - Why are classical states points in a cotangent bundle? What does the symplectic form represent? Why is time evolution a symplectomorphism? Must time evolution always be a diffeomorphism or a homeomorphism?
- At some point we realized that to give a satisfactory answer to those questions, we would have to better understand topological spaces on their own merit
  - What physical concept is captured by a topology? What do open sets and continuous functions correspond to?
- We believe we have found the answer: a topology keeps track of what can be distinguished through experimentation
  - It seems fitting that topology maps to such a fundamental concept for an experimental science

Overview

- Experimental observation
  - Observations are statements combined with a way to experimentally verify them. We'll define a Boolean-like algebra on them which is similar to topological structure.
- Experimental distinguishability
  - Study observations that can identify an object within a set of possibilities. This will lead to Hausdorff and second countable topological spaces.
- Experimental relationships
  - Study relationships between experimentally distinguishable objects. This will lead to continuous functions and homeomorphisms.

Keeping track of what is experimentally verifiable

EXPERIMENTAL OBSERVATIONS

Experimental observations

- In science, something is true if and only if it can be experimentally verified
- It is not enough to claim something
  - E.g. “Bob likes chocolate” “The ball is moving at about 1 m/s” “Birds descend from Dinosaurs”
- We must provide a clear procedure such that the result can be independently replicated
- Let’s see if we can capture this requirement more precisely

Experimental observations

- Def: an **experimental test** e is a **repeatable procedure** (i.e. can be stopped, restarted, executed as many times as needed) that, if successful, **terminates in finite time** (i.e. $0 < \Delta t(e) < +\infty$)
  - For example:
    1. Find a swan
    2. Check the color
    3. If black terminate successfully otherwise go to 1
- Def: an **experimental observation** is a tuple $o =< s, e >$ where s is a **statement that can be verified** by the experimental test e: s is true if and only if the experimental test e is successful
  - For example < “There are black swans”, “Find a swan, …”>

Algebra of experimental observations

- Now we want to understand how experimental observations behave under logical operations:
  - Negation/logical NOT
  - Conjunction/logical AND
  - Disjunction/logical OR

Negation/Logical NOT

- **Note: the negation of an experimental observation is not necessarily an experimental observation**
  - Being able to verify a statement in finite time does not imply the ability to verify its negation in finite time
  - Non-verification is not verification of the negation. Not finding black swans does not verify “there are no black swans”
- This idea has been intuitively present in the scientific community
  - James Randi’s “You can’t prove a negative”: pushing a few reindeer off the Empire State Building doesn’t prove they can’t fly
  - “Absence of evidence is not evidence of absence”
- This formalizes that intuition more precisely

Negation/Logical NOT

- But negation still gives us something!
- Def: an **experimental counter-observation** is a tuple $o^C = < s, e >^C$ where s is a **statement that can be refuted** by the experimental test e: s is false if and only if the experimental test e is successful
- **The negation of an experimental observation is an experimental counter-observation**
  - Being able to verify a statement s allows us to refute the statement $\neg s$
  - The negation of a negation is the original observation
- In this sense, observations and counter-observations are dual to each other, so we can concentrate on the former

Conjunction/Logical AND

- Def: the **conjunction of a finite number of observations** $\bigwedge_{i=1}^{n} o_i = \bigwedge_{i=1}^{n} < s_i, e_i > = < s, e >$ is the experimental observation where
  - $s = \bigwedge_{i=1}^{n} s_i$, the **conjunction of the statements**
  - $e = e_{\wedge}(\{e_i\}_{i=1}^{n})$, the experimental test that runs all tests and is successful if and only if all tests are successful
- The overall test is successful only if all sub-tests are successful
  - Note: we cannot extend to countable conjunction as we would never terminate

Disjunction/Logical OR

- Def: the **disjunction of a countable number of observations** $\bigvee_{i=1}^{\infty} o_i = \bigvee_{i=1}^{\infty} < s_i, e_i > = < s, e >$ is the experimental observation where
  - $s = \bigvee_{i=1}^{\infty} s_i$, the **disjunction of the statements**
  - $e = e_{\vee}(\{e_i\}_{i=1}^{\infty})$, the experimental that successfully terminates once one test successfully terminates
- Here we can have countably many observations because we can terminate once one test is successful
- As unsuccessful tests may not terminate, though, we need to be clever in the implementation of $e_{\vee}$

Disjunction/Logical OR

- The idea is to run one test for one second, then two tests for two seconds and so on
  1. initialize n to 1
  2. for each i=1…n
     a) run test e_i for n seconds
     b) if e_i terminated successfully, terminate successfully
  3. increment n and go to step 2
- All tests are eventually run for an arbitrary length of time. If one test is successful, it will eventually be run and it will terminate $e_\vee$ in finite time

Algebra of experimental observations

- Experimental observations are
  - Not closed under negation/logical NOT
  - Closed under finite conjunction/logical AND (but not under countable)
  - Closed under countable disjunction/logical OR

Things we can do with this algebra

- We can define **mutually exclusive observations** if verifying one implies the other will never be verified. We can define the empty/zero observation as the one that is never verified.
- Given a set of experimental observations (sub-basis), we can always **close** it **under finite conjunction and countable disjunction**
- We can **define a basis** for such a set
  - A set of experimental observations that we can use to verify all other experimental observations
- That is: we can take many ideas from set theory and topology and apply them to experimental observations!

Experimental domain

- Note: if we have a set of observations and we want (at least in the infinite time limit) to be able to find all experimental observations that are verified, then we must have a countable basis
  - If there does not exist a countable basis, there will be observations we'll never be able to test
- Def: an **experimental domain** is a set of experimental observations, **closed under finite conjunction and countable disjunction**, that allows a **countable basis**
  - This represents the enumeration of all possible answers to a question that can be settled experimentally

Using experimental observations to identify elements from a set

EXPERIMENTAL DISTINGUISHABILITY

Observations and identifications

- Many experimental observations are about identifying an element from a set of possibilities
  - E.g. “Bob’s illness is malaria” “The position of the ball is 5.1±0.05 meters” “This fossilized animal was a bird”
- Let’s look more carefully at how this works

Experimental identification

- Suppose we have a **set $\boldsymbol{X}$ of** all possible elements (which we call **possibilities**) among which we want to identify an object.
- Def: a **verifiable set** $U \subseteq X$ is a subset of possibilities for which there exists an **associated experimental observation** $o =<$“The object is in U”, $e_{\in}(U) >$ where $e_{\in}(U)$ is an experimental test that succeeds if and only if the object to identify is an element of U. We call such an observation an **experimental identification**.
- Conversely: a **refutable set** $U \subseteq X$ is a subset of possibilities for which there exists an **associated experimental counter-observation** $o^C =<$“The object is in U”, $e_{\notin}(U) >^C$ where $e_{\notin}(U)$ is an experimental test that succeeds if and only if the object to identify is not an element of U.

Experimental identification

- And so:
  - the **complement of a verifiable set is a refutable set and vice-versa**
  - the **finite intersection of verifiable sets is a verifiable set**
  - the **countable union of verifiable sets is a verifiable set**
- For example, negation can be shown as:
  - Suppose $U \subseteq X$ is a verifiable set
  - By definition, there exists an $o =<$"The object is in U", $e_{\in}(U) >$
  - Take the negation $\neg o =< \neg$"The object is in U", $e_{\in}(U) >^C$: this is an experimental counter-observation
  - $\neg o =<$"The object is in U^C", $e_{\notin}(U^C) >^C$
  - U^C is a refutable set because it is associated with a counter-observation of the correct form

Experimental distinguishability

- Def: a set of elements X is **experimentally distinguishable** if the set of all possible experimental identifications forms an **experimental domain** where given two elements we **can always find** two **mutually exclusive observations** such that each element is compatible with only one observation
  - E.g. {“Cat”, “Sparrow”} -> {“x is a mammal”, “x is a bird”}

Hausdorff and second countable topology

- **The set $T(X)$ of all verifiable sets associated to a set X of experimentally distinguishable elements is a Hausdorff and second countable topology on X**
  - Since an experimental domain has a countable basis, $\mathrm{T}(X)$ has a countable basis
  - Since an experimental domain is closed under finite conjunction and countable disjunction, $\mathrm{T}(X)$ is closed under finite intersection and arbitrary union
    - Arbitrary unions can be written as countable disjunctions using the basis
  - Since the experimental domain contains at least two mutually exclusive experimental observations, $\mathrm{T}(X)$ contains the empty set
  - Since each possibility is at least compatible with one experimental observation, the union of all basis elements is the verifiable set X
  - Since for each two elements we can find two mutually exclusive observations, each compatible with one, $\mathrm{T}(X)$ is Hausdorff

Cardinality of the elements

- This already has a very general implication: the cardinality of possibilities among which we can experimentally distinguish is at most that of the continuum
  - Euclidean space $\mathbb{R}^n$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$, all open sets in $\mathbb{R}$, are all mathematical objects that can represent experimentally distinguishable objects
  - All functions from $\mathbb{R}$ to $\mathbb{R}$, all subsets of $\mathbb{R}$, are not objects that can represent experimentally distinguishable objects
- Naturally, not everything with the right cardinality corresponds to experimentally distinguishable elements: one needs to find an experimentally meaningful topology

Establishing experimental relationships between elements

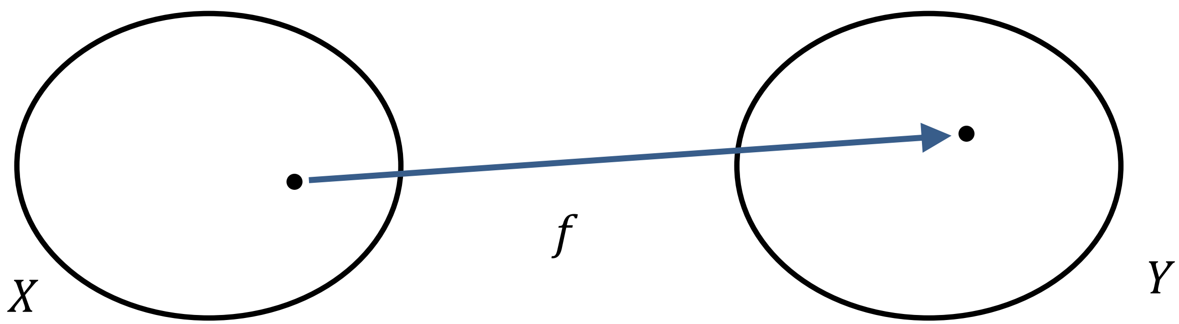
RELATIONSHIPS AND EXPERIMENTAL DISTINGUISHABILITY

Relationships between experimentally distinguishable elements

- Another important category of experimental observations is one that relates two different elements
  - E.g. "The person Bob is 1.74 ±0.005 m tall" "$E = \frac{1}{2}mv^2$" "The dinosaur Tyrannosaurus rex lived between 65 and 70 million years ago"
  - In fact, the real aim of scientific inquiry is finding such relationships
- We need to:
  - Define and study relationships
    - there are two ways and we show they are equivalent
  - We need to make sure the relationships are themselves experimentally distinguishable (or we can't verify them)

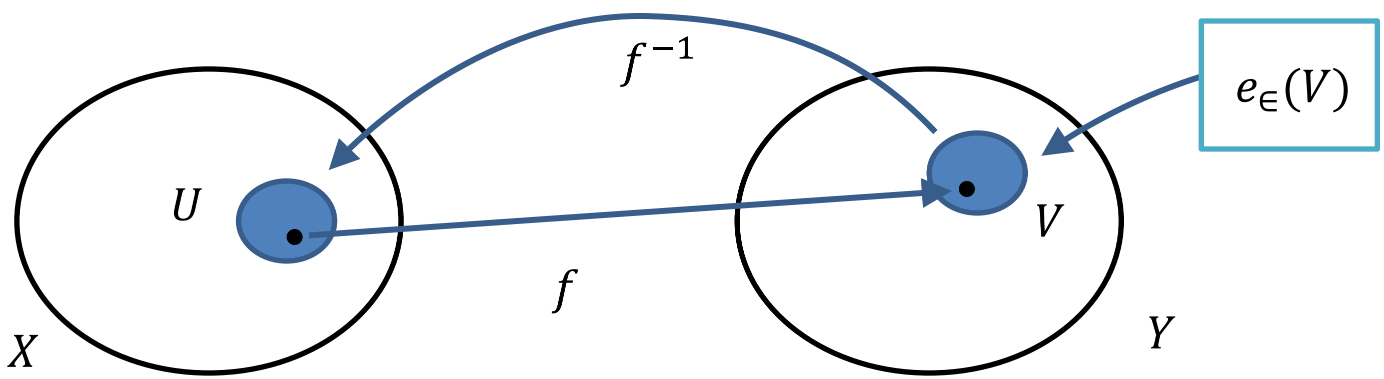
Relationships between experimentally distinguishable elements

- Suppose we have two experimentally distinguishable sets X and Y and a map between them $f: X \to Y$ that represents an experimental relationship (i.e. it can be used in an experimental test)



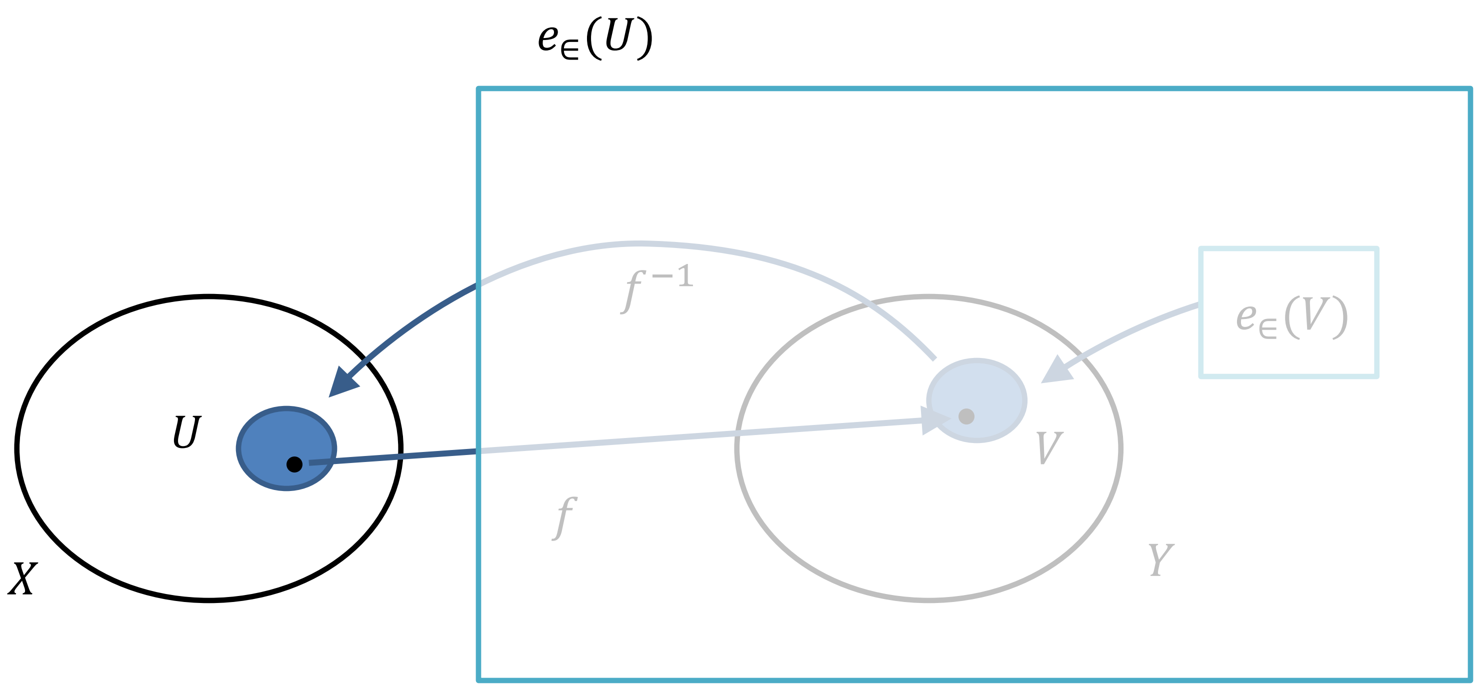
Relationships between experimentally distinguishable elements

- If we are able to test if $y \in V \subseteq Y$, then we can test if $x \in U = f^{-1}(V) \subseteq X$
  - First map x to $y = f(x)$ then check $y \in V \subseteq Y$



Relationships between experimentally distinguishable elements

- If V is a verifiable set, $U = f^{-1}(V)$ is a verifiable set: f is a continuous function!
  - Only continuous functions can properly represent experimental relationships as they preserve experimental distinguishability



Relationships between experimentally distinguishable elements

- The previous definition is straightforward, but relies on the elements. We want to define the relationship based on the observations.
  - If “the height of the mercury column is between 24 and 25 mm” then “the temperature of the mercury column is between 24 and 25 degrees Celsius”
- We can define an experimental relationship between experimentally distinguishable elements as a function $g: \mathrm{T}(Y) \to \mathrm{T}(X)$ such that:
  - The relationship is compatible with conjunction/intersection and disjunction/union
  - $g(Y) = X$ and $g(\emptyset) = \emptyset$

Relationships between experimentally distinguishable elements

- Under those conditions, given $g\colon \mathrm{T}(Y) \to \mathrm{T}(X)$, one can show that there exists a unique continuous function $f\colon X \to Y$ such that $g(V) = f^{-1}(V)$ for all $V \in \mathrm{T}(Y)$
  - The two definitions are equivalent
- The main idea of the proof is that using Hausdorff we take intersections of open sets to pin down specific points
  - Extend g to the Borel algebra
  - Look at images of singletons

Continuity in physics

- This tells us why continuity is so important in physics: it preserves experimental distinguishability!
- A dynamical system that preserves experimental distinguishability is a continuous map
- A reversible dynamical system that preserves experimental distinguishability is a homeomorphism

Experimental distinguishability of experimental relationships

- Now we need to prove that experimental relationships are themselves experimentally distinguishable
- Let X and Y be two topological spaces.
  Let $C(X,Y)$ be set of continuous functions from X to Y.
  Let $\mathcal{B}(X) \subseteq T(X)$ and $\mathcal{B}(Y) \subseteq T(Y)$ be two bases of the respective spaces.
- We define the basis-to-basis topology $T(C(X,Y), \mathcal{B}(X), \mathcal{B}(Y))$ the topology generated by all sets of the form $V(U_X, U_Y) = \{f \in C(X,Y) \mid f(U_X) \subset U_Y\}$ where $U_X \in \mathcal{B}(X)$ and $U_Y \in \mathcal{B}(Y)$

Basis-to-basis topology preserves "Hausdorff and second countable"

- If X and Y are Hausdorff, the basis-to-basis topology is Hausdorff
- If X and Y are second countable, the basis-to-basis topology is second countable
  - If X and Y are second countable, the sub-basis that generates the basis-to-basis topology is countable and generates a countable basis

Putting it all together

- Sets of experimentally distinguishable elements are Hausdorff and second countable topological spaces
- Relationships between experimentally distinguishable elements are continuous functions and form themselves a set of experimentally distinguishable elements
- We can recursively create relationships of relationships: they too will be experimentally distinguishable and form Hausdorff and second countable topological spaces.
- The universe of discourse is closed!!!

Dictionary

Math concept	**Physical meaning**
Hausdorff, second-countable topological space	Space of experimentally distinguishable elements, whose points are the possibilities.
Open set	Verifiable set. We can verify experimentally that an object is within that set of possibilities.
Closed set	Refutable set. We can verify experimentally that an object is not within that set of possibilities.
Basis of a topology	A minimum set of observations we need to test in order to test all the others.
Discrete topological space	Set of possibilities that can be individually verified or refuted.
Standard topology on $\mathbb{R}$	The value can be measured only with finite precision.
Continuous transformation	A function that preserves experimental distinguishability.
Homeomorphism	A perfect equivalence between experimentally distinguishable spaces.

Conclusion

- **The application of topology in science is to capture experimental distinguishability**
- This insight allows us to understand why topological spaces and continuous functions are pervasive in physics and other domains
- The hope is that we can build upon these ideas to understand why other mathematical concepts (e.g. differentiability, measures, symplectic forms) are also fundamental in science
  - A better understanding of the concepts of today may lead to the new ideas of tomorrow